ORIGINAL PAPER

ON THE BERTRAND B-PAIR CURVES IN 3-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. *The aim of this paper is to study the Bertrand B-curve and the Bertrand B-pair curves according to BishopII frame in Euclidean space. First, we define Bertrand B-curve, Bertrand B-pair curves and study the properties of the Bertrand B-curves by using BishopII frame. Second, we examine the relationships between the Bishop curvatures of the Bertrand B-pair curves with respect to each other.*
Keywords: *BishopII frame, Bertrand curve, Bertrand B-curve.*

1. INTRODUCTION

In differential geometry, the theory of curves examines the geometric property of the plane and spae curves by means of calculus methods. The Frenet frame is most popular structureused in analyzing this calculas of curves. This structure, in the simplest sense, reveals the kinematic of the particle moving along a curve. It performs this movement thanks to the curvature and torsion of the curve, which measure how a curve bends [1]. Therefore, for researchers to estimate the curvature and torsion of a curve and to establish a relationship between this curvature has been of critical importance. Bertrand curves, especially, are one of the most important species of curves studied in this sense [2-6]. These curves were first found in 1850 by Joseph Louis François Bertrand who a French mathematician. Geometrically, Bertrand curve is defined its the principal normal vector as a curve that is shared with other special curve which is called Bertrand pair [7]. It is well-known that a curve α in E^3 with the curvature κ and the torsion τ is a Bertrand curve if and only if α is it satisfies that $a\kappa + b\tau = 1$ for constant a and b. In the literature, many authors have studied Bertrand curve in space with both Riemann and pseudo-Riemann metrics; in Lorentz-Minkowski space $\mathbb{R}_1^3$, in Riemann-Otsuki spaces, in the Galilean space G_3, in semi-Euclidean space $\mathbb{R}_\upsilon^{n+1}$ p [8-11].

In 1975, L.Bishop introduced an alternative frame on curves. This frame have a mutual vector field called the tangent vector with Serret-Frenet frame. Even if the second derivative of the curve is zero, Bishop frame is meaningful according to Frenet frame [12]. In recent years, many researchers were published many article that have a ralationship with this frame in various spaces [13, 14]. On the other hand, In 2010, Yılmaz and Turgut have defined Bishop II frame using the binormal vector B which is a mutual vector field conjunction with Serret-Frenet frame and have obtained several interesting results [15].

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In this work, we give similar the definition of Bertrand B-curve in the Bishop II frame to the concept of Bertrand curve existing on the Frenet frame and obtain relationships between Bertrand B-pair curve. Also, to the authors' knowledge, all studies related to Bertrand curves can also be made for the notion of Bertrand B-curve in space forms of three or higher dimensions with Riemann and pseudo-Riemann metrics.

2. PRELIMINARIES

First, we review some basic elements of the theory of curves in 3-dimensional Euclidean space E^3 (For details [1]). The three-dimensional Euclidean space E^3 is the vector space endowed with the metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 ,$$

where (x_1, x_2, x_3) is the coordinate system of E^3. For a vector $x \in E^3$, the norm of x is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. In addition, x is called a unit vector if its norm is equal to 1. For vectors $x, y \in E^3$ it is said to be ortoghonal if and only if $\langle x, y \rangle = 0$. The sphere of radius $r > 0$ is given by

$$S^2 = \{ x = (x_1, x_2, x_3) \in E^3 : \langle x, x \rangle = r \}.$$

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ are vectors in E^3 . Then the cross product of vectors x and y is defined by

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Yılmaz and Turgut [15] introduced a new type of Bishop frame by using binormal vector of a regular curve as the common vector field. This new Bishop frame is called Type-2 Bishop frame. The Type-2 Bishop Frame is expressed as

$$\begin{bmatrix} \zeta_1' \\ \zeta_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\in_1 \\ 0 & 0 & -\in_2 \\ \in_1 & \in_2 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix}, \tag{1}$$

the set $\{\zeta_1, \zeta_2, B\}$ is called Type-2 Bishop trihedra and $\in_1, \in_2$ are called type-2 bishop curvatures. If we denote the moving Frenet frame along the curve by $\{T, N, B\}$ where T, N and B are the tangent, the principal normal and the binormal vector of the curve, respectively. Then we can express the relation between The Type-2 Bishop frame and The Frenet frame by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix},$$

where $\theta = \arctan\left(\frac{\in_2}{\in_1}\right)$, $\kappa = \theta'(s)$ and $\tau = \sqrt{\in_1^2 + \in_2^2}$.

3. RESULTS

Definition 1: Let α, α^* be curves in 3-dimansional Euclidean space. We denote BishopII frame along the curves α and α^* at the points $\alpha(s)$ and $\alpha^*(s^*)$ by $\{\xi_1, \xi_2, B\}$ and $\{\xi_1^*, \xi_2^*, B^*\}$, respectively. If B and B^* are linearly dependant for every $s \in I$, then α curve called a Bertrand B-curve and α^* curve called a Bertrand B-mate curve. Nevertheless, (α, α^*) called a Bertrand B-pair curve.

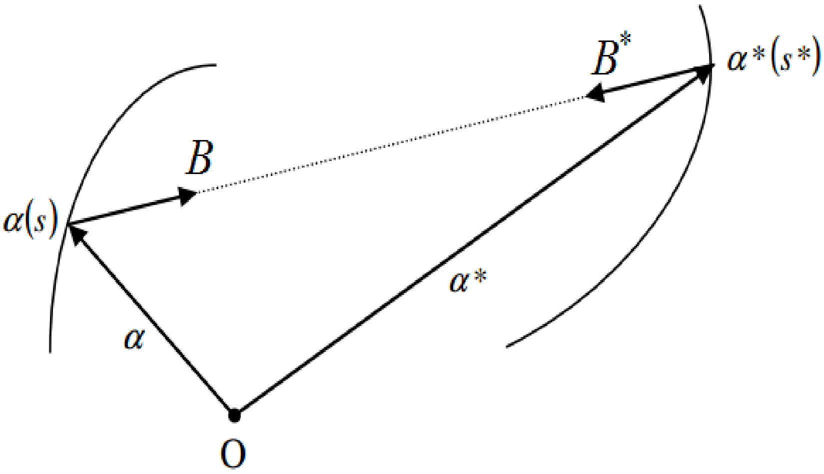

Figure 1. Bertrand B-Pair Curve.

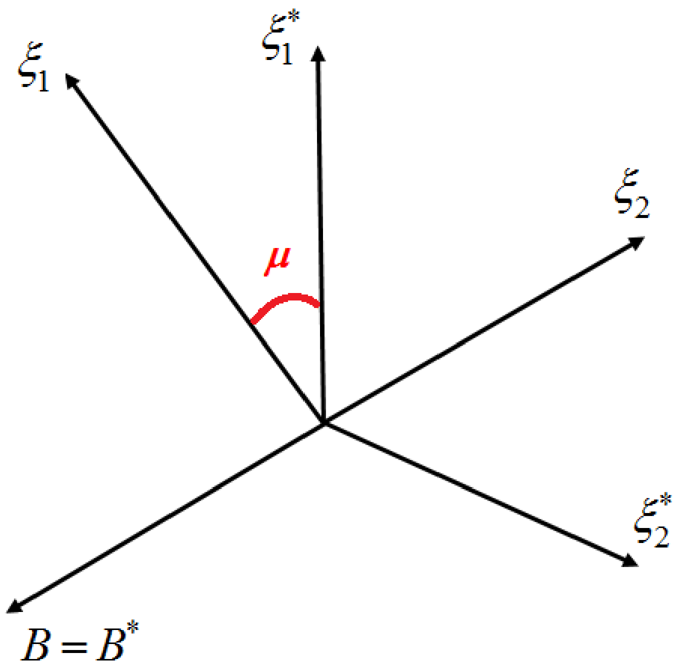

Figure 2. The Relationship Between Bertrand B-Pair Curves

Theorem 1: The distance between corresponding points of the Bertrand B-pair in E^3 is constant.

Proof: From the Fig. 1, we can write

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)B(s). \tag{2}$$

By taking the derivative of equation (2) with respect to s and using equation (1), we obtain

$$T^*(s^*)\frac{ds^*}{ds}=T(s)+\lambda'(s)B+\lambda(s)\{\varepsilon_1\xi_1+\varepsilon_2\xi_2\}. \tag{3}$$

Using a relationship between Frenet and Bishop frame, we get

$$\{\sin\theta^*(s^*)\xi_1^*-\cos\theta^*(s^*)\xi_2^*\}\frac{ds^*}{ds}=\{\sin\theta(s)+\lambda(s)\varepsilon_1\}\xi_1+\lambda'(s)B(s)+\{\lambda(s)\varepsilon_2-\cos\theta(s)\}\xi_2$$

.

Since B and B^* are linearly dependent, we have $\langle\xi_1^*,B\rangle=\langle\xi_2^*,B\rangle=0$. Then, we have

$$\lambda'=0.$$

This means that λ is a nonzero constant. Subsequently, from the distance function between two points, we obtain

$$\begin{aligned} d\left(\alpha^*\left(s^*\right),\alpha\left(s\right)\right)&=\left\|\alpha\left(s\right)-\alpha^*\left(s^*\right)\right\| \\ &=\left\|\lambda B^*\right\| \\ &=\left|\lambda\right| . \end{aligned}$$

In a word, $d\left(\alpha^*\left(s^*\right),\alpha\left(s\right)\right)=$ constant.

Theorem 2: The angle between corresponding points of both ξ_1,ξ_1^* and ξ_2,ξ_2^* vectors of a Bertarand B-pairs is constant.

Proof: Let's prove to the vector ξ_1. In terms of angle definition, we know that

$$\langle\xi_1,\xi_1^*\rangle=\|\xi_1\|.\|\xi_1^*\|\cos\mu .$$

By both side derivative with respect to s, we have

$$\left\langle\frac{d\xi_1}{ds},\xi_1^*\right\rangle+\left\langle\xi_1,\frac{d\xi_1^*}{ds}\right\rangle=\frac{d}{ds}\cos\mu .$$

Additionally, using derivative equations, we get

$$-\varepsilon_1\langle B,\xi_1^*\rangle-\varepsilon_1^*\frac{ds^*}{ds}\langle\xi_1,B^*\rangle=-\sin\mu\frac{d\mu}{ds} .$$

Since B and B^* are linearly dependent, we have both $\langle B, \xi_1^* \rangle = 0$ and $\langle \xi_1, B^* \rangle = 0$. As a result, we obtain

$$\frac{d\mu}{ds} = 0 .$$

This means that the angle between corresponding points of ξ_1 and ξ_1^* is a constant. Analogously, the angle between corresponding points of ξ_2 and ξ_2^* is also a constant.

Theorem 3: Let $\{\alpha, \alpha^*\}$ be a Bertrand B-pair in E^3. Then, there exist the relationship between the curvatures of the curves α and α^* such that

$$\cot\left(\arctan(\frac{\varepsilon_2^*}{\varepsilon_1^*}) + \mu \right) = \frac{\varepsilon_1 - \lambda\varepsilon_1\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{\varepsilon_2 + \lambda\varepsilon_1\sqrt{\varepsilon_1^2 + \varepsilon_2^2}} .$$

Proof: Considering in equation (3) as constant of λ and using a relationship between frames , we get

$$\left\{ \sin\theta^*(s^*)\xi_1^* - \cos\theta^*(s^*)\xi_2^* \right\} \frac{ds^*}{ds} = \left\{ \sin\theta(s) + \lambda\varepsilon_1 \right\} \xi_1 + \left\{ \lambda\varepsilon_2 - \cos\theta(s) \right\} \xi_2 . \tag{4}$$

Also, from Fig. 2, we know that

$$\begin{aligned} \xi_1^* &= \cos\theta\xi_1 + \sin\theta\xi_2 \\ \xi_2^* &= -\sin\theta\xi_1 + \cos\theta\xi_2 \end{aligned} \tag{5}$$

where μ is the angle between ξ_1 and ξ_1^* at the corresponding points of α and α^*. By taking into considering equations (4) and (5), we have

$$\cos(\theta^*(s^*) + \mu) = \frac{ds}{ds^*}(\cos\theta(s) - \lambda\varepsilon_2) \text{ and } \sin(\theta^*(s^*) + \mu) = \frac{ds}{ds^*}(\sin\theta(s) + \lambda\varepsilon_1) . \tag{6}$$

Proportioning the equations (6) and arranging this equations, we get

$$\cot(\theta^*(s^*) + \mu) = \frac{\cos\theta(s) - \lambda\varepsilon_2}{\sin\theta(s) + \lambda\varepsilon_1} .$$

Finally, if we write to place $\theta(s), \theta^*(s^*)$ and μ statements, then the desired expression is obtained.

Corollary 1: If $\mu = \frac{\pi k}{2}, k \in \mathbb{Z}$ and the curve α is a planar, then the relationship between the curvatures of the curves α and α^* as follows:

$$\varepsilon_1^* \varepsilon_1 + \varepsilon_2^* \varepsilon_2 = 0 .$$

Corollary 2: If $\mu = \pi k, k \in \mathbb{Z}$ and the curve α is a planar, then the relationship between the curvatures of the curves α and α^* as follows:

$$\varepsilon_1^* \varepsilon_2 - \varepsilon_2^* \varepsilon_1 = 0 .$$

Theorem 4: Let $\{\alpha, \alpha^*\}$ be Bertrand B-pairs in E^3. If respectively $\varepsilon_1, \varepsilon_2$ and $\varepsilon_1^*, \varepsilon_2^*$ the curvatures of the curves α and α^*, then the distance function between these curves is

$$\lambda = \frac{\left(\varepsilon_1 \varepsilon_2^* + \varepsilon_2 \varepsilon_1^*\right)\left(\sqrt{\varepsilon_1^{*2} + \varepsilon_2^{*2}} - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)}{\left(\varepsilon_1 \varepsilon_1^* - \varepsilon_2 \varepsilon_2^*\right)\sqrt{\varepsilon_1^2 + \varepsilon_2^2}\sqrt{\varepsilon_1^{*2} + \varepsilon_2^{*2}}} .$$

Proof: From equation (6), we have

$$\cos(\theta^*(s^*) + \mu) = \frac{ds}{ds^*}(\cos\theta(s) - \lambda\varepsilon_2)$$

and

$$\sin(\theta^*(s^*) + \mu) = \frac{ds}{ds^*}(\sin\theta(s) + \lambda\varepsilon_1) .$$

Also, since $\{\alpha, \alpha^*\}$ is the Bertrand B-pairs, we write

$$\alpha(s) = \alpha^*(s^*) - \lambda B^*(s^*) .$$

Using the process in the proof of theorem (3), we get

$$\cos(\theta(s) - \mu) = \frac{ds^*}{ds}(\cos\theta^*(s^*) + \lambda\varepsilon_2^*)$$

and

$$\sin(\theta(s) - \mu) = \frac{ds^*}{ds}(\sin\theta^*(s^*) - \lambda\varepsilon_1^*)$$

Multiplying the equations (7)-(9) and (8)-(10) and using trigonometric equations, we obtain the distance λ.

Theorem 5: Let $\{\alpha,\alpha^*\}$ be Bertrand B-pairs in E^3. Then, there exist the relationship between the curvatures of the curves α and α^* such that

i. $\varepsilon_1 = \cos\theta \dfrac{ds}{ds^*}\varepsilon_1^* + \sin\theta \dfrac{ds}{ds^*}\varepsilon_2^*$

ii. $\varepsilon_2 = -\sin\theta \dfrac{ds}{ds^*}\varepsilon_1^* + \cos\theta \dfrac{ds}{ds^*}\varepsilon_2^*$

iii. $\varepsilon_1^* = \cos\theta \dfrac{ds^*}{ds}\varepsilon_1 - \sin\theta \dfrac{ds^*}{ds}\varepsilon_2$

iv. $\varepsilon_2^* = \sin\theta \dfrac{ds^*}{ds}\varepsilon_1 + \cos\theta \dfrac{ds^*}{ds}\varepsilon_2$

Proof: (i) Since B and B^* are linearly dependent, we have $\left\langle \xi_1, B^* \right\rangle = 0$. By taking the derivative with respect to s, we get

$$\left\langle -\varepsilon_1 B, B^* \right\rangle + \left\langle \xi_1, \varepsilon_1^* \xi_1^* \frac{ds}{ds^*} + \varepsilon_2^* \xi_2^* \frac{ds}{ds^*} \right\rangle = 0 .$$

Using equation (5), we obtain

$$\varepsilon_1 = \cos\theta \frac{ds}{ds^*}\varepsilon_1^* + \sin\theta \frac{ds}{ds^*}\varepsilon_2^* .$$

Analogously, considering equations $\left\langle \xi_2, B^* \right\rangle, \left\langle \xi_1^*, B \right\rangle, \left\langle \xi_2^*, B \right\rangle$, the proof of the statement (ii), (iii), (iv) is obvious.

In the theorem 5, from the statement both (i), (ii) and (iii), (iv), we can give in the following result:

Corollary 3: Let $\{\alpha,\alpha^*\}$ be Bertrand B-pairs in E^3. Then, there exist the relationship between the arc-parameters of the curves α and α^* such that

$$s = \int \frac{\varepsilon_1^2 + \varepsilon_2^2}{\varepsilon_1^{*2} + \varepsilon_2^{*2}} ds^* .$$

Theorem 6: Let $\{\alpha,\alpha^*\}$ be Bertrand B-pairs in E^3. Then, for the curvature centers M and M^* at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ of the curves α and α^*, the ratio

$$\frac{\left\|\alpha^*\left(s^*\right)M\right\|}{\left\|\alpha\left(s\right)M\right\|}:\frac{\left\|\alpha^*\left(s^*\right)M^*\right\|}{\left\|\alpha\left(s\right)M^*\right\|}$$

is not constant.

Proof: For the curvature center M at the point $\alpha\left(s\right)$ of the curve α, we write

$$\left\|\alpha\left(s\right)M\right\|=\frac{1}{\varepsilon_1}.$$

Similarly, we get

$$\left\|\alpha\left(s\right)M^*\right\|=\sqrt{\lambda^2+\frac{1}{\varepsilon_1^{*2}}}$$

$$\left\|\alpha^*\left(s^*\right)M^*\right\|=\frac{1}{\varepsilon_1^*}$$

$$\left\|\alpha^*\left(s^*\right)M\right\|=\sqrt{\lambda^2+\frac{1}{\varepsilon_1^2}}.$$

Hence, we obtain the ratio such that

$$\frac{\left\|\alpha^*\left(s^*\right)M\right\|}{\left\|\alpha\left(s\right)M\right\|}:\frac{\left\|\alpha^*\left(s^*\right)M^*\right\|}{\left\|\alpha\left(s\right)M^*\right\|}=\sqrt{\left(1+\lambda^2\varepsilon_1^2\right)\left(1+\lambda^2\varepsilon_1^{*2}\right)}$$

$$\neq \text{ constant.}$$

Result 1: Mannheim's theorem is invalid for the Bertrand B-pair curves.

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