ORIGINAL PAPER

A NEW EPIMORPHISM OF MATRIX COALGEBRAS

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Abstract. *In this paper we construct a new morphism of matrix coalgebras, than we prove that it is an epimorphism and also that its kernel is a coideal in the matrix coalgebra. Finally we give a generalization on this type of morphism between two matrix coalgebras.*
Keywords: *coalgebra, matrix coalgebra, coideal, morphism, kernel*

1. INTRODUCTION AND PRELIMINARIES

Let k be a field. The aim of this paper is to construct an epimorphism between two coalgebras, namely the matrix coalgebra $M^C(n,k)$ and the coalgebra $T_n(k)$, the second is obtained in a similar way as the first one, where the comultiplication and the counity are those from the incidence coalgebra associated to the totally ordered set $X = \{1 < 2 < ... < n\}$ of all positive integers less than n, where $n \in \mathbf{N}^*$ is a positive integer.

Remember that [2] if $(X, \leq)$ is an intervally finite partially ordered set, means that the interval $[x, y] = \{z / x \leq z \leq y\}$ is a finite set for any $x \leq y$, then the incidence k - coalgebra of X is the k – linear space denoted by $IC(X)$ is a space with basis $\{f_{x,y} \,/\, x, y \in X, x \leq y\}$, where $f_{x,y} = [x, y]$, having the comultiplication Δ and the counity ε :

$$\Delta(f_{x,y}) = \sum_{x \leq z \leq y} f_{x,z} \otimes f_{z,y} \Leftrightarrow \Delta([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y], \text{ respectively}$$

$$\varepsilon(f_{x,y}) = \delta_{x,y} \Leftrightarrow \varepsilon([x, y]) = \delta_{x,y}.$$

Also, in [1] is presented the matrix coalgebra $M^C(n,k)$, where for any considered positive integer n, $M^C(n,k)$ is a k – linear space of dimension n^2 with basis $(e_{ij})_{1\leq i,j\leq n}$, where $e_{ij} \in M_n(k)$ is the matrix having 1 on the intersection of the line i and the column j as for the rest 0. The comultiplication Δ_n and the counity ε_n in $M^C(n,k)$ are defined by:

$$\Delta_n(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj} \text{ and } \varepsilon_n : M^C(n,k) \to k \;, \varepsilon_n(e_{ij}) = \delta_{ij}.$$

2. THE CONSTRUCTION OF A NEW EPIMORPHISM OF MATRIX COALGEBRAS

Now, let's consider an arbitrary positive integer $n \in \mathbf{N}^*$ and $X = \{1,2,..., n\}$ the set of all positive integers less than n ordered by $\leq$, so X is a totally ordered set. Much more, to any

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pair (i,j) with $i \le j$, $i, j \in X$, we can associate an interval denoted by f_{ij}, $f_{ij} := [i,j]$, from the set S of all intervals of X, where $[i, j] = \{k \in X \,/\, i \le k \le j\}$. With all these notations we can associate to X a matrix below:

$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ 0 & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f_{nn} \end{pmatrix}.$$

Also, let $T_n(k)$ be the k – linear space of dimension $\frac{n(n+1)}{2}$ with the base $f_{ij} = [i, j]$, where $i \le j$ in X. Then, the triplet $(T_n(k), \Delta, \varepsilon)$ is a coalgebra, where the comultiplication and the counity are those from the incidence coalgebra of X :

$$\Delta(f_{ij}) = \Delta([i,j]) = \sum_{i \le k \le j} [i,k] \otimes [k,j] = \sum_{i \le k \le j} f_{ik} \otimes f_{kj} \text{ and } \varepsilon(f_{ij}) = \varepsilon([i,j]) = \delta_{ij}.$$

Now, we can define the map $\varphi : M^C(n,k) \to T_n(k)$ by $\varphi(e_{ij}) = \begin{cases} f_{ij}, & if \ \ i \le j \\ 0, & else \end{cases}$.

Proposition. The map φ is an epimorphism of coalgebras.

Proof. From the way it is defined φ it is obvious that φ is a k - linear application. Let's prove now that φ is a morphism of coalgebras, means that we have to prove the relations:

$$(\varphi \otimes \varphi) \circ \Delta_n = \Delta \circ \varphi \text{ and } \varepsilon \circ \varphi = \varepsilon_n.$$

So, for every $i \le j$ we have:

$$(\varphi \otimes \varphi)(\Delta_n(e_{ij})) = (\varphi \otimes \varphi)(\sum_{k=1}^{n} e_{ik} \otimes e_{kj}) = \sum_{k=1}^{n} \varphi(e_{ik}) \otimes \varphi(e_{kj}) =$$

$$= \sum_{k=i}^{j} f_{ik} \otimes f_{kj} = \sum_{i \le k \le j} [i,k] \otimes [k,j] = \Delta([i,j]) = \Delta(f_{ij}) = \Delta(\varphi(e_{ij})).$$

In the case $i > j$ it is obvious that $(\varphi \otimes \varphi)(\Delta_n(e_{ij})) = 0 = \Delta(\varphi(e_{ij}))$.

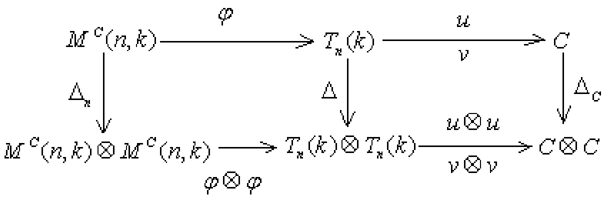
So we have the first relation $(\varphi \otimes \varphi) \circ \Delta_n = \Delta \circ \varphi$.

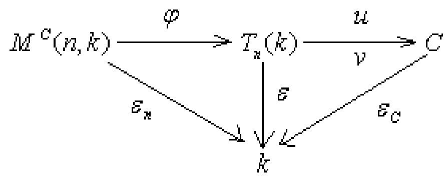
Also, for every $i \le j$ we have $\varepsilon(\varphi(e_{ij})) = \varepsilon(f_{ij}) = \varepsilon([i,j]) = \delta_{ij} = \varepsilon_n(e_{ij})$, and for $i > j$, $\varepsilon(\varphi(e_{ij})) = \varepsilon(0) = 0 = \delta_{ij} = \varepsilon_n(e_{ij})$. In this way we obtained the second relation, $\varepsilon \circ \varphi = \varepsilon_n$.

In conclusion, φ is a morphism of coalgebras.

Now, let's prove that φ an epimorphism, which is equivalent to prove that for any coalgebra $(C,\Delta_C,\varepsilon_C)$ and for any two morphisms $u,v:T_n(k)\to C$ with the property $u\circ\varphi=v\circ\varphi$, we have that $u=v$.

For this we have the following two diagrams:





Let consider $i\le j$ in *P*, and from the first diagram we obtain:

$$(\Delta_C\circ u\circ\varphi)(e_{ij})=((u\otimes u)\circ(\varphi\otimes\varphi)\circ\Delta_n)(e_{ij})\Leftrightarrow$$

$$(\Delta_C\circ u)(f_{ij})=(u\otimes u)(\varphi\otimes\varphi)(\sum_{i\le k\le j}e_{ik}\otimes e_{kj})\Leftrightarrow$$

$$\sum u(f_{ij})_1\otimes u(f_{ij})_2=\sum_{i\le k\le j}u(f_{ik})\otimes u(f_{kj}).$$

In the same way, for the map *v* we have:

$\sum v(f_{ij})_1\otimes v(f_{ij})_2=\sum_{i\le k\le j}v(f_{ik})\otimes v(f_{kj})$, but $u\circ\varphi=v\circ\varphi$, then $(\Delta_C\circ u\circ\varphi)(e_{ij})=(\Delta_C\circ v\circ\varphi)(e_{ij})$,

then

$$\sum u(f_{ij})_1\otimes u(f_{ij})_2=\sum v(f_{ij})_1\otimes v(f_{ij})_2\Leftrightarrow\sum_{i\le k\le j}u(f_{ik})\otimes u(f_{kj})=\sum_{i\le k\le j}v(f_{ik})\otimes v(f_{kj}).$$

From the second diagram we obtain:

$(\varepsilon_C\circ u)(f_{ij})=\varepsilon(f_{ij})=\delta_{ij}=(\varepsilon_C\circ v)(f_{ij})$ and $(\varepsilon_C\circ u\circ\varphi)(e_{ij})=\varepsilon_n(e_{ij})=\delta_{ij}=(\varepsilon_C\circ v\circ\varphi)(e_{ij})$.

In conclusion, we have that $u=v$, so the map φ is an epimorphism of coalgebras between the matrix coalgebra $M^C(n,k)$ and the new coalgebra $T_n(k)$.

Remark. From the way it was defined φ, we obtain that it's kernel is $Ker(\varphi)=\sum_{j<i}k\cdot e_{ij}$.

Consequence. If we denote $K=Ker(\varphi)$, we have that K is a coideal in the matrix coalgebra $M^C(n,k)$.

Proof. Remind that for any $k-$coalgebra (C,Δ,ε), a $k-$subspace I of C is a coideal if:

$$\Delta(I)\subseteq C\otimes I+I\otimes C \text{ and } \varepsilon(I)=0.$$

It is obvious that, for any $j<i$, we have $\varepsilon_n(e_{ij})=\delta_{ij}=0$ and $\varepsilon(K)=0$ (1)

More, for $j<i$ we have:

$$\Delta_n(e_{ij})=\sum_{k=1}^{n}e_{ik}\otimes e_{kj}=\sum_{k<i}e_{ik}\otimes e_{kj}+\sum_{k>i}e_{ik}\otimes e_{kj}\in K\otimes M^C(n,k)+M^C(n,k)\otimes K ,$$

then

$$\Delta_n(K)\subseteq K\otimes M^C(n,k)+M^C(n,k)\otimes K . \quad (2)$$

From the relations (1) and (2) we have that $K = Ker(\varphi)$ is a coideal in the matrix coalgebra $M^C(n,k)$.

Generalization. Let $M_N(k)$ be the $k-$linear space of all matrices with finite columns having the base $(e_{ij})_{\substack{j\in N\\ i\in\Lambda_j}}$, where Λ_j is a finite set, for any positive integer $j\in$**N**. On this linear space we define a comultiplication $\Delta_N : M_N(k)\to M_N(k)\otimes M_N(k)$ by $\Delta_N(e_{ij})=\sum_{k\in\Lambda_j}e_{ik}\otimes e_{kj}$, and a counity $\varepsilon_N : M_N(k)\to k$ by $\varepsilon_N(e_{ij})=\delta_{ij}$. In this way, $M_N(k)$ becomes a coalgebra, named *the matrix coalgebra with finite columns.*

Like above, we associate to the set of all positive integers ***N*** a matrix coalgebra $T_N(k)$ of base $(f_{ij})_{\substack{j\in N\\ i\in\Lambda_j}}$, where $f_{ij}:=[i,j]$, where the comultiplication and the counity are the same with those of the incidence coalgebra associated to **N,** considered as a totally ordered set.

In these conditions, and in a similar way, we have an epimorphism between the matrix coalgebra $M_N(k)$ and the coalgebra $T_N(k)$.

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