ORIGINAL PAPER

THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A SPACELIKE CURVE WITH NULL BINORMAL IN MINKOWSKI 3-SPACE

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Abstract. *In this study, we dealt with the natural lift curves of the spherical indicatrices of a spacelike curve with null binormal.Furthermore, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle* $T(S_1^2)$ *and* $T(\Lambda)$.

Keywords: *Natural Lift, Geodesic Sprays.*

1. INTRODUCTION

Thorpe gave the concepts of the natural lift curve and geodesic spray in 1979. Thorpe provied the natural lift $\bar{\alpha}$ of the curve α is an integralcurve of the geodesic spray iff α is an geodesic on M in 1979. Çalışkan, Sivridağ and Hacısalihoğlu studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in 1984.They gave some interesting results about the original curve were obtaied, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$ in 1984. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in 2011.The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in 2011. Walrave characterized the curve with constant curvature in Minkowski 3-space in 1995.

Let Minkowski 3-space IR_1^3 be the vector space IR^3 equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2,$$

where $X = (x_1, x_2, x_3) \in IR^3$.

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A vector $X = (x_1, x_2, x_3) \in$ IR^3 is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in IR_1^3 where t is a pseudo-arclength parameter, can be locally timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathrm{IR}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (*resp.* $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (*resp.* $x_1 < 0$). The norm of a vector X is defined by [1]

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$

The Lorentzian sphere of radius 1 in IR_1^3 is given by [1]

$$S_1^2 = \{X = (x_1, x_2, x_3) \in \mathrm{IR}_1^3 \ : \ g(X,X) = 1\}$$

The lightlike (null) cone in IR_1^3 is given by [1]

$$\Lambda = \{X = (x_1, x_2, x_3) \in \mathrm{IR}_1^3 - \{0\} : \ g(X,X) = 0\}$$

We denote the moving Frenet frame along the curve α by $\{T(t), N(t), B(t)\}$, where T, N and B are the tangent, the principal normal and the binormal vector of the curve α , respectively.

(i) Let α be a unit speed timelike space curve with curvature κ and torsion τ and Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then, Frenet formulas are given by [5]

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = -\tau N.$$

(ii) Let α be a unit speed spacelike space curve with a spacelike binormal. For the Frenet vector fields we assume that T and B are spacelike vector fields and N is a timelike vector field .Then, Frenet formulas are given by [5]

$$\dot{T} = \kappa N, \dot{N} = \kappa T + \tau B,\ \dot{B} = \tau N.$$

(iii) Let α be a unit speed spacelike space curve with a timelike binormal. We assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by [5]

$$\dot{T} = \kappa N, \dot{N} = -\kappa T + \tau B,\ \dot{B} = \tau N.$$

(iv) Let α be a unit speed spacelike space curve with a null binormal. We assume that T is spacelike vector fields and B and N are null vector field. Then, Frenet formulas are given by [5]

$$\dot{T} = \kappa N, \dot{N} = \tau N, \; \dot{B} = -\kappa T - \tau B.$$

(**v**) Let α be a unit speed null space curve. We assume that T and B are null vector fields and N is a spacelike vector field. Then, Frenet formulas are given by [5]

$$\dot{T} = \kappa N, \dot{N} = \tau T - \kappa B, \dot{B} = -\tau N.$$

Theorem 1.1. Let α be a unit speed spacelike space curve with a null binormal. Then, we have

(1) $\kappa = 0$ if and only if α is a part of a spacelike straight line,

(2) $\kappa = 1$ and $\tau = 0$ if and only if α is a part of a planer curve with parameterrisation,

$$\alpha(s) = \left(\frac{s^2}{2}, s, \frac{s^2}{2} \right),$$

(3) $\kappa = 1$ and $\tau = constant \neq 0$ if and only if α is a part of a planer curve with parameterrisation,

$$\alpha(s) = \frac{1}{\tau^2}\left(\cosh(\tau s) + \sinh(\tau s), \tau^2 s, \cosh(\tau s) + \sinh(\tau s)\right), [5].$$

Definition 1.1. Let M be a hypersurface in IR_1^3 and $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \; (\text{for all } t \in I),$$

where X is a smooth tangent vector field on M, [1]. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M),$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition 1.2. For any parametrized curve $\alpha : I \to M$, $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t) \right) = \dot{\alpha}(t)|_{\alpha(t)}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)|_{\alpha(t)} \right) = D_{\dot{\alpha}(t)} \dot{\alpha}(t),$$

where D is the Levi-Civita connection on IR_1^3, [4].

Definition 1.3. A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$

$$X(V) = +\varepsilon g(S(V), V)N,$$

where $\varepsilon = g(N,N)$,[4] .

Theorem 1.2. The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M , [4].

2. THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A SPACELIKE CURVE WITH NULL BINORMAL IN MINKOWSKI 3-SPACE

Let D , $\bar{D}$ and $\tilde{\tilde{D}}$ be Levi-Civita connections on IR_1^3 , S_1^2 and Λ respectively and ξ be a unit normal vector field of S_1^2 and Λ . Then Gauss Equations are given by the followings

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)\xi, \quad D_X Y = \tilde{\tilde{D}}_X Y + \varepsilon g(S(X), Y)\xi,$$

where $\varepsilon = g(\xi,\xi)$ and S is the shape operator of S_1^2 and Λ and

$$S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let the natural lift curve of the spherical indicatrix of a unit speed spacelike curve α with null binormal.

The natural lift of the spherical indicatrix of the tangent vector of α

Let α_T be the spherical indicatrix of tangent vectors of α and $\overline{\alpha_T}$ be the natural lift of the curve α_T . If $\overline{\alpha_T}$ is an integral curve of the geodesic spray, then from **Theorem 1.2** we have

$$\bar{D}_{\dot{\alpha}_T} \dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = \bar{D}_{\dot{\alpha}_T} \dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)\xi \, ,$$

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = \bar{D}_{\dot{\alpha}_T} \dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)T$$

$$\varepsilon = g(\xi,\xi) = g(T,T) = 1$$

$$g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right) = -g\left(\dot{\alpha}_T, \dot{\alpha}_T\right) = -[g(\kappa N, \kappa N)] = -\kappa^2 g(N,N) = 0$$

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = 0$$

$$D_{\dot{\alpha}_T}(\kappa N) = \frac{ds}{ds_T}\left[(\dot{\kappa} + \kappa\tau)N\right] = 0$$

$$\frac{ds}{ds_T} = \frac{1}{\|\kappa N\|}, g(N, N) = 0$$

Proposition 2. 1. Since the natural lift $\overline{\alpha_T}$ of α_T is an integral curve of the geodesic on the tangent bundle $T(S_1^3)$, therefore there is no such curve satisfying this condition.

The natural lift of the spherical indicatrix of the principal normal vector of α

Let α_N be the spherical indicatrix of tangent vectors of α and $\overline{\alpha_N}$ be the natural lift of the curve α_N . If $\overline{\alpha_N}$ is an integral curve of the geodesic spray, then because of **Theorem 1.2** we have

$$\tilde{\tilde{D}}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \tilde{\tilde{D}}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right), \dot{\alpha}_N\right)\xi ,$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \tilde{\tilde{D}}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right), \dot{\alpha}_N\right)N$$

$$\varepsilon = g(\xi, \xi) = g(N, N) = 0$$

$$g\left(S\left(\dot{\alpha}_N\right), \dot{\alpha}_N\right) = -g\left(\dot{\alpha}_N, \dot{\alpha}_N\right) = -\left[g(\tau N, \tau N)\right] = -\tau^2 g(N, N) = 0$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

$$D_{\dot{\alpha}_N}(\tau N) = \frac{ds}{ds_N}\left[(\dot{\tau} + \tau^2)N\right] = 0$$

$$\frac{ds}{ds_N} = \frac{1}{\|\tau N\|}, g(N, N) = 0$$

Proposition 2.2. Since the natural lift $\overline{\alpha_N}$ of α_N is an integral curve of the geodesic on the tangent bundle $T(\Lambda)$, therefore there is no such curve satisfying this condition.

The natural lift of the spherical indicatrix of the binormal vector of α

Let α_B be the spherical indicatrix of tangent vectors of α and $\overline{\alpha_B}$ be the natural lift of the curve α_B . If $\overline{\alpha_B}$ is an integral curve of the geodesic spray, then by using **Theorem 1.2** we have

$$\tilde{\tilde{D}}_{\dot{\alpha}_B}\dot{\alpha}_B = 0$$

that is

$$D_{\dot{\alpha}_B}\dot{\alpha}_B = \tilde{\tilde{D}}_{\dot{\alpha}_B}\dot{\alpha}_B + \varepsilon g\left(S\left(\dot{\alpha}_B\right),\dot{\alpha}_B\right)\xi,$$

$$D_{\dot{\alpha}_B}\dot{\alpha}_B = \tilde{\tilde{D}}_{\dot{\alpha}_B}\dot{\alpha}_B + \varepsilon g\left(S\left(\dot{\alpha}_B\right),\dot{\alpha}_B\right)B$$

$$\varepsilon = g(\xi,\xi) = g(B,B) = 0$$

$$g\left(S\left(\dot{\alpha}_B\right),\dot{\alpha}_B\right) = -g\left(\dot{\alpha}_B,\dot{\alpha}_B\right) = -\left[g(-\kappa T-\tau B,-\kappa T-\tau B)\right]$$

$$= -\left[\kappa^2 g(T,T)+\tau^2 g(B,B)\right] = -\kappa^2$$

$$D_{\dot{\alpha}_B}\dot{\alpha}_B = 0$$

$$D_{\dot{\alpha}_B}(-\kappa T-\tau B) = \frac{ds}{ds_B}\left[-\dot{\kappa}T-\kappa(\kappa N)-\dot{\tau}B-\tau(-\kappa T-\tau B)\right]$$

$$= \left[(-\dot{\kappa}+\kappa\tau)T+\kappa^2 N+\left(-\dot{\tau}+\tau^2\right)B\right]\frac{ds}{ds_B}$$

$$= 0$$

$$\frac{ds}{ds_B} = \frac{1}{\|(-\kappa T-\tau B)\|} = \frac{1}{\kappa}, g(B,B)=0, g(T,T)=1$$

$$\frac{-\dot{\kappa}+\kappa\tau}{\kappa}T+\frac{\kappa^2}{\kappa}N+\left(\frac{-\dot{\tau}+\tau^2}{\kappa}\right)B = 0$$

Proposition 2.3. Since the natural lift $\bar{\alpha}_B$ of α_B is an integral curve of the geodesic on the tangent bundle $T(\Lambda)$, therefore there is no such curve satisfying this condition.

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