ORIGINAL PAPER

SOME REFINEMENTS OF HARDY-TYPE INEQUALITIES

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Abstract. *Using some new generalizations of Young's inequality we apply a new refinement of Holder's inequalities in order to give several new variants of Hardy-type inequalities following the method of Pachpatte.*

Keywords: *Young's inequality, Holder's inequality, Hardy's inequality, Pachpatte's inequality.*

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1. INTRODUCTION

We recall a new Young-type inequality for positive and real numbers, a definition and a result in order to use them below in section 2.

Theorem 1[1] Let λ, ν and τ be real numbers with a, $b > 0$ and $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda}$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

We also recall the definition of the isotonic linear functionals which appears in [2].

Definition 1. ([2]) Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:

(L1) If $f, g \in L$ and $a, b \in \mathbb{R}$ then $(af + bg) \in L$.

(L2) $1 \in L$ i.e. if $f(t) = 1$ for all $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbb{R}$ then $A(af + bg) = aA(f) + bA(g)$

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$ then $A(f) \geq 0$.

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The mapping *A* is said to be normalized if

(A3) $A(1)=1$.

Now a new generalization of Holder's inequality for isotonic linear functionals. see [3], is given below staring from a new Young-type inequality given in [1].

Theorem 2. If L satisfies conditions L1, L2, A satisfies A1, A2 on the set E and $f^p, g^q, fg, f^{\frac{p}{p1}}, g^{\frac{q}{q1}} \in L$, $A(f^p)>0$, $A(g^q)>0$, $q=\frac{p}{p-1}$, p_1, q_1 with $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $1<p_1<p$ then we have:

$$\frac{p_1}{p}\left[1-\frac{A\left(f^{\frac{p}{p_1}}g^{\frac{q}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(f^p\right)A^{\frac{1}{q_1}}\left(g^q\right)}\right]<1-\frac{A(fg)}{A^{\frac{1}{p}}\left(f^p\right)A^{\frac{1}{q}}\left(g^q\right)}<\frac{q_1}{q}\left[1-\frac{A\left(f^{\frac{p}{p_1}}g^{\frac{q}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(f^p\right)A^{\frac{1}{q_1}}\left(g^q\right)}\right]$$

when *f* and *g* are positive functions.

The classical integral inequality due to Hardy states that for $f(x)\geq 0$ and $p>1$

$$\int_0^{\infty}\left\{\frac{1}{x}F(x)\right\}^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx,$$

where $F(x)=\int_0^x f(t)dt$.

If we suppose that all the integral exist on the respective domains of their definition, then a generalization of Hardy's inequality which was given by Pachpatte in [8] is the following:

Theorem A. [8] Let $p>1$, $m>1$ be two constants and *f* a nonnegative and integrable function on $(0,a)$, $0<a<\infty$. If $F(x)$ is defined by

$$F(x)=\int_{\frac{x}{2}}^{x}\frac{1}{t}\left\{\int_{\frac{t}{2}}^{t}\frac{f(s)}{s}ds\right\}dt,\quad x\in(0,a)$$

then

$$\int_0^a x^{-m}F^p(x)dx \leq \left(\frac{p}{m-1}\right)^{2p}\int_0^a x^{-m}\left|f(x)-f\left(\frac{x}{4}\right)\right|^p dx$$

2. SEVERAL VARIANTS OF HARDY-PACHPATTE-COPSON'S INEQUALITIES

As a consequence of Theorem 2 for the isotonic linear functional $A(f)=\int_a^b f(x)dx$ we have the following inequality:

Proposition 1. Let f and g be two positive functions and $q=\frac{p}{p-1}$, p_1, q_1 with $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $1<p_1<p$. We suppose that all the integral exist on the respective domains of their definition. Then we have:

$$\frac{p_1}{p}\left(1-\frac{\int_a^b f^{\frac{p}{p_1}}(x)g^{\frac{q}{q_1}}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p_1}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q_1}}}\right)<1-\frac{\int_a^b f(x)g(x)dx}{\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}}<$$

$$<\frac{q_1}{q}\left(1-\frac{\int_a^b f^{\frac{p}{p_1}}(x)g^{\frac{q}{q_1}}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p_1}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q_1}}}\right)$$

The following result is a new variant of Hardy-Pachpatte's inequality from [8].

We suppose like in [8] that all the integral exist on the respective domains of their definition and that $q=\frac{p}{p-1}$, p_1, q_1 with $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $1<p_1<p$ in all then further theorems, without further mention.

Theorem 3. Let $p>1$, $m>1$, p_1, q_1 be constants like before and f a nonnegative and integrable function on $(0,a), 0<a<\infty$. If $F(x)$ is defined by

$$F(x)=\int_{\frac{x}{2}}^{x}\frac{1}{t}\left\{\int_{\frac{t}{2}}^{t}\frac{f(s)}{s}ds\right\}dt,\ x\in(0,a)$$

then

$$\int_0^a x^{-m}F^p(x)dx\le\left(\frac{p}{m-1}\right)^{2p}\int_0^a x^{-m}\left|f(x)-f\left(\frac{x}{4}\right)\right|^p dx\ \times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{q1}}(x)\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^{\frac{p}{p_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{p_1}}\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right]^p\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}\left[\left|f(x)\right|-\left|f\left(\frac{x}{4}\right)\right|\right]^{\frac{p}{p_1}}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^{\frac{p}{q_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{q_1}}\left(\int_0^a x^{-m}\left[\left|f(x)\right|-\left|f\left(\frac{x}{4}\right)\right|\right]^p dx\right)^{\frac{1}{p_1}}}\right)\right]^p$$

Proof: We use the same method as in [8], but the classical Holder's inequality will be replaced by an improvement given in [3]. By integrating the left side of the inequality from Theorem 3 by parts we obtain:

$$\int_0^a x^{-m}F^p(x)dx$$
$$= -\frac{a^{-m+1}}{m-1}F^p(a)$$
$$+\frac{p}{m-1}\int_0^a x^{-m+1}F^{p-1}(x)\left[\frac{1}{x}\int_{\frac{x}{2}}^{x}\frac{f(s)}{s}ds-\frac{1}{2}\frac{2}{x}\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{f(s)}{s}ds\right]dx$$

and therefore

$$\int_0^a x^{-m}F^p(x)dx \le \frac{p}{m-1}\int_0^a x^{-m}F^{p-1}(x)\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}dx.$$

Using now the inequality from Proposition 1 with indices $q=\frac{p}{p-1}$ and p like below, we find that

$$\int_0^a x^{-m}F^p(x)dx \le \frac{p}{m-1}\int_0^a x^{-m}F^{p-1}(x)\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}dx =$$
$$=\frac{p}{m-1}\int_0^a\left[\{x^{-m}\}^{\frac{1}{q}}F^{p-1}(x)\right]\left[\{x^{-m}\}^{\frac{1}{p}}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}\right]dx \le$$
$$\le\frac{p}{m-1}\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{p}}\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{q}}\times$$
$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{q_1}}(x)\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^{\frac{p}{p_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{p_1}}\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right].$$

Now, dividing in previous inequality by the first integral factor and taking the p^th power of both sides we obtain:

$$\int_0^a x^{-m}F^p(x)dx \le \left(\frac{p}{m-1}\right)^p\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\times$$

$$\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{q_1}}(x)\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^{\frac{p}{p_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{p_1}}\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right]^p.$$

We take into account the expression $\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx$ and we use the same method like before. Integrating by parts this expression and using that the first term is negative, we will have:

$$\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx \le \frac{p}{m-1}\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^{p-1}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]dx.$$

Using again inequality from Proposition 1 with $q=\frac{p}{p-1}$ and p we find by calculus that

$$\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx \le$$

$$\le \frac{p}{m-1}\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{q}}\left(\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^p dx\right)^{\frac{1}{p}}\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^{\frac{p}{p_1}}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^{\frac{p}{q_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{q_1}}\left(\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^p dx\right)^{\frac{1}{p_1}}}\right)\right].$$

Dividing again both sides of last inequality by the first integral factor and the taking p^th power we obtain:

$$\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx \le \left(\frac{p}{m-1}\right)^p\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^p dx\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^{\frac{p}{p_1}}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^{\frac{p}{q_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{q_1}}\left(\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{4}\right)\right|\right]^p dx\right)^{\frac{1}{p_1}}}\right)\right]^p.$$

Taking into account the obtained inequalities we get the desired inequality.

A slightly different version of Theorem 3 will be also stated below and is an extension of Theorem 3 from [8].

Theorem 4. Let p, m, p_1, q_1 and f be defined as in Theorem 3. If

$$F(x)=\int_{\frac{x}{2}}^{x}\frac{1}{t}\left\{\int_0^t\frac{f(s)}{s}ds\right\}dt, x\in(0,a)$$

then

$$\int_0^a x^{-m}F^p(x)dx\le\left(\frac{p}{m-1}\right)^{2p}\int_0^a x^{-m}\left|f(x)-f\left(\frac{x}{2}\right)\right|^p dx\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{q_1}}(x)\left\{\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right\}^{\frac{p}{p_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{p_1}}\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right]^p\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{2}\right)\right|\right]^{\frac{p}{p_1}}\left\{\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right\}^{\frac{p}{q_1}}dx}{\left(\int_0^a x^{-m}\left\{\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right)^{\frac{1}{q_1}}\left(\int_0^a x^{-m}\left[|f(x)|-\left|f\left(\frac{x}{2}\right)\right|\right]^p dx\right)^{\frac{1}{p_1}}}\right)\right]^p.$$

Proof: We use the same method as in Theorem 3.

In $\mathbb{R}$ we have the following two theorems which are new improvements of the variant of the Hardy's inequality given by Izumi and Izumi in [6], Theorem 2 and [9], Theorems 3 and 4.

Theorem 5. Let $p>1, m>1, p_1, q_1$ be constants and $f(x)$ be a nonnegatiue and integrable function on $(0,b)$ where $b>0$ is a constant. Let $h(x)$ be a positive continuous function on $(0,b)$ and let $H(x)=\int_0^x h(t)dt$, for $x\in(0,b)$.

Let also $w(x)$ and $r(x)$ be positive and absolutely continuous functions on $(0,b)$. If

$$1-\frac{1}{m-1}\frac{H(x)}{h(x)}\frac{w'(x)}{w(x)}+\frac{p}{m-1}\frac{H(x)}{h(x)}\frac{r'(x)}{r(x)}\ge\frac{1}{\gamma}$$

for almost all $x\in(0,b)$ and some positive constant γ and $G(x)$ is defined by

$$G(x)=\frac{1}{r(x)}\int_{\frac{x}{2}}^{x}r(t)h(t)f(t)dt,$$

for $x\in(0,b)$, then

$$\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx\le\left(\gamma\frac{p}{m-1}\right)^p\int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)A^p(x)dx.$$

$$\cdot\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^b w(x)h^{1-\frac{p}{p_1}}(x)H^{\frac{p}{p_1}-m}(x)G^{\frac{p}{q_1}}(x)A^{\frac{p}{p_1}}(x)dx}{\left(\int_0^b w(x)h^{-(p-1)}(x)H^{p-m}(x)A^p(x)dx\right)^{\frac{1}{p_1}}\left(\int_0^b w(x)h(x)H^{-m}(x)G^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right]^p$$

where $A(x)=\frac{1}{r(x)}\left|r(x)h(x)f(x)-\frac{1}{2}r\left(\frac{x}{2}\right)h\left(\frac{x}{2}\right)f\left(\frac{x}{2}\right)\right|$.

Proof: We use the same method like in the proof of Theorem 3 from [9] and Proposition 1.

Theorem 6. Let $p, m, f, h, H, p_1, q_1, w$ and r be like in Theorem 5 and

$$1-\frac{1}{m-1}\frac{H(x)}{h(x)}\frac{w'(x)}{w(x)}-\frac{p}{m-1}\frac{H(x)}{h(x)}\frac{r'(x)}{r(x)}\geq\frac{1}{\delta}$$

for almost all $x\in(0,b)$ and some positive constant δ and $G(x)$ is defined by

$$G(x)=r(x)\int_{\frac{x}{2}}^{x}\frac{h(t)f(t)}{r(t)}dt,$$

for $x\in(0,b)$. Then we have

$$\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx\leq\left(\delta\frac{p}{m-1}\right)^p\int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)B^p(x)dx.$$

$$\cdot\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^b w(x)h^{1-\frac{p}{p_1}}(x)H^{\frac{p}{p_1}-m}(x)G^{\frac{p}{q_1}}(x)B^{\frac{p}{p_1}}(x)dx}{\left(\int_0^b w(x)h^{-(p-1)}(x)H^{p-m}(x)B^p(x)dx\right)^{\frac{1}{p_1}}\left(\int_0^b w(x)h(x)H^{-m}(x)G^p(x)dx\right)^{\frac{1}{q_1}}}\right)\right]^p$$

where $B(x)=r(x)\left|\frac{h(x)f(x)}{r(x)}-\frac{1}{2}\frac{h\left(\frac{x}{2}\right)f\left(\frac{x}{2}\right)}{r\left(\frac{x}{2}\right)}\right|$.

Proof: We use the same method like in the proof of Theorem 4 from [9] and inequality from Proposition 1.

Theorem 7. Let $a<b<R, p>1, q<1, \alpha>0, p_1$ and q_1 be constants with $\frac{1}{p_1}+\frac{1}{q_1}=1$, $1<p_1<p$ and $w(x), r(x)$ be positive and locally absolutely continuous in (a,b). Let $h(x)$ be a positive continuous function and $H(x)=\int_a^x h(t)dt$, for $x\in(a,b)$. Also, let $f(x)$ be nonnegative and measurable on (a,b) and

$$1-\frac{1}{1-q}\frac{H(x)}{h(x)}\frac{w'(x)}{w(x)}\log\frac{H(R)}{H(x)}+\frac{p}{1-q}\frac{H(x)}{h(x)}\frac{r'(x)}{r(x)}\log\frac{H(R)}{H(x)}\geq\frac{1}{\alpha},$$

for almost all x in (a,b). If $F(x)$ is defined by

$$F(x)=\frac{1}{r(x)}\int_a^x r(t)h(t)f(t)dt,$$

for all $x \in (a,b)$, then

$$\int_a^b w(x)H^{-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{-q} F^p(x)dx \leq$$
$$\leq \left(\frac{\alpha p}{1-q}\right)^p \int_a^b w(x)H^{p-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{p-q} f^p(x)dx.$$
$$.\left\{1-\frac{p_1}{p}[1-C]\right\}^p,$$

where

$$C = \frac{\int_a^b w(x)h(x)\left(H^{-1}(x)\right)^{1-\frac{p}{p_1}}\left(\log\frac{H(R)}{H(x)}\right)^{\frac{p}{p_1}-q} f^{\frac{p}{p_1}}(x)F^{\frac{p}{q_1}}(x)dx}{\left(\int_s^b w(x)h(x)f^p(x)\left(H^{-1}(x)\right)^{-(p-1)}\left(\log\frac{H(R)}{H(x)}\right)^{p-q}dx\right)^{\frac{1}{p_1}}\left(\int_a^b w(x)H^{-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{-q} F^p(x)dx\right)^{\frac{1}{q_1}}}$$

Proof: Also, we use the same method like in the proof of Theorem 7 from [9] and inequality from Proposition 1.

Theorem 8. Let $a<b<R, p>1, q>1, \beta>0$, p_1 and q_1 be constants with $\frac{1}{p_1}+\frac{1}{q_1}=1,\ 1<p_1<p$ and w,r,h,H,f be as in previous theorem. If

$$1+\frac{1}{q-1}\frac{H(x)}{h(x)}\frac{w'(x)}{w(x)}\log\frac{H(R)}{H(x)}-\frac{p}{q-1}\frac{H(x)}{h(x)}\frac{r'(x)}{r(x)}\log\frac{H(R)}{H(x)} \geq \frac{1}{\beta},$$

for almost all $x \in (a,b)$ and if $F(x)$ is defined by

$$F(x)=\frac{1}{r(x)}\int_x^b r(t)h(t)f(t)dt,$$

for all $x \in (a,b)$, then

$$\int_a^b w(x)H^{-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{-q} F^p(x)dx \leq$$
$$\leq \left(\frac{\beta p}{q-1}\right)^p \int_a^b w(x)H^{p-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{p-q} f^p(x)dx.$$
$$.\left\{1-\frac{p_1}{p}[1-C]\right\}^p,$$

where

$$C = \frac{\int_a^b w(x)h(x)\left(H^{-1}(x)\right)^{1-\frac{p}{p_1}}\left(\log\frac{H(R)}{H(x)}\right)^{\frac{p}{p_1}-q} f^{\frac{p}{p_1}}(x)F^{\frac{p}{q_1}}(x)dx}{\left(\int_s^b w(x)h(x)f^p(x)\left(H^{-1}(x)\right)^{-(p-1)}\left(\log\frac{H(R)}{H(x)}\right)^{p-q} dx\right)^{\frac{1}{p_1}}\left(\int_a^b w(x)H^{-1}(x)h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{-q} F^p(x)dx\right)^{\frac{1}{q_1}}}$$

Proof: The same method like in the proof of Theorem 7 and 8 from [9] and Proposition 1 will be used.

Using the function f_r, the Riemann-Lioville integral of f of order r, we give an extension of Theorem 2.3 from [7] which proves an extension of Hardy's inequality in two dimensions.

Theorem 9. Let $f, g \geq 0, p > 1, r > 0, \frac{1}{p} + \frac{1}{q} = 1$ and p_1 and q_1 be constants with $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $1 < p_1 < p$. We assume that $k(x.y)$ is a non-negative homogeneous functions of degree -2 λ with $\lambda > 1$ and that $\alpha_p = p(\lambda - 1) + 1$. If
$f_r(x) = \frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1} f(t)dt$, and $g_r(y) = \frac{1}{\Gamma(r)}\int_0^y (y-t)^{r-1}\, g(t)dt$,
then

$$\int_0^\infty\int_0^\infty \left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}} \left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}} k(x,y)dxdy <$$

$$< A^{\frac{1}{p}}(\lambda)B^{\frac{1}{q}}(\lambda)\Theta(p)\Theta(q)\left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times\left[1 - \frac{p_1}{p}\left(1 - \frac{\int_0^\infty\int_0^\infty (f_r(x))^{\frac{\alpha_p}{p_1}}(g_r(y))^{\frac{\alpha_q}{q_1}} k(x,y)x^{-\frac{(\lambda-1)(rp-1)+(r-1)}{p_1}} y^{-\frac{(\lambda-1)(rq-1)+(r-1)}{q_1}} dxdy}{A^{\frac{1}{p_1}}(\lambda)B^{\frac{1}{q_1}}(\lambda)\left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p_1}}\left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q_1}}}\right)\right]$$

where $A(\lambda) = \int_0^\infty t^{\lambda-1}\, k(1,t)dt, B(\lambda) = \int_0^\infty t^{\lambda-1}k(t,1)dt$
and

$$\Theta(s) = \left(\frac{\Gamma(1-\frac{1}{\alpha_s})}{\Gamma(r+1-\frac{1}{\alpha_s})}\right)^{\frac{\alpha_s}{s}}.$$

Proof: We use the same technique as in the proof of Theorem 2.3 from [7] and the following inequality

$$\int_0^\infty\int_0^\infty f(x,y)g(x,y)dxdy \le \left(\int_0^\infty\int_0^\infty f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty g^q(x,y)dxdy\right)^{\frac{1}{q}} \times$$
$$\left[1-\frac{p_1}{p}\left(\frac{\int_0^\infty\int_0^\infty f^{\frac{p}{p_1}}(x,y)\, g^{\frac{q}{q_1}}(x,y)dxdy}{\left(\int_0^\infty\int_0^\infty f^p(x,y)dxdy\right)^{\frac{1}{p_1}}\left(\int_0^\infty\int_0^\infty g^q(x,y)dxdy\right)^{\frac{1}{q_1}}}\right)\right]$$

which is a consequence of Proposition 1 when $\lambda = 1, p > 1, \frac{1}{p}+\frac{1}{q}=1$ and p_1 and q_1 be constants with $\frac{1}{p_1}+\frac{1}{q_1}=1,\ 1<p_1<p$.

Remark 1.

(a) Now, if we take $k(x,y)=\frac{1}{(x+y)^{2\lambda}}$ in previous theorem and $r=1$ then we also obtain a generalization of Theorem 1 from [10] as in [7], and therefore we have:

$$\int_0^\infty\int_0^\infty \frac{F^{\frac{\alpha_p}{p}}(x)G^{\frac{\alpha_q}{q}}(y)}{(x+y)^{2\lambda}}dxdy <$$
$$<\beta(\lambda,\lambda)\left(\frac{\alpha_p}{\alpha_p-1}\right)^{\frac{\alpha_p}{p}}\left(\frac{\alpha_q}{\alpha_q-1}\right)^{\frac{\alpha_q}{q}}\left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}}\times$$
$$\left[1-\frac{p_1}{p}\left(1-\frac{\int_0^\infty\int_0^\infty (F(x))^{\frac{\alpha_p}{p_1}}(G(y))^{\frac{\alpha_q}{q_1}}\frac{x^{-\frac{(\lambda-1)(p-1)}{p_1}}y^{-\frac{(\lambda-1)(q-1)}{q_1}}}{(x+y)^{2\lambda}}dxdy}{\beta(\lambda,\lambda)\left(\int_0^\infty\left(\frac{F(x)}{x}\right)^{\alpha_p}dx\right)^{\frac{1}{p_1}}\left(\int_0^\infty\left(\frac{G(y)}{y}\right)^{\alpha_q}dy\right)^{\frac{1}{q_1}}}\right)\right].$$

(b) We assume that $A,B>0$ and we take the kernel $k(x,y)=\frac{1}{(Ax+By)^{2\lambda}}$ in previous theorem and then a new generalization of Theorem 1, see [10] occur:

$$\int_0^\infty\int_0^\infty \frac{\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}}\left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}}{(Ax+By)^{2\lambda}}dxdy <$$
$$<\frac{\beta(\lambda,\lambda)}{(AB)^\lambda}\Theta(p)\Theta(q)\left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}}\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{A^{\lambda}B^{\lambda}\int_0^{\infty}\int_0^{\infty}\frac{(f_r(x))^{\frac{\alpha_p}{p_1}}(g_r(y))^{\frac{\alpha_q}{q_1}}}{(Ax+By)^{2\lambda}}x^{-\frac{(\lambda-1)(rp-1)+(r-1)}{p_1}}y^{-\frac{(\lambda-1)(rq-1)+(r-1)}{q_1}}dxdy}{\beta(\lambda,\lambda)\left(\int_0^{\infty}\left(\frac{f_r(x)}{x^r}\right)^{\alpha_p}dx\right)^{\frac{1}{p_1}}\left(\int_0^{\infty}\left(\frac{g_r(y)}{y^r}\right)^{\alpha_q}dy\right)^{\frac{1}{q_1}}}\right)\right].$$

(c) As in [7] if we take the kernel $k(x,y)=\frac{1}{x^{2\lambda}+y^{2\lambda}}$ in previous theorem we get:

$$\int_0^{\infty}\int_0^{\infty}\frac{\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}}\left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}}{x^{2\lambda}+y^{2\lambda}}dxdy<$$

$$<\frac{\pi\Theta(p)\Theta(q)}{2\lambda}\left(\int_0^{\infty}f^{\alpha_p}(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^{\alpha_q}(y)dy\right)^{\frac{1}{q}}\times$$

$$\times\left[1-\frac{p_1}{p}\left(1-\frac{2\lambda\int_0^{\infty}\int_0^{\infty}\frac{(f_r(x))^{\frac{\alpha_p}{p_1}}(g_r(y))^{\frac{\alpha_q}{q_1}}}{(x^{2\lambda}+y^{2\lambda})}x^{-\frac{(\lambda-1)(rp-1)+(r-1)}{p_1}}y^{-\frac{(\lambda-1)(rq-1)+(r-1)}{q_1}}dxdy}{\pi\left(\int_0^{\infty}\left(\frac{f_r(x)}{x^r}\right)^{\alpha_p}dx\right)^{\frac{1}{p_1}}\left(\int_0^{\infty}\left(\frac{g_r(y)}{y^r}\right)^{\alpha_q}dy\right)^{\frac{1}{q_1}}}\right)\right].$$

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