ORIGINAL PAPER

ON 4-DIMENSIONAL GOLDEN-WALKER STRUCTURES

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Abstract. *In this paper, we show a method of pure metrics construction on a semi-Riemannian 4-manifold of neutral signature with respect to Golden structures. As an illustration, by applying the method, we exhibit explicitly pure metrics on Walker 4-manifolds. Moreover, we present some examples for 4-dimensional Golden-Walker structures.*
Keywords: *Golden structure, Walker metric, Neutral metric, Pure metric.*

1. INTRODUCTION

Let (M_{2n}, g) be a $2n-$dimensional semi-Riemannian smooth manifold with the metric tensor field g, which is necessarily of neutral signature (n,n), and let $\Im_s^r(M_{2n})$ be the tensor field of M_{2n}, i.e. the field of all tensors of type (r,s) on M_{2n}.

A Golden structure on M_{2n} is an affinor field φ on M_{2n}, such that $\varphi^2 = \varphi + I$, where I is the identity tensor field on $\Im_0^1(M_{2n})$. For Golden structure φ, defined on a semi-Riemannian manifold (M_{2n}, g), if

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1}$$

for all $X, Y \in \Im_0^1(M_{2n})$, the semi-Riemannian metric g is called φ-compatible. The triple (M_{2n}, φ, g) is named a Golden semi-Riemannian manifold [3,5-7,15-18]. Moreover, such metrics are expressed as pure (or anti-Hermitian) metrics [4,8-11,14,19-24].

For a neutral metric g on a 4-dimensional manifold M_4, if there exists a 2-dimensional null plane and parallel distribution D on M_4 with respect to a neutral metric g, then a neutral metric g on M_4 is said to be a *Walker metric*. From Walker's theorem [25], there is, locally, a system of suitable coordinates (x^1, x^2, x^3, x^4) around any point of M_4, so that g has the following canonical form:

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$$g=(g_{ij})=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&a&c\\0&1&c&b\end{pmatrix} \qquad (2)$$

where a,b and c are some functions of the coordinates (x^1,x^2,x^3,x^4). The metric in Eq. (2) is the most generic form of Walker metrics on M_4. Hereafter, we show the coordinate tangent vectors by $\partial_i=\partial/\partial x^i$ $(i=1,2,3,4)$ and we use a subscript for the partial derivative, i.e. $h_i=\dfrac{\partial h}{\partial x^i}$, for any function h depending on (x^1,x^2,x^3,x^4). Such Walker metrics have been intensively investigated, as in [1,2,10,12,13,20,22].

The purpose of this paper is to show a method of Golden-Walker metric construction for the given Golden structure on a neutral 4-manifold. In Sec. 2, we review the basic information about a Golden structure for a neutral metric on a 4-manifold. In Sec. 3, we show how to construct pure metrics. In Sec. 4, we apply the proposed method to construct Golden-Walker structures in 2 different cases, which are instructive examples, that will be helpful for applying our method to various other neutral 4-manifolds.

2. GOLDEN STRUCTURES ON A NEUTRAL 4-MANIFOLD

In this section, we focus our attention on 4-dimensional semi-Riemannian manifolds of neutral signature $(++--)$. For the next step, it is appropriate to state a neutral metric g and the Golden structure φ in terms of an orthonormal frame $\{e_i\}$ $(i=1,\dots,4)$ of vectors, and its dual frame $\{e^j\}$ $(j=1,\dots,4)$ of 1-forms. Actually, the metric g can be given by;

$$g=\left(g(e_i,e_j)\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \qquad (3)$$

Two Golden structures, $\underset{1}{\varphi}$ and $\underset{2}{\varphi}$, can be written as

$$\underset{1}{\varphi}=(\underset{1}{\varphi}{}_i^j)=\begin{pmatrix}\frac{1}{2}(1+\sqrt{5})&0&0&0\\0&\frac{1}{2}(1+\sqrt{5})&0&0\\0&0&\frac{1}{2}(1-\sqrt{5})&0\\0&0&0&\frac{1}{2}(1-\sqrt{5})\end{pmatrix} \qquad (4)$$

$$\underset{2}{\varphi} = (\underset{2}{\varphi}_i^j) = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & 1 & 0 & -1 \\ 1 & \frac{1}{2}(1-\sqrt{5}) & 1 & 0 \\ 0 & -1 & \frac{1}{2}(1+\sqrt{5}) & 1 \\ 1 & 0 & 1 & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}. \quad (5)$$

Furthermore, note that the neutral metric g in Eq. (3) is pure with respect to the Golden structures $\underset{1}{\varphi}$ and $\underset{2}{\varphi}$, given by Eqs. (4) and (5).

3. CONSTRUCTION OF PURE METRICS

In this section, we determine the forms of pure metrics with respect to a pair $(\underset{1}{\varphi}, \underset{2}{\varphi})$ of Golden structures. Let (M_4, g) be a 4-dimensional semi-Riemannian manifold of neutral signature $(++--)$, which has the associated Golden structures $(\underset{1}{\varphi}, \underset{2}{\varphi})$. Hereafter, we turn our attention to the search for a new metric g that can be pure.

The pure metric g, which is given in Eq. (1), can be written as the matrix equation:

$$\varphi^T g = g\varphi \quad (6)$$

where φ^T is a transpose matrix of the matrix φ. We put:

$$g = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \quad (7)$$

and determine the components $g_{ij} = g(e_i, e_j)$ with respect to the orthonormal frame $\{e_i\}$ $(i = 1,...,4)$. With the substitution of Eqs. (4) and (7) into Eq. (6), for the metric g in Eq. (7), we obtain:

$$g_{13} = g_{14} = g_{23} = g_{24} = 0$$

and the other components are not zero. In this instance, a new pure metric, $g\ (= \underset{1}{g})$, is:

$$\underset{1}{g} = (\underset{1}{g}_{ij}) = \begin{pmatrix} \alpha & \delta & 0 & 0 \\ \delta & \beta & 0 & 0 \\ 0 & 0 & \gamma & \mu \\ 0 & 0 & \mu & \theta \end{pmatrix}$$

where $\underset{1}{g}_{11}=\alpha,\ \underset{1}{g}_{22}=\beta,\ \underset{1}{g}_{33}=\gamma,\ \underset{1}{g}_{44}=\theta,\ \underset{1}{g}_{12}=\delta,\ \underset{1}{g}_{34}=\mu$ and $\det \underset{1}{g}\neq 0$.

Next, we have:

Theorem 1. A metric of signature $(++--)$ on M_4 is pure if and only if it is of the form

$$\underset{1}{g}=(\underset{1}{g}_{ij})=\begin{pmatrix}\alpha & \delta & 0 & 0\\ \delta & \beta & 0 & 0\\ 0 & 0 & \gamma & \mu\\ 0 & 0 & \mu & \theta\end{pmatrix},\quad \det \underset{1}{g}\neq 0. \tag{8}$$

with respect to the orthonormal basis $\{e_i\}$ $(i=1,\dots,4)$.

With the substitution of Eqs. (5) and (7) into Eq. (6), one of the new pure metrics, $g(=\underset{2}{g})$, is obtained as:

$$\underset{2}{g}=(\underset{2}{g}_{ij})=\begin{pmatrix}0 & \alpha & -\frac{\sqrt{5}}{2}(\alpha-\beta) & \beta\\ \alpha & 0 & \beta & -\frac{\sqrt{5}}{2}(\alpha+\beta)\\ -\frac{\sqrt{5}}{2}(\alpha-\beta) & \beta & 0 & \alpha\\ \beta & -\frac{\sqrt{5}}{2}(\alpha+\beta) & \alpha & 0\end{pmatrix} \tag{9}$$

where $\underset{2}{g}_{12}=\underset{2}{g}_{34}=\alpha,\ \underset{2}{g}_{14}=\underset{2}{g}_{23}=\beta,\quad \underset{2}{g}_{13}=-\frac{\sqrt{5}}{2}(\alpha-\beta),\ \underset{2}{g}_{24}=-\frac{\sqrt{5}}{2}(\alpha+\beta),$ and $\det \underset{2}{g}\neq 0$ $(\alpha\neq\mp\beta)$.

Next, we have:

Theorem 2. A metric of signature $(++--)$ on M_4 is pure if and only if it is of the form:

$$\underset{2}{g}=(\underset{2}{g}_{ij})=\begin{pmatrix}0 & \alpha & -\frac{\sqrt{5}}{2}(\alpha-\beta) & \beta\\ \alpha & 0 & \beta & -\frac{\sqrt{5}}{2}(\alpha+\beta)\\ -\frac{\sqrt{5}}{2}(\alpha-\beta) & \beta & 0 & \alpha\\ \beta & -\frac{\sqrt{5}}{2}(\alpha+\beta) & \alpha & 0\end{pmatrix} \tag{10}$$

with respect to the orthonormal basis $\{e_i\}$ $(i=1,...,4)$ and where $\det \underset{2}{g} \neq 0$ $(\alpha \neq \mp\beta)$.

3.1. SOME EXAMPLES OF PURE METRICS

It is worthwhile to give some simple examples of pure metrics. From Eq. (8), two typical examples of pure metrics are written as follows:

$$\underset{1}{g} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{11}$$

or

$$\underset{1}{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{12}$$

The former corresponds to the case of trivial $\underset{1}{g}_{11} = \underset{1}{g}_{22} = \underset{1}{g}_{33} = \underset{1}{g}_{44} = 0,$ and nontrivial $\underset{1}{g}_{12} = \underset{1}{g}_{34} = 1$, and the latter to the other case of trivial $\underset{1}{g}_{12} = \underset{1}{g}_{33} = \underset{1}{g}_{44} = 0,$ and nontrivial $\underset{1}{g}_{11} = \underset{1}{g}_{22} = \underset{1}{g}_{34} = 1.$

Similarly, from Eq. (10), two typical examples of pure metrics are written as follows:

$$\underset{2}{g} = \begin{pmatrix} 0 & 1 & -\frac{\sqrt{5}}{2} & 0 \\ 1 & 0 & 0 & -\frac{\sqrt{5}}{2} \\ -\frac{\sqrt{5}}{2} & 0 & 0 & 1 \\ 0 & -\frac{\sqrt{5}}{2} & 1 & 0 \end{pmatrix} \tag{13}$$

or

$$\underset{2}{g}=\begin{pmatrix} 0 & 0 & \frac{\sqrt{5}}{2} & 1 \\ 0 & 0 & 1 & -\frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & 1 & 0 & 0 \\ 1 & -\frac{\sqrt{5}}{2} & 0 & 0 \end{pmatrix}. \tag{14}$$

The former corresponds to the case of trivial $\underset{2}{g}_{14}=\underset{2}{g}_{23}=0$ and nontrivial $\underset{2}{g}_{12}=\underset{2}{g}_{34}=1$, $\underset{2}{g}_{13}=\underset{2}{g}_{24}=-\frac{\sqrt{5}}{2}$, and the latter to the other case of trivial $\underset{2}{g}_{12}=\underset{2}{g}_{34}=0$, and nontrivial $\underset{2}{g}_{14}=\underset{2}{g}_{23}=1$, $\underset{2}{g}_{13}=-\underset{2}{g}_{24}=\frac{\sqrt{5}}{2}$.

4. GOLDEN-WALKER STRUCTURES

4.1. CHANGE OF COORDINATES MATRIX CHOICE 1

Let (M_4,g) be a Walker 4-manifold with the Walker metric g, which is given in Eq. (2). If $\{e_1,e_2,e_3,e_4\}$ and $\{\partial_1,\partial_2,\partial_3,\partial_4\}$ are two orthonormal frames, matrix $A=\left(A_j^i\right)$ of the change of coordinates satisfies:

$$g=A^T g' A \tag{15}$$

where matrix A^T is the transpose matrix of matrix A.

With the substitution of Eqs. (2) and (3) into Eq. (15), one of the matrices, A, which we apply in the present analysis, is:

$$A=\left(A_j^i\right)=\begin{pmatrix} 0 & -\frac{1}{2}(1-a) & 0 & \frac{1}{2}(1+a) \\ \frac{1}{2}(1-b) & c & -\frac{1}{2}(1+b) & c \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{16}$$

Moreover, for affinors, matrix $A=\left(A_j^i\right)$ of the change of coordinates satisfies:

$$\varphi=A^{-1}\varphi' A \tag{17}$$

where matrix A^{-1} is the inverse matrix of matrix A.

The inverse of the matrix in Eq. (16), A^{-1}, is given by:

$$A^{-1}=\begin{pmatrix} 0 & 1 & c & \frac{1}{2}(1+b) \\ -1 & 0 & -\frac{1}{2}(1+a) & 0 \\ 0 & -1 & -c & \frac{1}{2}(1-b) \\ 1 & 0 & -\frac{1}{2}(1-a) & 0 \end{pmatrix}. \qquad (18)$$

With the substitution of Eqs. (4), (16) and (18) into Eq. (17), the Golden structure in Eq. (4) has the following form:

$$\underset{1}{\varphi'}=(\underset{1}{\varphi'}{}_i^j)=\begin{pmatrix} \frac{1}{2}(1-\sqrt{5}a) & 0 & \frac{\sqrt{5}}{4}(1-a^2) & 0 \\ -\sqrt{5}c & \frac{1}{2}(1-\sqrt{5}b) & -\frac{\sqrt{5}}{2}c(a+b) & \frac{\sqrt{5}}{4}(1-b^2) \\ \sqrt{5} & 0 & \frac{1}{2}(1+\sqrt{5}a) & 0 \\ 0 & \sqrt{5} & \sqrt{5}c & \frac{1}{2}(1+\sqrt{5}b) \end{pmatrix}. \qquad (19)$$

Similarly, with the substitution of Eqs. (5), (16) and (18) into Eq. (17), the Golden structure in Eq. (5) has the following form:

$$\underset{2}{\varphi'}=(\underset{2}{\varphi'}{}_i^j)=\begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & 0 & 0 & a \\ -2 & \frac{1}{2}(1+\sqrt{5}) & \sqrt{5}c-a & 2c \\ 0 & 0 & \frac{1}{2}(1-\sqrt{5}) & -2 \\ 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix} \qquad (20)$$

Even though we can obtain a form of pure metrics easily on the basis of Theorem 1, we now exhibit the typical examples given in Eqs. (11) and (12).

Proposition 1. With matrix $A=\left(A_j^i\right)$ in Eq. (16), the typical examples of pure metrics in Eqs. (11) and (12) on a Walker 4 −manifold are written, respectively, as follows:

$$\underset{1}{g'}=\begin{pmatrix} 0 & -2 & -2c & -b \\ -2 & 0 & -a & 0 \\ -2c & -a & -2ac & -\frac{1}{2}(1+ab) \\ -b & 0 & -\frac{1}{2}(1+ab) & 0 \end{pmatrix} \quad (21)$$

$$\underset{1}{g'}=\begin{pmatrix} 1 & -1 & \frac{1}{2}(1+a)-c & \frac{1}{2}(1-b) \\ -1 & 1 & \frac{1}{2}(1-a)+c & \frac{1}{2}(1+b) \\ \frac{1}{2}(1+a-2c) & \frac{1}{2}(1-a+2c) & \frac{1}{4}(1+a)^2+c^2+c(1-a) & \frac{1}{2}c(1+b)-\frac{1}{4}(1-a)(1-b) \\ \frac{1}{2}(1-b) & \frac{1}{2}(1+b) & \frac{1}{2}c(1+b)-\frac{1}{4}(1-a)(1-b) & \frac{1}{4}(1+b)^2 \end{pmatrix} \quad (22)$$

Proof. The proof is straightforward using Eqs. (15) and (16). □

Now, we have two examples for $(\underset{1}{\varphi'},\underset{1}{g'})$ Golden structures on a Walker 4-manifold (for simplicity, Golden-Walker structures); one obtained from Eqs. (19) and (21) and the other from Eqs. (19) and (22).

Similarly, we have the simple forms of pure metrics in Eqs. (13) and (14).

Proposition 2. With matrix $A=\left(A_j^i\right)$ in Eq. (16), the typical examples of pure metrics in Eqs. (13) and (14) on a Walker 4-manifold are written, respectively, as follows:

$$\underset{2}{g'}=\begin{pmatrix} \sqrt{5} & -2 & \frac{\sqrt{5}}{2}a-2c & -b \\ -2 & \sqrt{5} & \sqrt{5}c-a & \frac{\sqrt{5}}{2}b \\ \frac{\sqrt{5}}{2}a-2c & \sqrt{5}c-a & \sqrt{5}c^2-2ac-\frac{\sqrt{5}}{4}(1-a^2) & \frac{\sqrt{5}}{2}bc-\frac{1}{2}(1-ab) \\ -b & \frac{\sqrt{5}}{2}b & \frac{\sqrt{5}}{2}bc-\frac{1}{2}(1-ab) & -\frac{\sqrt{5}}{4}(1-b^2) \end{pmatrix} \quad (23)$$

$$\underset{2}{g'}=\begin{pmatrix} \sqrt{5} & 2 & \frac{\sqrt{5}}{2}a+2c & b \\ 2 & -\sqrt{5} & -\sqrt{5}c+a & -\frac{\sqrt{5}}{2}b \\ \frac{\sqrt{5}}{2}a+2c & -\sqrt{5}c+a & -\sqrt{5}c^2+2ac-\frac{\sqrt{5}}{4}(1-a^2) & -\frac{\sqrt{5}}{2}bc-\frac{1}{2}(1-ab) \\ b & -\frac{\sqrt{5}}{2}b & -\frac{\sqrt{5}}{2}bc-\frac{1}{2}(1-ab) & \frac{\sqrt{5}}{4}(1-b^2) \end{pmatrix}. \quad (24)$$

Thus, we have two other examples for $(\underset{2}{\varphi}', \underset{2}{g}')$ Golden-Walker structures; one obtained from Eqs. (20) and (23), and the other from Eqs. (20) and (24).

4.2. CHANGE OF COORDINATES MATRIX CHOICE 2

We give another example of Golden-Walker structure constructed in terms of a different matrix A as follows:

$$A = \left(A_j^i\right) = \begin{pmatrix} c & \frac{1}{2}(1-a) & c & -\frac{1}{2}(1+a) \\ \frac{1}{2}(1-b) & -c & -\frac{1}{2}(1+b) & -c \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \qquad (25)$$

The inverse of the matrix in Eq. (25), A^{-1}, is obtained as:

$$A^{-1} = \begin{pmatrix} 0 & 1 & c & \frac{1}{2}(1+b) \\ 1 & 0 & \frac{1}{2}(1+a) & -c \\ 0 & -1 & -c & \frac{1}{2}(1-b) \\ -1 & 0 & \frac{1}{2}(1-a) & c \end{pmatrix}. \qquad (26)$$

With the substitution of Eqs. (4), (25) and (26) into Eq. (17), the Golden structure in Eq. (4) has the following form:

$$\underset{1}{\varphi}'' = (\underset{1}{\varphi}''^{j}_{i}) = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}a) & \sqrt{5}c & \sqrt{5}c^2 + \frac{\sqrt{5}}{4}(1-a^2) & \frac{\sqrt{5}}{2}c(a+b) \\ -\sqrt{5}c & \frac{1}{2}(1-\sqrt{5}b) & -\frac{\sqrt{5}}{2}c(a+b) & \sqrt{5}c^2 + \frac{\sqrt{5}}{4}(1-b^2) \\ \sqrt{5} & 0 & \frac{1}{2}(1+\sqrt{5}a) & -\sqrt{5}c \\ 0 & \sqrt{5} & \sqrt{5}c & \frac{1}{2}(1+\sqrt{5}b) \end{pmatrix}. \qquad (27)$$

Similarly, with the substitution of Eqs. (5), (25) and (26) into Eq. (17), the Golden structure in Eq. (5) has the following form:

$$\underset{2}{\varphi''} = (\underset{2}{\varphi''}{}_i^j) = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & 0 & 0 & \sqrt{5}c - a \\ 2 & \frac{1}{2}(1+\sqrt{5}) & \sqrt{5}c + a & -4c \\ 0 & 0 & \frac{1}{2}(1-\sqrt{5}) & 2 \\ 0 & 0 & 0 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}. \quad (28)$$

As in the previous subsection, we show simple forms constructed by change of coordinates choice for matrix A on pure metrics.

Proposition 3. With matrix $A = \left(A_j^i\right)$ in Eq. (25), the typical examples of pure metrics in Eqs. (11) and (12) on a Walker 4-manifold are written, respectively, as follows:

$$\underset{1}{g''} = \begin{pmatrix} 0 & 2 & 2c & b \\ 2 & 0 & a & -2c \\ 2c & a & 2ac & -2c^2 + \frac{1}{2}(1+ab) \\ b & -2c & -2c^2 + \frac{1}{2}(1+ab) & -2cb \end{pmatrix} \quad (29)$$

$$\underset{1}{g''} = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+a)+c & -\frac{1}{2}(1-b)-c \\ 1 & 1 & -\frac{1}{2}(1-a)+c & \frac{1}{2}(1+b)-c \\ \frac{1}{2}(1+a)+c & -\frac{1}{2}(1-a)+c & \frac{1}{4}(1+a)^2 + c^2 - c(1-a) & \frac{1}{2}c(b-a) + \frac{1}{4}(1-a)(1-b) - c^2 \\ -\frac{1}{2}(1-b)-c & \frac{1}{2}(1+b)-c & \frac{1}{2}c(b-a) + \frac{1}{4}(1-a)(1-b) - c^2 & c^2 + \frac{1}{4}(1+b)^2 + c(1-b) \end{pmatrix}. \quad (30)$$

Proof. The proof is straightforward using Eqs. (15) and (25). □

In that case, we have two new examples for $(\underset{1}{\varphi''}, \underset{1}{g''})$ Golden-Walker structures; one obtained from Eqs. (27) and (29), and the other from Eqs. (27) and (30).

Similarly, we have the typical examples of pure metrics in Eqs. (13) and (14).

Proposition 4. With matrix $A = \left(A_j^i\right)$ in Eq. (25), the typical examples of pure metrics in Eqs. (13) and (14) on a Walker 4-manifold are written, respectively, as follows:

$$\underset{2}{g''}=\begin{pmatrix} \sqrt{5} & 2 & \frac{\sqrt{5}}{2}a+2c & -\sqrt{5}c+b \\ 2 & \sqrt{5} & \sqrt{5}c+a & \frac{\sqrt{5}}{2}b-2c \\ \frac{\sqrt{5}}{2}a+2c & \sqrt{5}c+a & \sqrt{5}c^2+2ac-\frac{\sqrt{5}}{4}(1-a^2) & \frac{\sqrt{5}}{2}c(b-a)-2c^2+\frac{1}{2}(1+ab) \\ -\sqrt{5}c+b & \frac{\sqrt{5}}{2}b-2c & \frac{\sqrt{5}}{2}c(b-a)-2c^2+\frac{1}{2}(1+ab) & \sqrt{5}c^2-2cb-\frac{\sqrt{5}}{4}(1-b^2) \end{pmatrix} \quad (31)$$

$$\underset{2}{g''}=\begin{pmatrix} \sqrt{5} & -2 & \frac{\sqrt{5}}{2}a-2c & -\sqrt{5}c-b \\ -2 & -\sqrt{5} & -\sqrt{5}c-a & \frac{\sqrt{5}}{2}b-2c \\ \frac{\sqrt{5}}{2}a-2c & -\sqrt{5}c-a & -\sqrt{5}c^2-2ac-\frac{\sqrt{5}}{4}(1-a^2) & -\frac{\sqrt{5}}{2}c(a+b)+2c^2+\frac{1}{2}(1-ab) \\ -\sqrt{5}c-b & -\frac{\sqrt{5}}{2}b+2c & -\frac{\sqrt{5}}{2}c(a+b)+2c^2+\frac{1}{2}(1-ab) & \sqrt{5}c^2+2cb+\frac{\sqrt{5}}{4}(1-b^2) \end{pmatrix}. \quad (32)$$

Next, we have two other examples for $(\underset{2}{\varphi''},\underset{2}{g''})$ Golden-Walker structures: one obtained from Eqs. (28) and (31), and the other from Eqs. (28) and (32).

We thus have shown how to construct Golden-Walker structures on neutral 4-manifolds.

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