ORIGINAL PAPER

ON INTEGRAL INVARIANTS OF RULED SURFACES GENERATED BY THE DARBOUX FRAMES OF THE TRANSVERSAL INTERSECTION CURVE OF TWO SURFACES IN E^3

ENGIN AS[1], AYHAN SARIOĞLUGİL[1]

Manuscript received: 18.05.2016; Accepted paper: 02.06.2016;
Published online: 30.06.2016.

Abstract. *In this paper, the some characteristic properties of ruled surfaces which are generated by the Darboux frame of the transversal intersection curve of two surfaces were given in 3-dimensional Euclidean space* E^3*. Also, the relations between the integral invariants of the closed ruled surfaces were showned. Finally, the examples for parametric-parametric and imlicit-implicit surfaces were given.*

Keywords. *Transversal intersection curve, ruled surface, geodesic curvature, Darboux frame.*

Mathematics Subject Classification. *53A04, 53A05.*

1. INTRODUCTION

It is well known that the curvatures of a curve given by the parametric equation in 3-dimensional Euclidean space can be found easily. If a curve is an intersection curve of two surfaces, representing the curve as parametric is usually imposible and computing the curvatures of this curve is hard. Intersections of geometric structures are also important in the representation of the design of complex shapes or in computer animation. So some methods and formulas are developed. Since the surfaces have parametric or implicit forms, there are three aspects for the intersection of the surfaces.

1- Parametric- parametric
2- Implicit- implicit
3- Parametric- implicit.

Here, the main purpose is to determine the intersection curve of the surfaces. So determining the geometric properties as the tangent vector, the curvature, the torsion is su¢ cient for determining the intersection curve. Two surfaces can intersect to each other as the transversal or the tangential. If the normal vector fields of the surfaces are linearly dependent at the intersection point, the intersection is called tangential intersection. If the normal vector fields of the surfaces are linearly independent at the intersection points, the intersection is called transversal intersection. The unit tangent vector of the transversal intersection curve can be easily computed via the vectoral product of the the unit normal vectors of the surfaces. So, there are many studies on the geometric properties of the transversal intersection curves. Willmore (1959) studied the Frenet vector .elds of the transversal intersection curve for two implicit surfaces. The formulas related to the curvatures of the intersection curves for all types (parametric-parametric, ...,etc.) were developed by Hartman (1996). Aléssio (2006)

[1] Ondokuz Mayıs University, Faculty of Science And Arts, Department of Mathematics, 55139 Kurupelit, Samsun, Turkey. E-mail: engin.as@hotmail.com; sarioglugil@gmail.com.

introduced a method to compute the Frenet vector fields and the curvatures of the transversal intersection curves on implicit surfaces. Aléssio and Guadalupe (2007) gave results related to the geodesic curvature and the geodesic torsion of the intersection curve of two space-like surfaces in Lorentzian 3- space. Aléssio (2009) studied the intersection curve of three implicit surfaces in IR^4 by using implicit function theorem. Goldman (2005) found the curvature and the torsion of intersection curve by using the classical di¤erential geometry methods. Ye and Maekawa (1999) computed the curvatures and the Frenet vectors of the tangential and transversal intersection curve for the three cases of two surfaces. Also, they computed the normal curvature in the direction of the tangent vector of the intersection curve and gave the formulas to compute higher order derivatives. For three parametric surfaces in E^4, the curvatures and the Frenet vectors of the intersection curve were given by Düldül M. (2010). Çalışkan and Düldül B. (2010) studied the geodesic curvature and the geodesic torsion of the intersection curve for impilicit-implicit and parametric- parametric surfaces. Also, they gave the curvature and the curvature vector of intersection curve by using the normal vectors of surfaces. Sarıoğlugil and Tutar (2007) studied the geodesic curvature and the fundamental forms of the regular surfaces in E^3.

In this paper, the relation between the Darboux frames of the intersection curve at the intersection point for two surfaces was given. Also, the relations between the geodesic curvatures, the geodesic torsions and the normal curvatures were investigated. The apex angles, the pitches and tha dralls were computed for the closed ruled surfaces generated by the Darboux frames and the relations between each other were showned. Finally, the examples were given for the parametric-parametric, implicit-implicit surfaces.

2. PRELIMINARIES

In this section; firstly, we will review some basic concepts in E^3 for later use. Let $\alpha : I \to E^3$ be a differentiable curve with arc-length parameter s and $\{t, n, b\}$ be the Frenet frame of α at the point $\alpha(s)$, where

$$t(s) = \alpha'(s), \quad n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad b(s) = t(s) \wedge n(s).$$

The Frenet formulas of α are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{1}$$

If α is a curve and x is a generator vector, then the ruled surface $X(s,v)$ has the following parameter representation

$$X(s,v) = \alpha(s) + v x(s).$$

Namely, a ruled surface is a surface generated by the motion of a straight line x along α. Furthermore, if α is a closed curve, then this surfaces is called closed ruled surface. Moreover, the drall P_x, the apex angle λ_x and the pitch l_x of the closed ruled surface are defined by

$$P_x = \frac{\det(\alpha', x, x')}{\|x'\|^2}, \quad \lambda_x = \langle D, x \rangle, \quad l_x = \langle V, x \rangle \tag{2}$$

respectively. Here, D and V are Steiner rotation vector and Steiner translation vector, respectively. The Steiner translation vector V and Steiner rotation vector D are given as follows:

$$V = \oint_{(\alpha)} dx = t\oint_{(\alpha)} ds \tag{3}$$

$$D = \oint_{(\alpha)} w = t\oint_{(\alpha)} \tau + b\oint_{(\alpha)} \kappa \tag{4}$$

where

$$w = n \wedge n' = \tau t + \kappa b \tag{5}$$

is called Darboux vector, (Fig 1).

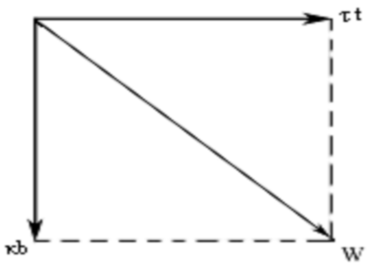

Figure 1. Darboux vector of a curve.

If the Frenet vectors t,n,b are the straigth lines of the closed ruled surface, then we have

$$\begin{cases} \lambda_t = \oint_{(\alpha)} \tau ds \\ \lambda_n = 0 \\ \lambda_b = \oint_{(\alpha)} \kappa ds \end{cases}, \quad \begin{cases} l_t = \oint_{(\alpha)} ds \\ l_n = 0 \\ l_b = 0 \end{cases}, \quad \begin{cases} P_t = 0 \\ P_n = \dfrac{\tau}{\kappa^2 + \tau^2} \\ P_b = \dfrac{1}{\tau} \end{cases} \tag{6}$$

(Hacısalioğlu, H.H., 1983).

Definition 1. Let M be an oriented surface in E^3 and α be a unit speed curve on M : If t is the unit tangent vector of α ; N is the unit normal vector of M and $g = N \wedge t$ at the point $\alpha(s)$; then $\{t, g, N\}$ is called the Darboux frame of α at that point. Thus, Darboux formulas are

$$\begin{bmatrix} t' \\ g' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} t \\ g \\ N \end{bmatrix} \tag{7}$$

where ψ is the angle between N and the unit principal normal n of α. Here $\kappa_n = \kappa\cos\psi,\ \kappa_g = \kappa\sin\psi$ and $\tau_g = \tau + \dfrac{d\psi}{ds}$ are called the normal curvature, the geodesic curvature and the geodesic torsion of α respectively, (Kühnel, W., 1950).

Definition 2. Let M be an oriented surface in E^3 and t be the tangent vector of M at the point P. Then, the value of $\kappa_n = \langle S(t), t\rangle$ is called the normal curvature of M in the direction of t, where S is the shape operator on M, (O.Neill, B., 1966). Namely, the normal curvature of M at $P \in M$ is the curvature of the projection of α on the plane which is spanned by the normal vector and the tangent vector of M at $P \in M$.

Definition 3. Let M be an oriented surface in E^3 and α be a differentiable curve on M. If the normal curvature of α is zero, then α is called an asymptotic line, (O.Neill, B., 1966).

Definition 4. Let M be an oriented surface in E^3 and α be a differentiable curve on M. The curvature of the projection curve on the tangent plane of M along α is called the geodesic curvature of α and denoted by κ_g (Graustein, W. C., 1966).

Definition 5. Let M be an oriented surface in E^3 and α be a differentiable curve on M. If the geodesic curvature of α is zero, then α is called a geodesic curve, (Struik, D. J., 1961).

Theorem 1. Let M be an oriented surface in E^3 and α be a differentiable curve on M. Then, the relation between the curvature κ, the geodesic curvature κ_g and the normal curvature κ_n of α is given as follows

$$\kappa = \kappa_g^2 + \kappa_n^2$$

(O.Neill, B., 1966).

Definition 6. Let M be an oriented surface in E^3 and α be a differentiable curve on M. If the geodesic torsion of α is zero, then α is called a principal line, (Struik, D. J., 1961).

Theorem 2. Let M be a ruled surface in E^3 and $\alpha : I \to M$ be the leading curve of M. The relation between the Darboux frame $\{t, g, N\}$ and the Frenet frame $\{t, n, b\}$ of α is given as follows:

$$\begin{bmatrix} t \\ g \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \\ 0 & \cos\psi & -\sin\psi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{8}$$

where ψ is the angle between N and n, (Şenatalar, M., Diferensiyel Geometri 1978).

3. INTERSECTION CURVE OF TWO SURFACES

Let A and B be two parametric surfaces which have $\alpha = X^A(u_A, v_A)$ and $\beta = X^B(u_B, v_B)$ parametric representations. The unit normal of the parametric surface $\alpha = X(u,v)$ is

$$N = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}$$

and the unit normal of the implicit surface is

$$N = \frac{\nabla f}{\|\nabla f\|}.$$

The curves $u = u(s)$ and $v = v(s)$ define the curve $\alpha = \alpha(s) = X(u(s), v(s))$ on the parametric surface $X(u,v)$. Similarly, the curves $x = x(s)$, $y = y(s)$, $z = z(s)$ define the curve $f(x(s), y(s), z(s)) = 0$ on the implicit surface $f(x,y,z) = 0$. If the normal vectors of A and B

are linear independent (linear dependent) at the intersection points, then the intersection curve is called transversal intersection curve (tangential intersection curve), (Ye, X. and Maekawa, T., 1999).

3.1. TRANSVERSAL INTERSECTION CURVE OF PARAMETRIC-PARAMETRIC SURFACES

Let A and B be two regular surfaces which have $X^A(u_A,v_A)$ and $X^B(u_B,v_B)$ parametric representations, respectively α be the transversal intersection curve of A and B with arc-length parameter, t be the unit tangent vector at the intersection point $P = X^A(u_A,v_A) = X^B(u_B,v_B)$ and N^A and N^B be the normal vectors of A and B at the point P, respectively. Then, we have

$$N_A = \frac{X_{u_A} \wedge X_{v_A}}{\left\| X_{u_A} \wedge X_{v_A} \right\|},\ N_B = \frac{X_{u_B} \wedge X_{v_B}}{\left\| X_{u_B} \wedge X_{v_B} \right\|} \tag{9}$$

Since A and B intersect transversally, N^A and N^B is not parallel at the point P. Also, since the unit tangent vector t of the intersection curve lies on the tangent planes of A and B,

$$t = \frac{N_A \wedge N_B}{\left\| N_A \wedge N_B \right\|}. \tag{10}$$

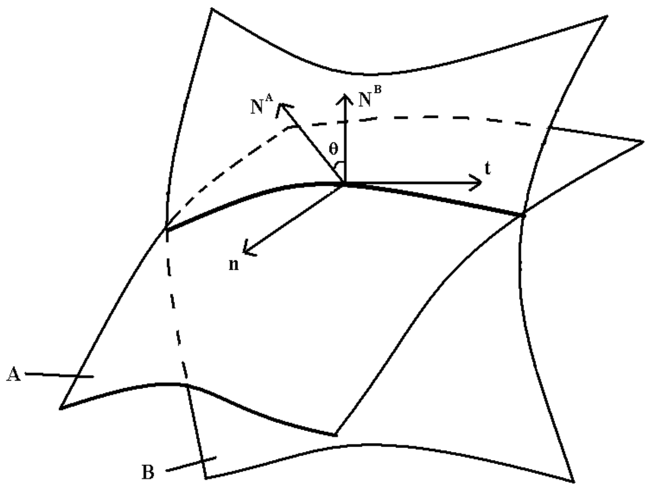

Figure 2. Intersection curve of two surfaces.

Let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be Darboux frame α at the point P where $g^A = N^A \wedge t$ and $g^B = N^B \wedge t$. Then, Darboux formolus are

$$\begin{bmatrix} t \\ (g^A)' \\ (N^A)' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^A & \kappa_n^A \\ -\kappa_g^A & 0 & \tau_g^A \\ -\kappa_n^A & -\tau_g^A & 0 \end{bmatrix} \begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} t \\ \left(g^B\right)' \\ \left(N^B\right)' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^B & \kappa_n^B \\ -\kappa_g^B & 0 & \tau_g^B \\ -\kappa_n^B & -\tau_g^B & 0 \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{12}$$

respectively, (Çalışkan, M. and Düldül U., B., 2010).

Theorem 3. Let A and B be two regular surfaces which have $X(u,v)$ and $Y(p,q)$ parametric representations, respectively and α be the transversal intersection curve of A and B. Then, the geodesic torsion τ_g^A of α with respect to the surface A is

$$\tau_g^A = \frac{1}{\sqrt{EG-F^2}}\left\{(EM-FL)(u')^2 + (EN-GL)u'v' + (FN-GM)(v')^2\right\} \tag{13}$$

Here, $E=\langle X_u, X_u\rangle$, $F=\langle X_u, X_v\rangle$ and $G=\langle X_v, X_v\rangle$ are the coefficients of the first fundamental form of the surface A and $L=\langle X_{uu}, N^A\rangle$, $M=\langle X_{uv}, N^A\rangle$ and $N=\langle X_{vv}, N^A\rangle$ are the coefficients of the second fundamental form of the surface A, (Çalışkan, M. and Düldül U., B., 2010), where

$$\begin{cases} u' = \dfrac{1}{\sqrt{EG-F^2}}\left(G\langle t, X_u\rangle - F\langle t, X_v\rangle\right) \\ v' = \dfrac{1}{\sqrt{EG-F^2}}\left(E\langle t, X_v\rangle - F\langle t, X_u\rangle\right) \end{cases} \tag{14}$$

(Ye, X. and Maekawa, T., 1999).

Theorem 4. Let A and B be two regular surfaces which have $X(u,v)$ and $Y(p,q)$ parametric representations, respectively and α be the transversal intersection curve of A and B. Then, the geodesic curvature κ_g^A of α with respect to the surface A is

$$\kappa_g^A = \frac{1}{\sqrt{EG-F^2}}\left\{\begin{array}{l} \left[\left(F_u - \dfrac{E_v}{2}\right)\langle t, X_u\rangle - \dfrac{E_u}{2}\langle t, X_v\rangle\right](u')^2 \\ +\left[G_u\langle t, X_u\rangle - E_v\langle t, X_v\rangle\right]u'v' \\ +\left[\dfrac{G_v}{2}\langle t, X_u\rangle - \left(F_v - \dfrac{G_u}{2}\right)\langle t, X_v\rangle\right](v')^2 \end{array}\right\} + \sqrt{EG-F^2}\,(u'v'' - v'u'') \tag{15}$$

(Çalışkan, M. and Düldül U., B., 2010). Here,

$$\begin{cases} u'' = \dfrac{1}{|\sin\theta|\sqrt{EG-F^2}}\left\{\langle \Lambda, N^B\rangle\langle t, X_v\rangle + \langle N^B, X_v\rangle\left[\langle t, X_{uu}\rangle(u')^2 + 2\langle t, X_{uv}\rangle u'v' + \langle t, X_{vv}\rangle(v')^2\right]\right\} \\ v'' = \dfrac{-1}{|\sin\theta|\sqrt{EG-F^2}}\left\{\langle \Lambda, N^B\rangle\langle t, X_u\rangle + \langle N^B, X_u\rangle\left[\langle t, X_{uu}\rangle(u')^2 + 2\langle t, X_{uv}\rangle u'v' + \langle t, X_{vv}\rangle(v')^2\right]\right\} \end{cases} \tag{16}$$

where

$$\Lambda = Y_{pp}(p')^2 + 2Y_{pq}p'q' + Y_{qq}(q')^2 - X_{uu}(u')^2 - 2X_{uv}u'v' - X_{vv}(v')^2 .$$

Theorem 5. Let A and B be two regular surfaces which have $X(u,v)$ and $Y(p,q)$ parametric representations, respectively and α be the transversal intersection curve of A and B. Then, the normal curvature κ_n^A of α with respect to the surface A is

$$\kappa_n^A = L(u')^2 + 2Mu'v' + N(v')^2, \tag{17}$$

(Ye, X. and Maekawa, T., 1999).

3.2. TRANSVERSAL INTERSECTION CURVE OF IMPLICIT-IMPLICIT SURFACES

Let $f(x,y,z)=0$ and $g(x,y,z)=0$ be the implicit equations of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the unit tangent vector of α is

$$t = \frac{\nabla f \wedge \nabla g}{\|\nabla f \wedge \nabla g\|}, \tag{18}$$

where

$$N^A = \frac{\nabla f}{\|\nabla f\|},\ N^B = \frac{\nabla g}{\|\nabla g\|} \tag{19}$$

are the normal vector fields of A and B, respectively, (Çalışkan, M. and Düldül U., B., 2010).

Theorem 6. Let $f(x,y,z)=0$ and $g(x,y,z)=0$ be the implicit equations of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the geodesic torsion τ_g^A of α with respect to the surface A is

$$\tau_g^A = \left(\frac{1}{\|\nabla f \wedge \nabla g\|} \nabla G - \frac{\cot\theta}{\|\nabla f\|^2} \nabla F \right) \phi T, \tag{20}$$

where

$$\nabla F = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix},\ \nabla G = \begin{bmatrix} g_x & g_y & g_z \end{bmatrix},\ T = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix},\ \phi = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}$$

(Çalışkan, M. and Düldül U., B., 2010).

Theorem 7. Let $f(x,y,z)=0$ and $g(x,y,z)=0$ be the implicit equations of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the geodesic curvature κ_g^A of α with respect to the surface A is

$$\kappa_g^A = \frac{1}{\|\nabla f\|} \left[(y'z'' - y''z') f_x + (z'x'' - z''x') f_y + (x'y'' - x''y') f_z \right] \tag{21}$$

Here, x'', y'' and z'' can be obtained by the following linear equaiton systems:

$$\begin{cases} x'x'' + y'y'' + z'z'' = 0 \\ f_x x'' + f_y y'' + f_z z'' = -f_{xx}(x')^2 - f_{yy}(y')^2 - f_{zz}(z')^2 - 2\left(f_{xy}x'y' + f_{xz}x'z' + f_{yz}y'z'\right) \\ g_x x'' + g_y y'' + g_z z'' = -g_{xx}(x')^2 - g_{yy}(y')^2 - g_{zz}(z')^2 - 2\left(g_{xy}x'y' + g_{xz}x'z' + g_{yz}y'z'\right) \end{cases} \tag{22}$$

(Çalışkan, M. and Düldül U., B., 2010).

Theorem 8. Let $f(x,y,z)=0$ and $g(x,y,z)=0$ be the implicit equations of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the normal curvature κ_n^A of α with respect to the surface A is

$$\kappa_n^A = \frac{f_x x'' + f_y y'' + f_z z''}{\sqrt{f_x^2 + f_y^2 + f_z^2}} \tag{23}$$

(Ye, X. and Maekawa, T., 1999).

3.3. TRANSVERSAL INTERSECTION CURVE OF PARAMETRIC-IMPLICIT SURFACES

Let $X(u,v)$ and $g(x,y,z)=0$ be the parametric representation and the implicit equation of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the normal vector field of the surfaces A and B are

$$N^A = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}, \quad N^B = \frac{\nabla g}{\|\nabla g\|} \tag{24}$$

respectively and the unit tangent vector of α is

$$t = \frac{(X_u \wedge X_v)\wedge \nabla g}{\|(X_u \wedge X_v)\wedge \nabla g\|}, \tag{25}$$

(Çalışkan, M. and Düldül U., B., 2010).

Theorem 9. Let $X(u,v)$ and $g(x,y,z)=0$ be the parametric representation and the implicit equation of the surfaces A and B, respectively and α be the transversal intersection curve of A and B. Then, the geodesic torsion τ_g^B of α with respect to the surface B is

$$\tau_g^B = \frac{1}{\|(X_u \wedge X_v)\wedge \nabla g\|\|\nabla g\|^2}\det(\langle \nabla g, X_u\rangle X_v - \langle \nabla g, X_v\rangle X_u, \nabla g, (\nabla g)'), \tag{26}$$

(Çalışkan, M. and Düldül U., B., 2010).

4. ON INTEGRAL INVARIANTS OF RULED SURFACES GENERATED BY THE DARBOUX FRAMES OF THE TRANSVERSAL INTERSECTION CURVE OF TWO SURFACES IN E^3

Theorem 10. Let α be the transversal intersection curve of the surfaces A and B, with arc length parameter and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then the relation between the Darboux frames of the surfaces A and B, is given as follows:

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{27}$$

where θ is the angle between N^A and N^B.

Proof. Let α be the transversal intersection curve of the surfaces A and B. By using (8), the Darboux frame $\{t, g^A, N^A\}$ and Frenet frame $\{t, n, b\}$ can be writen

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \\ 0 & \cos\psi & -\sin\psi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{28}$$

where ψ is the angle between n and N^A. Similarly, by using (8), the Darboux frame $\{t, g^B, N^B\}$ and Frenet frame $\{t, n, b\}$ can be writen

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\overline{\psi} & \cos\overline{\psi} \\ 0 & \cos\overline{\psi} & -\sin\overline{\psi} \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{29}$$

where $\overline{\psi}$ is the angle between n and N^B. Substituting (29) into (28) we have

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\psi - \overline{\psi}\right) & \sin\left(\psi - \overline{\psi}\right) \\ 0 & -\sin\left(\psi - \overline{\psi}\right) & \cos\left(\psi - \overline{\psi}\right) \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{30}$$

or

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix}$$

where

$$\theta = \overline{\psi} - \psi \tag{31}$$

Thus, the following corollary can be given.

Corollary 1. . Let α be the transversal intersection curve of the surfaces A and B, with arc length parameter and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. If N^A and N^B are orthogonal to each other along the curve α, then the relation between the Darboux frames of A and B is given as follows:

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{32}$$

Proof. The proof is clear.

Theorem 11. Let α be the closed transversal intersection curve of the surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then the relations between the apex angles of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$\begin{cases} \lambda_t = \oint\limits_{(\alpha)} \tau ds \\ \lambda_{N^B} = (\cos\theta + \sin\theta \cot\psi)\lambda_{N^A} \\ \lambda_{g^B} = (\cos\theta - \sin\theta \tan\psi)\lambda_{g^A} \end{cases} \tag{33}$$

Proof. Let ψ be the angle between n and N^A and let $\overline{\psi}$ be the angle between n and N^B. The apex angle of ruled surface which is generated by the unit tangent vector of the transversal intersection curve α is

$$\lambda_t = \langle D, t \rangle .$$

Substituting (4) into the last equation we obtain

$$\lambda_t = \oint\limits_{(\alpha)} \tau ds$$

The apex angle of ruled surface which is generated by the unit normal vector N^A of the surface A is

$$\lambda_{N^A} = \langle D, N^A \rangle .$$

Substituting (4) into the last equation we obtain

$$\lambda_{N^A} = -\sin\psi \oint\limits_{(\alpha)} \kappa ds .$$

Substituting (6) into the above equation we get

$$\lambda_{N^A} = -\sin\psi \lambda_b . \tag{34}$$

From the last equation we have

$$\lambda_b = -\frac{\lambda_{N^A}}{\sin\psi} \tag{35}$$

Similarly, the apex angle of ruled surface which is generated by the unit normal vector N^B of the surface B is

$$\lambda_b = -\frac{\lambda_{N^B}}{\sin\overline{\psi}} . \tag{36}$$

From (35) and (36) we get

$$\lambda_{N^B} = \frac{\sin\overline{\psi}}{\sin\psi}\lambda_{N^A}$$

Substituting (31) into the last equation we obtain

$$\lambda_{N^B} = \left(\cos\theta + \sin\theta \cot\psi\right)\lambda_{N^A}.$$

The apex angle of ruled surface which is generated by the unit vector g^A of the surface A is

$$\lambda_{g^A} = \left\langle D, g^A \right\rangle.$$

Substituting (4) into the last equation we obtain

$$\lambda_{g^A} = \cos\psi \oint_{(\alpha)} \kappa ds$$

Substituting (6) into the above equation we get

$$\lambda_{g^A} = \cos\psi\lambda_b. \quad (37)$$

From the last equation we obtain

$$\lambda_b = \frac{\lambda_{g^A}}{\cos\psi} \quad (38)$$

Similarly, the apex angle of ruled surface which is generated by the unit vector g^B of the surface B is

$$\lambda_b = \frac{\lambda_{g^B}}{\cos\overline{\psi}}. \quad (39)$$

From (38) and (39) we get

$$\lambda_{g^B} = \frac{\cos\overline{\psi}}{\cos\psi}\lambda_{g^A}.$$

Subtituting (31) into the last equation we obtain

$$\lambda_{g^B} = \left(\cos\theta - \sin\theta \tan\psi\right)\lambda_{g^A}.$$

Thus, the following corollary can be writen.

Corollary 2. Let α be the closed transversal intersection curve of the surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. . If N^A and N^B are orthogonal to each other along the curve α, then the relation between the apex angles of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$\begin{cases} \lambda_t = \oint\limits_{(\alpha)} \tau ds \\ \lambda_{N^B} = \cot\psi \lambda_{N^A} \\ \lambda_{g^B} = \tan\psi \lambda_{g^A} \end{cases} \tag{40}$$

Proof. The proof is clear.

Theorem 12. Let α be the closed transversal intersection curve of the surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then the relations between the pitches of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$l_t = \oint\limits_{(\alpha)} ds, \quad l_{N^A} = l_{N^B} = l_{g^A} = l_{g^B} = 0 \tag{41}$$

Proof. The pitch of ruled surface which is generated by the unit tangent vector of α is

$$l_t = \langle V.t \rangle . \tag{42}$$

Substituting (3) into the last equation, we have

$$l_t = \oint\limits_{(\alpha)} ds.$$

Similarly, the pitches of ruled surfaces which are generated by vectors N^A, N^B, g^A and g^B are

$$l_{N^A} = l_{N^B} = l_{g^A} = l_{g^B} = 0 .$$

Theorem 13. Let α be the closed transversal intersection curve of the surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Let τ_g^A be the geodesic torsion of α with respect to the surface A and let τ_g^B be the geodesic torsion of α with respect to the surface B. Then, the relation between the geodesic torsions is given as follows:

$$\tau_g^A = \tau_g^B - \frac{d\theta}{ds} . \tag{43}$$

Proof. From (27) we know that

$$g^A = \cos\theta g^B - \sin\theta N^B . \tag{44}$$

Differentiating this equation, we obtain

$$\left(g^A\right)' = -\theta' \sin\theta g^B + \cos\theta \left(g^B\right)' - \theta' \cos\theta N^B - \sin\theta \left(N^B\right)' .$$

From the Darboux formulas, we get

$$-\kappa_g^A t + \tau_g^A N^A = \left(-\cos\theta\kappa_g^B + \sin\theta\kappa_n^B\right)t + \left(-\theta'\sin\theta + \sin\theta\tau_g^B\right)g^B + \left(-\theta'\cos\theta + \cos\theta\tau_g^B\right)N^B.$$

Multiplying the last equation with N^A, we have

$$\tau_g^A = \tau_g^B - \frac{d\theta}{ds} \tag{46}$$

Theorem 14. Let α be transversal intersection curve of surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Let κ_g^A and κ_n^A be the geodesic curvature and the normal curvature of α with respect to the surface A , respectively and let κ_g^B and κ_n^B be the geodesic curvature and the normal curvature of α with respect to the surface B, respectively. Then, the relations between the geodesic curvatures and the geodesic torsions are given as follows:

$$\begin{cases} \kappa_g^A = \cos\theta\kappa_g^B - \sin\theta\kappa_n^B \\ \kappa_n^A = \cos\theta\kappa_n^B + \sin\theta\kappa_g^B \end{cases} \tag{47}$$

Proof. From (27) we know that

$$g^A = \cos\theta g^B - \sin\theta N^B. \tag{48}$$

Differentiating this equation we obtain

$$\left(g^A\right)' = -\theta'\sin\theta g^B + \cos\theta\left(g^B\right)' - \theta'\cos\theta N^B - \sin\theta\left(N^B\right)'.$$

From the Darboux formulas we get

$$-\kappa_g^A t + \tau_g^A N^A = \left(-\cos\theta\kappa_g^B + \sin\theta\kappa_n^B\right)t + \left(-\theta'\sin\theta + \sin\theta\tau_g^B\right)g^B + \left(-\theta'\cos\theta + \cos\theta\tau_g^B\right)N^B.$$

Multiplying the last equation with t, we have

$$\kappa_g^A = \cos\theta\kappa_g^B - \sin\theta\kappa_n^B. \tag{50}$$

Also, from (27) we know that

$$N^A = \sin\theta g^B + \cos\theta N^B. \tag{51}$$

Differentiating this equation we obtain

$$-\kappa_n^A t + \tau_g^A g^A = \left(-\sin\theta\kappa_g^B - \cos\theta\kappa_n^B\right)t + \left(\theta'\cos\theta - \cos\theta\tau_g^B\right)g^B + \left(-\theta'\sin\theta + \sin\theta\tau_g^B\right)N^B. \tag{52}$$

Multiplying the last equation with t, we have

$$\kappa_n^A = \cos\theta\kappa_n^B + \sin\theta\kappa_g^B \quad . \tag{53}$$

Corollary 3. From (50) and (53) it can be easily seen that

$$\left(\kappa_n^A\right)^2 + \left(\kappa_g^A\right)^2 = \left(\kappa_n^B\right)^2 + \left(\kappa_g^B\right)^2 . \tag{54}$$

Corollary 4. Let α be transversal intersection curve of surfaces A and B. If α is the geodesic curve for the surfaces A and B, then the relation between normal curvatures of α is

$$\kappa_n^A = \pm\kappa_n^B . \tag{55}$$

Corollary 5. Let α be transversal intersection curve of surfaces A and B. If α is the asymptotic curve for the surfaces A and B, then the relation between geodesic curvatures of α is

$$\kappa_g^A = \pm\kappa_g^B . \tag{56}$$

Corollary 6. Let α be transversal intersection curve of surfaces A and B. If N^A and N^B are orthogonal to each other along the intersection curve α, then the relation between the geodesic curvatures and the geodesic torsions are

$$\begin{cases} \kappa_g^A = -\kappa_n^B \\ \kappa_n^A = \kappa_g^B \\ \tau_g^A = \tau_g^B \end{cases} \tag{57}$$

Theorem 15. Let α be transversal intersection curve of surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. If the angle between N^A and N^B is constant along the curve α, then the relations between the dralls of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{1}{P_{g^B}} + \frac{1}{P_{N^B}} \tag{58}$$

Proof. The drall of ruled surface which is generated by the unit tangent vector of α is

$$P_t = \frac{\det\left(\alpha', t, t'\right)}{\|t'\|^2} = 0$$

the drall of ruled surface which is generated by the unit normal vector N^A of the surface A is

$$P_{N^A} = \frac{\det\left(\alpha', N^A, \left(N^A\right)'\right)}{\left\|\left(N^A\right)'\right\|^2} = \frac{\tau_g^A}{\left(\kappa_n^A\right)^2 + \left(\tau_g^A\right)^2} \tag{59}$$

and the drall of ruled surface which is generated by the unit vector g^A of the surface A is

$$P_{g^A} = \frac{\det\left(\alpha', g^A, \left(g^A\right)'\right)}{\left\|\left(g^A\right)'\right\|^2} = \frac{\tau_g^A}{\left(\kappa_g^A\right)^2 + \left(\tau_g^A\right)^2} \tag{60}$$

Similarly, the dralls of ruled surfaces which are generated by N^B and g^B are

$$P_{N^B} = \frac{\tau_g^B}{\left(\kappa_n^B\right)^2 + \left(\tau_g^B\right)^2} \tag{61}$$

and

$$P_{g^B} = \frac{\tau_g^B}{\left(\kappa_g^B\right)^2 + \left(\tau_g^B\right)^2} \tag{62}$$

From here, we get

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{\left(\kappa_g^A\right)^2 + \left(\kappa_n^A\right)^2 + 2\left(\tau_g^A\right)^2}{\tau_g^A}.$$

Substituting (43) and (54) into the last equation we obtain

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{\left(\kappa_g^B\right)^2 + \left(\kappa_n^B\right)^2 + 2\left(\tau_g^B - \frac{d\theta}{ds}\right)^2}{\tau_g^B - \frac{d\theta}{ds}}.$$

Since θ is constant, it is seen that

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{\left(\kappa_g^B\right)^2 + \left(\tau_g^B\right)^2}{\tau_g^B} + \frac{\left(\kappa_n^B\right)^2 + \left(\tau_g^B\right)^2}{\tau_g^B},$$

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{1}{P_{g^B}} + \frac{1}{P_{N^B}}. \tag{63}$$

5. EXAMPLES

5.1. TRANSVERSAL INTERSECTION CURVE OF PARAMETRIC-PARAMETRIC SURFACES.

Example 1. Let the surface A be the sphere given by

$$X\left(u,v\right) = \left(\cos u \cos v, \sin u \cos v, \sin v\right),\ 0 \le u \le 2\pi,\ \ 0 \le v \le 2\pi$$

and the surface B be the cylinder given by

$$Y(p,q)=\left(\frac{1}{2}\cos p+\frac{1}{2},\ \frac{1}{2}\sin p,\ q\right),\ 0\le u\le 2\pi,\ \ -2\le q\le 2$$

respectively (Fig. 2). Let us find the geodesic curvatures and the normal curvatures of the intersection curve at the point $P=X\left(\frac{\pi}{4},\frac{\pi}{4}\right)=Y\left(\frac{\pi}{2},\frac{\sqrt{2}}{2}\right)=\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right)$ and show that theorem 14 is satisfied.

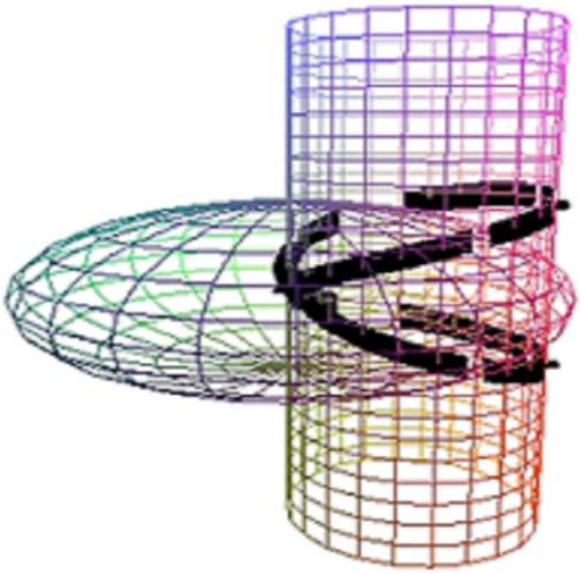

Figure 3. Intersection curve of parametric-parametric surfaces.

The partial derivatives of the surfaces A at the point P are

$$X_u=\left(-\frac{1}{2},-\frac{1}{2},0\right),\ X_v=\left(-\frac{1}{2},-\frac{1}{2},\frac{\sqrt{2}}{2}\right),\ X_{uu}=\left(-\frac{1}{2},-\frac{1}{2},0\right),\ X_{uv}=\left(\frac{1}{2},-\frac{1}{2},0\right),\ X_{vv}=\left(-\frac{1}{2},-\frac{1}{2},-\frac{\sqrt{2}}{2}\right).$$

From here, the unit normal of surface A at the point P and the coefficients of the first and second fundamental form are

$$N^A=\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right),\ E=-L=\frac{1}{2},\ G=1,\ E_v=N=-1,\ F=E_u=F_v=G_u=G_v=M=0,$$

respectively. Similarly, the partial derivatives of the surfaces B at the point P are

$$Y_p=\left(-\frac{1}{2},0,0\right),\ Y_q=(0,0,1),\ Y_{pp}=\left(0,-\frac{1}{2},0\right),\ Y_{pq}=Y_{qq}=0.$$

From here, the unit normal of the surface B at the point P and the coefficients of the first and second fundamental form are

$$N^B=(0,1,0),\ e=\frac{1}{4},\ g=1,\ l=-\frac{1}{2},\ f=e_p=e_q=f_p=f_q=g_p=g_q=m=n=0,$$

respectively. Since the unit tangent vector is $t=\left(\frac{-\sqrt{6}}{3},0,\frac{\sqrt{3}}{3}\right)$ at the point P and from (14) and (16), it is hold

$$u'=v'=\frac{\sqrt{6}}{3},\ p'=\frac{2\sqrt{6}}{3},\ q'=\frac{\sqrt{3}}{3},\ \Lambda=\left(0,0,\frac{\sqrt{2}}{3}\right),\ u''=v''=\frac{2}{9},\ p''=\frac{2}{9},\ q''=-\frac{\sqrt{2}}{9}.$$

Substituting the last equaiton into (15) and (17) we get

$$\kappa_g^A=\frac{5\sqrt{3}}{9},\ \kappa_g^B=-\frac{2\sqrt{3}}{9},\ \kappa_n^A=-1,\ \kappa_n^B=-\frac{4}{3}.$$

Using this equaiton, it can be easily seen that the theorem 14 is satisfied .

5.2. TRANSVERSAL INTERSECTION CURVE OF IMPLICIT-IMPLICIT SURFACES

Example 2. Let A and B be the surfaces are given by $f(x,y,z)=z-xy=0$ and $g(x,y,z)=x^2+y^2+z^2=3$ respectively. Let us find the geodesic curvatures and the normal curvatures of the intersection curve at the point $P=(1,-2,-2)$ and show that theorem 14 is satisfied.

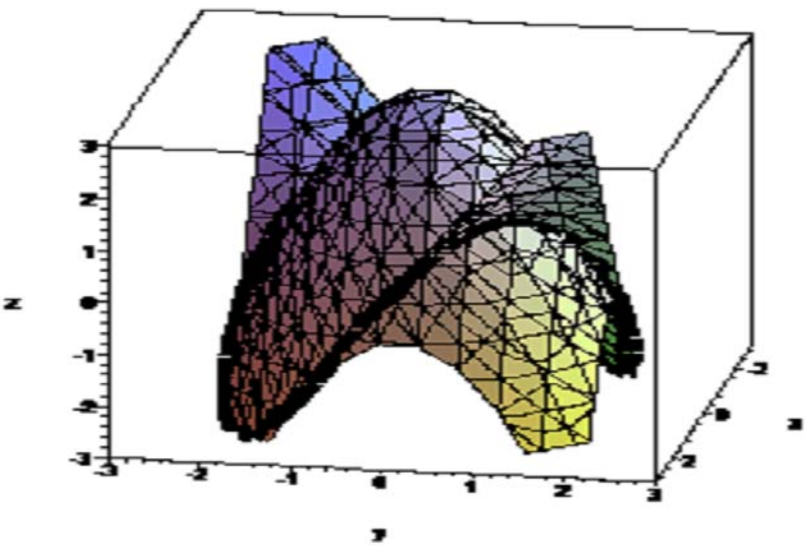

Figure 4. Intersection curve of implicit-implicit surfaces.

The normal vectors of the surfaces A and B at the point P are

$$\nabla f=(2,-1,1),\ \nabla g=(2,-4,1)$$

respectively. Moreover, at the point P we get

$$f_{xx}=f_{xz}=f_{yy}=f_{yz}=f_{zz}=0,\ f_{xy}=-1$$

and

$$g_{xy} = g_{xz} = g_{yz} = g_{zz} = 0,\ g_{xx} = g_{yy} = 2.$$

From (22) we have

$$x'' - 2z'' = 0$$
$$2x'' - y'' + z'' = 0$$
$$2x'' - 4y'' + z'' = -\frac{2}{5}.$$

Solving this linear equation system, we obtain

$$x'' = \frac{4}{75},\ y'' = \frac{2}{15},\ z'' = \frac{2}{75}.$$

Substituting x'', y'' and z'' in (21) and (23), we get

$$\kappa_g^A = \frac{2\sqrt{30}}{75},\ \kappa_g^B = \frac{2\sqrt{105}}{175},\ \kappa_n^A = 0,\ \kappa_n^B = -\frac{2\sqrt{21}}{105}.$$

Using this equaiton, it can be easily seen that the theorem 14 is satis.ed .

REFERENCES

[1] Aléssio, O., *TEMA Tend. Mat. Apl. Comput.*, **7**(2), 169, 2006.
[2] Aléssio, O., Guadalupe, I.V., *Hadronic Journal*, **30**(3), 315, 2007.
[3] Aléssio, O., *Comput. Aided Geom. Des.*, **26**(4), 455, 2009.
[4] Blaschke, W., *Diferensiyel Geometri Dersleri*, İstanbul Üniversitesi Yayınları (Translated by Ord. Prof. Dr. K. Erim), 1949.
[5] Çalışkan, M., Düldül, U.B., *Acta Universitatis Apulensis*, **24**, 161, 2010.
[6] Düldül, M., *Computer Aided Geometric Design*, **27**, 118, 2010.
[7] Graustein, W.C., *Differential Geometry*, Dover Publucations, Inc., New York, 1966.
[8] Goldman, R., *Computer Aided Geometric Design*, **22**, 632, 2005.
[9] Hacısalihoğlu, H.H., *Yüksek Boyutlu Uzaylarda Dönüşümler ve Geometriler*, İnönü Üniversitesi Temel Bilgiler Fakültesi Yayınları, Mat. No:1, İstanbul, 1980.
[10] Hacısalihoğlu, H.H., *Diferensiyel Geometri*, İnönü Üniversitesi Fen-Edebiyat Fakültesi Yayınları, Malatya, 1983.
[11] Hartmann, E., *The Visual Computer*, **12**, 181, 1996.
[12] Kühnel, W., *Differential Geometry, Curves-Surfaces-Manifolds* (Second Edition-Translated by Bruce Hunt), American Mathematical Society, USA, 1950.
[13] O.Neill, B., *Elementary Differential Geometry*, Academic Press, New York, 1966.
[14] Sarıoğlugil, A., Tutar, A., *Int. J. Contemp. Math. Sci.*, **2**(1), 1, 2007.
[15] Struik, D.J., *An Introduction to Differential Geometry*, Oxford University Press, 1961.
[16] Senatalar, M., *Diferensiyel Geometri* (Eğriler ve Yüzeyler Teorisi) İstanbul Devlet Mühendislik ve Mimarlık Akademisi Yayınları, İstanbul, 1978.
[17] Ye, X., Maekawa, T., *Computer Aided Geometric Design*, **16**, 767, 1999.
[18] Willmore, T.J., *An Introduction to Differential Geometry*, Oxford University Press, India, 1961.