ORIGINAL PAPER

EXACT SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS ARISING IN MATHEMATICAL PHYSICS BY $(G'/G, 1/G)$-EXPANSION METHOD

MUHAMMAD SHAKEEL[1], JAMSHAD AHMAD[2], QAZI MAHMOOD UL HASSAN[3], MUHAMMAD NAEEM[4]

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Abstract. *In this article one of the most reliable and effective method,* $(G'/G, 1/G)$*-expansion method has been employed to obtain exact traveling wave solutions of highly nonlinear partial differential equations (PDEs). The set of abundant exact traveling wave solutions of two very important nonlinear evolution equations of mathematical physics, i.e., modified Benjamin-Bona-Mahony (mBBM) and (2 + 1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equations are developed. The comparison of the obtained numerical results with the existing along with the graphical representation is presented. It is shown that the Bi variable* $(G'/G, 1/G)$*-expansion method is a potent and very concise mathematical technique for solving nonlinear problems.*

Keywords: $(G'/G, 1/G)$*-expansion method; mBBM equation; CBS equation; Solitary wave solutions; Exact solutions.*

MSC: *35C07; 35C08; 35P99*

1. INTRODUCTION

The study of Nonlinear evolution equations play a momentous role in various scientific and engineering fields, and is widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, propagation of shallow water waves, etc. In recent decades, there has been a great amount of activity aiming to find methods for solutions of NLPDEs and exact solutions of these NLPDEs have been investigated by many researchers [1-45] who are concerned in nonlinear physical phenomena. An extensive list of non-integrable nonlinear PDEs, there is a class of equations that can be referred to as the partially integrable, because these equations can be transformed to integrable form for some values of the involved parameters. There are many different techniques to derive for the exact solutions of these equations. The most famous algorithms are the truncated Painleve expansion method [1], the tanh-function method [2-6] and the Jacobi elliptic function expansion method [7-10]. There are other methods which can be found in [11-13]. For integrable nonlinear differential equations, the inverse scattering transform method [14], the Hirota method [15], truncated Painleve expansion method [16],

[1] Mohi-ud-Din Islamic University, Departmen of Mathematics, Nerian Sharif AJ&K, Pakistan.
[2] University of Gujrat, Faculty of Sciences, Department of Mathematics, Pakistan.
E-mail: jamshadahmadm@gmail.com.
[3] University of Wah, Department of Mathematics, Wah Cantt., Pakistan.
[4] UET Lahore, Faculty of Sciences, Department of Mathematics, Pakistan.

Backlund transform method [17] and the Exp-function method [18-20] are used for searching the exact solutions.

A direct and concise method called (G'/G)-expansion method was introduced by Wang et al [21] to look for traveling wave solutions of nonlinear partial differential equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$; λ and μ are arbitrary constants. For additional references see the articles [22-28]. It is to be highlighted that LI et al. [29] applied the $(G'/G, 1/G)$-expansion method on certain nonlinear evolution equations to attain their exact traveling wave solutions. They constructed the set of abundant traveling wave solutions of Zakharov equations with arbitrary parameters and solitary wave solutions when the parameters are replaced by special values. Zayed et al. [30, 31] also extended the application of $(G'/G, 1/G)$-expansion method to nonlinear (3+1)-dimensional Kadomtsev-Petviashvili, nonlinear KdV-mKdV for traveling wave solutions. A detail work on (G'/G)-expansion method [32, 33] may be witnessed.

Due to deep interest and motivated by the ongoing research in this hot area, we extended the approach $(G'/G, 1/G)$-expansion method to find the exact traveling wave solutions of two well known nonlinear evolution equations; modified Benjamin-Bona-Mahony (mBBM) and (2 +1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equations.

2. METHODOLOGY

In this section, we present main steps of the $(G'/G, 1/G)$-expansion method for constructing traveling wave solutions of nonlinear evolution equation. Suppose a nonlinear equation for $P(x, y, t)$ is defined as

$$P(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{1}$$

in which both nonlinear term(s) and higher order derivatives of $P(x, y, t)$ are all involved. In general, the left-hand side of (1) is a polynomial in ψ and its various derivatives. The $(G'/G, 1/G)$-expansion method for solving (1) proceed in the following steps:

Step 1: Look for traveling wave solution of (1) by taking

$$P = P(\xi), \xi = x + y - V t, \tag{2}$$

where V is nonzero constant, $P(\xi)$ the function of ξ. Substituting (2) into (1) yields an ordinary differential equation (ODE) for $P(\xi)$.

$$Q(u, -Vu', u', u'', V^2u'', \cdots) = 0. \tag{3}$$

Step 2: If possible, integrate (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) may be set to zero.

Step 3: According to the bi variable $(G'/G, 1/G)$-expansion method the solution $u(\xi)$ can be expressed by a finite power series in the form

$$u(\xi) = \sum_{n=0}^{M} a_n \varphi^n + \sum_{n=1}^{M} b_n \varphi^{n-1} \psi, \tag{4}$$

where a_n ($n = 1, 2, 3, \cdots, M$) and b_n ($n = 1, 2, 3, \cdots, M$) are constants to be determine later and $\varphi(\xi)$ and $\psi(\xi)$ are given by

$$\varphi(\xi) = \left(\frac{G'(\xi)}{G(\xi)} \right), \psi(\xi) = \left(\frac{1}{G(\xi)} \right), \tag{5}$$

which satisfies

$$G''(\xi) + \lambda G(\xi) = \mu \tag{6}$$

The equations (5) and (6) yields

$$\varphi' = -\varphi^2 + \mu\varphi - \lambda \qquad \psi' = \varphi\psi, \tag{7}$$

From the three cases of general solutions of (6), we have:

Case 1: When $\lambda < 0$ the general solution of (6) is

$$G(\xi) = A\sinh\left(\sqrt{-\lambda}\xi\right) + B\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda},$$

we have

$$\psi^2 = -\frac{\lambda}{\lambda^2\sigma + \mu^2}\left(\varphi^2 - 2\mu\psi + \lambda\right), \tag{8}$$

where A and B are two arbitrary constants and $\sigma = A^2 - B^2$.

Case 2: When $\lambda > 0$ the general solution of (6) is

$$G(\xi) = A\sin\left(\sqrt{\lambda}\xi\right) + B\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda},$$

we have

$$\psi^2 = \frac{\lambda}{\lambda^2\varepsilon - \mu^2}\left(\varphi^2 - 2\mu\psi + \lambda\right), \tag{9}$$

where A and B are two arbitrary constants and $\varepsilon = A^2 + B^2$.

Case 3: When $\lambda = 0$ the general solution of (6) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A\xi + B,$$

and we have

$$\psi^2 = \frac{1}{A^2 - 2\mu B}\left(\varphi^2 - 2\mu\psi\right), \qquad (10)$$

where A and B are two arbitrary constants.

Step 4: Determine M. This usually can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) obtained in step 2.

Step 5: Substituting (4) into (3) and using equations (7) and (8), will yield a polynomial in φ and ψ in which the degree of ψ is not larger than 1.Compare the like powers of φ^M and $\varphi^M\psi$ equal to zero, yields a set of algebraic equations for a_n $(n = 0,1,2,3,\cdots,M)$ and b_n $(n = 1,2,3,\cdots,M), \mu, \lambda, A, B$ and V.

Step 6: Solve the system which is obtained in step 5 for a_n $(n = 0,1,2,3,\cdots,M)$ and b_n $(n = 1,2,3,\cdots,M), \mu, \lambda, A, B$ and V with the help of symbolic computational software Maple, to determine these constants. Putting the values of these constants into (4), one can obtain the traveling wave solutions expressed by the hyperbolic functions of (2). We can obtain the more general type and new exact traveling wave solution of the nonlinear partial differential equation (1).

Step 7: Similarly substituting (4) into (3) and using equations (7) and (9) (or equations (7) and (10)) will yield a polynomial in φ and ψ in which the degree of ψ is not larger than 1.Compare the like powers of φ^M and $\varphi^M\psi$ equal to zero, yields a set of algebraic equations for a_n $(n = 0,1,2,3,\cdots,M)$ and b_n $(n = 1,2,3,\cdots,M), \mu, \lambda, A, B$ and V, we obtain traveling wave solutions of (1) which are expressed by trigonometric functions (or expressed by rational functions) as proceeding before.

3. NUMERICAL APPLICATIONS

In this section, we will exhibit the bi variable $(G'/G, 1/G)$-expansion method on two well-known nonlinear evolution equations, namely the modified Benjamin-Bona-Mahony (mBBM) equation and the (2 +1) – dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. Numerical results are very encouraging.

3.1 MODIFIED BENJAMIN-BONA-MAHONY EQUATION

Let us consider the nonlinear Modified Benjamin-Bona-Mahony (mBBM) Equation

$$u_t + u_x + u^2 u_x + u_{xxt} = 0, \tag{11}$$

which was first derived to describe an approximation for surface long waves in nonlinear dispersive media The equation can also characterize the hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals and acoustic-gravity waves in compressible fluids [34]. Khan et al. [35] used the modified simple equation method to find the exact solutions for the modified Benjamin-Bona-Mahony (mBBM) equation. Naher and Abdullah [36] applied the extended generalized Riccati equation mapping method to find the exact traveling wave solutions including solitons and periodic solutions of this equation. Aslan [37] find exact and explicit solutions to this equation by utilizing the (G'/G) -expansion method. Yusufoglu [38] find new solitonary solutions for the mBBM equation by using Exp-function method.

Equation (2) permits us to convert (11) into an ordinary differential equation,

$$\left(1-V\right) u' + u^2 u' - V u''' = 0, \tag{12}$$

where prime denotes the derivative with respect to ξ. Considering the homogeneous balancing principle between u''' and $u^2 u'$, we deduce that $M = 1$. Therefore the trail solution becomes

$$u = a_0 + a_1 \varphi + b_1 \psi, \tag{13}$$

where a_0, a_1 and b_1 are constants to be determined later. There are three cases to be discussed as follows:

Case 1: If $\lambda < 0$, substituting (13) into (12) and using equations (7) and (8), the left-hand side of (12) becomes a polynomial in φ and ψ . Setting the coefficients of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, \mu, \lambda, V$ and σ (which are not shown here for the sake of simplicity) as follows.

Solving the obtained system of algebraic equation by the symbolic software Maple, we have the following solution sets:

Result 1. We have

$$a_0 = \sqrt{-\frac{3\lambda}{(\lambda+2)\mu^2 + (1-\lambda)2\lambda^2\sigma}}\mu,\ b_1 = -\sqrt{-\frac{12}{(\lambda+2)\lambda\mu^2 + (1-\lambda)2\lambda^3\sigma}}\ (\lambda^2\sigma + \mu^2),$$
$$a_1 = 0,\ V = \frac{2(\lambda^2\sigma + \mu^2)}{(\lambda+2)\mu^2 + (1-\lambda)2\lambda^2\sigma},\ \sigma = A^2 - B^2. \tag{14}$$

Now, the traveling wave solution of (11) becomes:

$$u(x,t)=\sqrt{-\frac{3\lambda}{(\lambda+2)\mu^2+(1-\lambda)2\lambda^2\sigma}}\mu-\sqrt{-\frac{12}{(\lambda+2)\lambda\mu^2+(1-\lambda)2\lambda^3\sigma}}(\lambda^2\sigma+\mu^2) \times\left(\frac{\lambda}{A\sinh\left(\sqrt{-\lambda}\xi\right)\lambda+B\cosh\left(\sqrt{-\lambda}\xi\right)\lambda+\mu}\right), \quad (15)$$

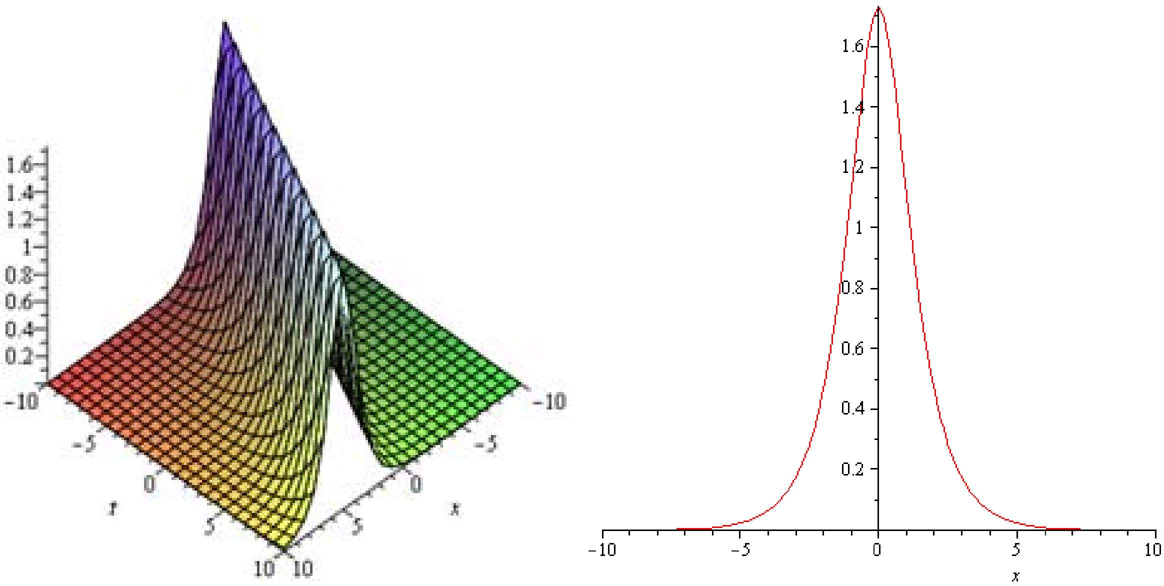
where

$$\xi=x-\left(\frac{2(\lambda^2\sigma+\mu^2)}{(\lambda+2)\mu^2+(1-\lambda)2\lambda^2\sigma}\right)t.$$

In particular, if we take $A=0, B>0$ and $\mu=0$ or $A>0, B=0$ and $\mu=0$ in (15), we have the solitary solutions

$$u(x,t)=\sqrt{\frac{6\lambda}{1-\lambda}}\operatorname{sec}h\left(\sqrt{-\lambda}\xi\right), \quad (16)$$

$$u(x,t)=-\sqrt{\frac{-6\lambda}{1-\lambda}}\operatorname{csc}h\left(\sqrt{-\lambda}\xi\right). \quad (17)$$



(a) (b)

Figure 1. (a) 3D and (b) 2D traveling wave solutions of Eq. (16) for different values of parameters.

Result 2. We have

$$a_0=0,\ \ a_1=\sqrt{\frac{3}{\lambda+2}},\ \ b_1=\sqrt{-\frac{3(\lambda^2\sigma+\mu^2)}{(\lambda+2)\lambda}},\ V=\frac{2}{\lambda+2},\ \ \sigma=A^2-B^2. \quad (18)$$

Now, the traveling wave solution of (11) becomes:

$$u(x,t)=-\sqrt{\frac{3}{\lambda+2}}\left(\frac{(-\lambda)^{3/2}\left(A\cosh\left(\sqrt{-\lambda}\xi\right)+B\sinh\left(\sqrt{-\lambda}\xi\right)\right)}{A\sinh\left(\sqrt{-\lambda}\xi\right)\lambda+B\cosh\left(\sqrt{-\lambda}\xi\right)\lambda+\mu}\right)+\sqrt{-\frac{3(\lambda^2\sigma+\mu^2)}{(\lambda+2)\lambda}}$$
$$\times\left(\frac{\lambda}{A\sinh\left(\sqrt{-\lambda}\xi\right)\lambda+B\cosh\left(\sqrt{-\lambda}\xi\right)\lambda+\mu}\right), \tag{19}$$

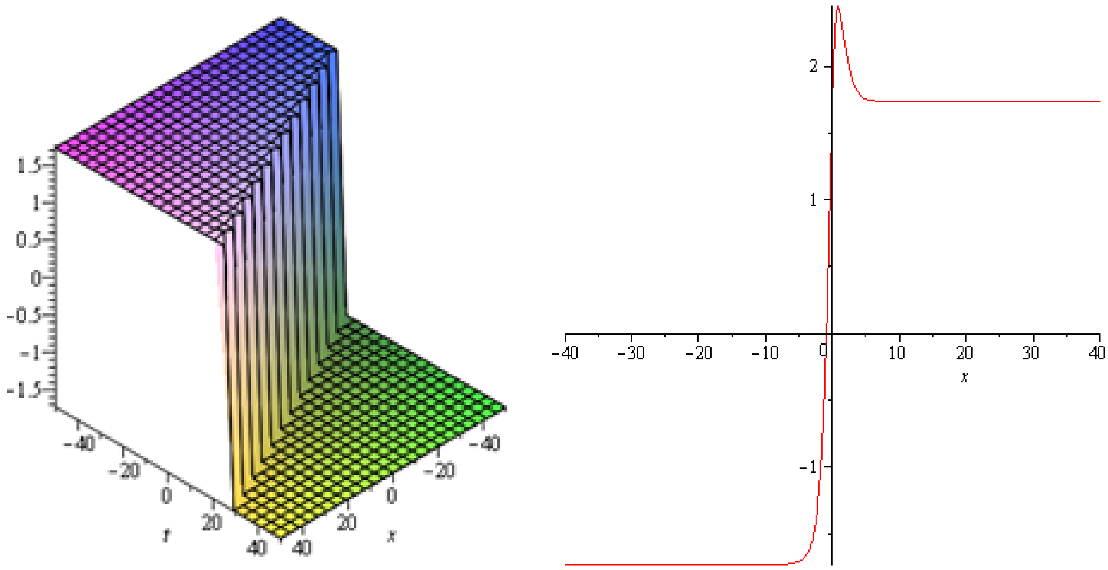
where

$$\xi=x-\left(\frac{2}{\lambda+2}\right)t.$$

In particular, by setting $A=0, B>0$ and $\mu=0$ or $A>0, B=0$ and $\mu=0$ in (19), we have the solitary wave solutions

$$u(x,t)=\sqrt{\frac{-3\lambda}{\lambda+2}}\tanh\left(\sqrt{-\lambda}\xi\right)+\sqrt{\frac{3\lambda}{\lambda+2}}\operatorname{sec}h\left(\sqrt{-\lambda}\xi\right), \tag{20}$$

$$u(x,t)=\sqrt{-\frac{3\lambda}{\lambda+2}}\left(\coth\left(\sqrt{-\lambda}\xi\right)+\operatorname{csc}h\left(\sqrt{-\lambda}\xi\right)\right). \tag{21}$$



(a) **(b)**

Figure 2. (a) 3D and (b) 2D kink solutions of Eq. (20) for different values of parameters.

Case 2: If $\lambda>0$, substituting (13) into (12) and using equations (7) and (9), the left-hand side of (12) becomes a polynomial in φ and ψ. Similar to case 1, after solving the system of algebraic equations, we obtain the following results.

Result 1. We have

$$a_0 = \sqrt{-\frac{3\lambda}{(\lambda+2)\mu^2+(\lambda-1)2\lambda^2\varepsilon}}\mu,\ b_1 = -\sqrt{-\frac{12}{(\lambda+2)\lambda\mu^2+(\lambda-1)2\lambda^3\varepsilon}}\ (\mu^2-\lambda^2\varepsilon),$$
$$a_1 = 0,\ V = \frac{2(\mu^2-\lambda^2\varepsilon)}{(\lambda+2)\mu^2+(\lambda-1)2\lambda^2\varepsilon},\ \ \varepsilon = A^2+B^2. \tag{22}$$

Now, the traveling wave solution of (11) becomes:

$$u(x,t) = \sqrt{-\frac{3\lambda}{(\lambda+2)\mu^2+(\lambda-1)2\lambda^2\sigma}}\mu - \sqrt{-\frac{12}{(\lambda+2)\lambda\mu^2+(\lambda-1)2\lambda^3\varepsilon}}\ (\mu^2-\lambda^2\varepsilon)$$
$$\times\left(\frac{\lambda}{A\sin\left(\sqrt{\lambda}\xi\right)\lambda+B\cos\left(\sqrt{\lambda}\xi\right)\lambda+\mu}\right), \tag{23}$$

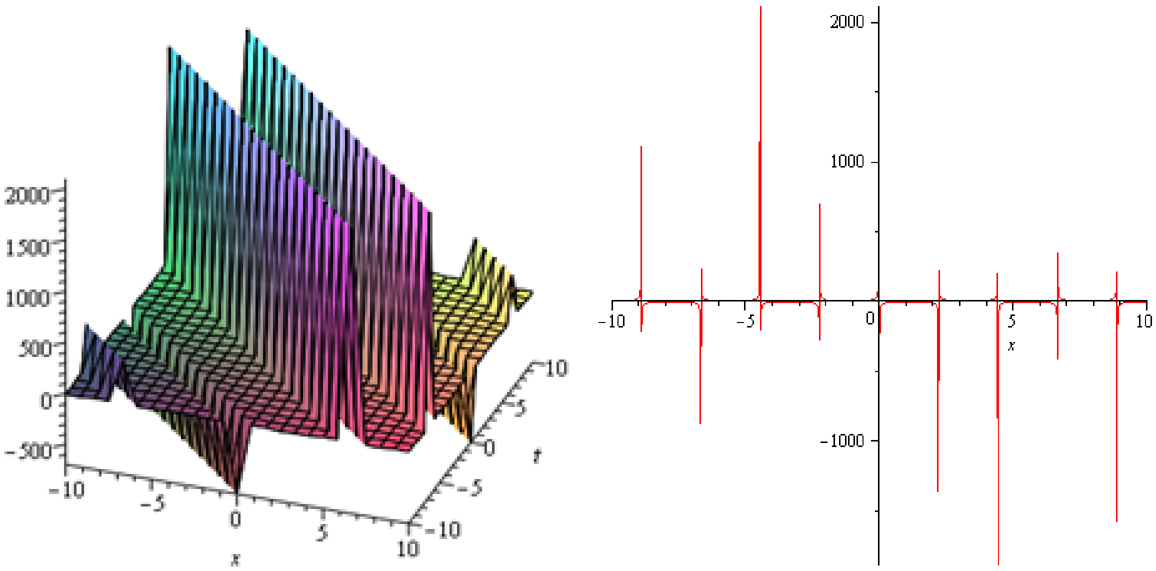
where

$$\xi = x - \left(\frac{2(\mu^2-\lambda^2\varepsilon)}{(\lambda+2)\mu^2+(\lambda-1)2\lambda^2\varepsilon}\right)t.$$

In particular, if we take $A>0, B=0$ and $\mu=0$ or $A=0, B>0$ and $\mu=0$ in (23), we have the solitary solutions:

$$u(x,\ t) = \sqrt{\frac{6\lambda}{1-\lambda}}\csc\left(\sqrt{\lambda}\xi\right), \tag{24}$$

$$u(x,\ t) = \sqrt{\frac{6\lambda}{1-\lambda}}\sec\left(\sqrt{\lambda}\xi\right). \tag{25}$$



(a) (b)

Figure 3. (a) 3D and (b) 2D represent the periodic solutions of Eq. (25) for different values of parameters.

Result 2. We have

$$a_0 = 0,\ \ a_1 = \sqrt{\frac{3}{\lambda+2}},\ \ b_1 = \sqrt{\frac{3(\lambda^2\varepsilon-\mu^2)}{\lambda(\lambda+2)}}\ ,\ V = \frac{2}{\lambda+2},\ \ \varepsilon = A^2 + B^2. \tag{26}$$

Now, in this result the traveling wave solution of (11) becomes:

$$u(x,t) = \sqrt{\frac{3}{\lambda+2}}\left(\frac{(\lambda)^{3/2}\left(A\cos\left(\sqrt{\lambda}\xi\right)-B\sin\left(\sqrt{\lambda}\xi\right)\right)}{A\sin\left(\sqrt{\lambda}\xi\right)\lambda+B\cos\left(\sqrt{\lambda}\xi\right)\lambda+\mu}\right)+\sqrt{\frac{3(\lambda^2\varepsilon-\mu^2)}{\lambda(\lambda+2)}} \times\left(\frac{\lambda}{A\sin\left(\sqrt{\lambda}\xi\right)\lambda+B\cos\left(\sqrt{\lambda}\xi\right)\lambda+\mu}\right), \tag{27}$$

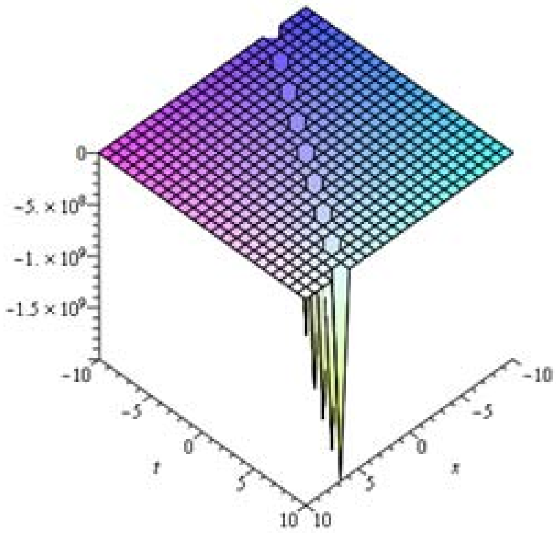
where

$$\xi = x - \left(\frac{2}{\lambda+2}\right)t.$$

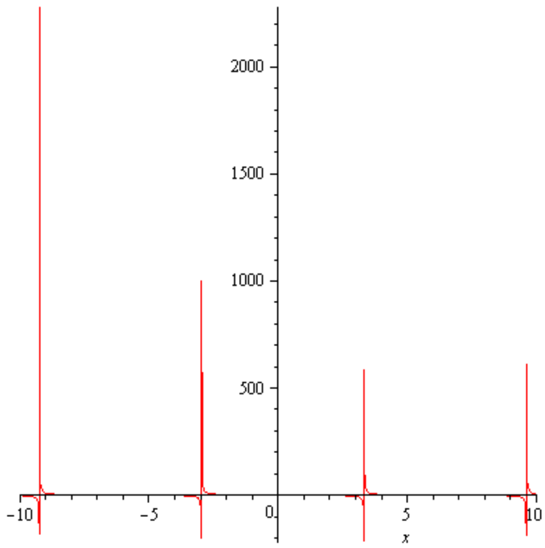
In particular, by setting $A=0, B>0$ and $\mu=0$ or $A>0, B=0$ and $\mu=0$ in (27), we have

$$u(x,t) = \sqrt{\frac{3\lambda}{\lambda+2}}\left(-\tan\left(\sqrt{\lambda}\xi\right)+\sec\left(\sqrt{\lambda}\xi\right)\right), \tag{28}$$

$$u(x,t) = \sqrt{\frac{3\lambda}{\lambda+2}}\left(\cot\left(\sqrt{\lambda}\xi\right)+\csc\left(\sqrt{\lambda}\xi\right)\right). \tag{29}$$



(a)



(b)

Figure 4. (a) 3D and (b) 2D represent the periodic traveling wave solutions of Eq. (29) for different values of parameters.

Case 3: When $\lambda = 0$, by analogous computations like case 1 and 2, we have

Result 1. We have

$$a_0 = \sqrt{\frac{3}{2A^2-4\mu B-3\mu^2}}\mu,\ \ b_1 = \sqrt{\frac{12}{2A^2-4\mu B-3\mu^2}}\ (A^2-2\mu B),$$
$$a_1 = 0,\ \ V = \frac{2(A^2-2\mu B)}{2A^2-4\mu B-3\mu^2}. \tag{30}$$

Now, the traveling wave solution of (11) becomes:

$$u(x,t)=\sqrt{\frac{3}{2A^2-4\mu B-3\mu^2}}\left(\mu+\frac{2A^2-4\mu B}{(\mu/2)\xi^2+A\xi+B}\right), \tag{31}$$

where

$$\xi=x-\left(\frac{2(A^2-2\mu B)}{2A^2-4\mu B-3\mu^2}\right)t.$$

Result 2. We have

$$a_0=0,\ \ a_1=\sqrt{\frac{3}{2}},\ b_1=\sqrt{\frac{3(A^2-2\mu B)}{2}}\ ,V=1. \tag{32}$$

Now, in the traveling wave solution of (11) becomes:

$$u(x,t)=\sqrt{\frac{3}{2}}\left(\frac{\mu\xi+A+\sqrt{A^2-2\mu B}}{(\mu/2)\xi^2+A\xi+B}\right), \tag{33}$$

where

$$\xi=x-t.$$

3.2 THE (2+1)-DIMENSIONAL CALOGERO-BOGOYAVLENSKII-SCHIFF (CBS) EQUATION

Now consider the (2+1)-dimensional (CBS) equation in the form

$$u_{xt}+\ u_{xxxy}+4u_xu_{xy}+2u_{xx}u_y=0, \tag{34}$$

The CBS equation was first investigated by Bogoyavlenskii and Schiff in different ways. Bogoyavlenskii used the modified Lax formalism, whereas Schiff derived the same equation by reducing the self-dual Yang-Mills equation [39-43]. Toda and Yu [44] constructed some new (2 +1)-dimensional integrable models using the Calogero method. They also derived the (2+1)-dimensional CBS equation from the Korteweg–de Vries equation. This equation is used to describe the interaction of a Riemann wave propagating along the *y*-axis with a long wave along the *x*-axis. The sine-cosine method was used by Najafi [45] to find the traveling wave solutions for CBS equation.

Consider the transformation (2) that converts (34) into ordinary differential equation

$$-V\,u''+u^{(4)}+6u'u''=0. \tag{35}$$

Integrating (35) with respect to ξ and setting the constant of integration to zero yields:

$$-V\,u'+u'''+3(u')^2=0, \tag{36}$$

where prime denotes the derivative with respect to ξ. Applying the homogeneous balancing principle between u''' and $(u')^2$ we have $M=1$. Therefore the trail solution is

$$u=\alpha_0+\alpha_1\varphi+\beta_1\psi, \tag{37}$$

where α_0,α_1 and β_1 are constants to be determined later. Now, we discuss the following cases:

Case 1: If $\lambda<0$, substituting (37) into (36) and by means of equations (7) and (8), the left-hand side of (36) becomes a polynomial in φ and ψ . Setting the coefficients of this polynomial to zero yields a system of algebraic equations (which are omitted here for simplicity) in $\alpha_0,\alpha_1,\beta_1,\ \mu,\lambda,V$ and σ.

Solving the system of algebraic equations, we have

$$\alpha_0=\alpha_0,\alpha_1=1,\ \beta_1=\sqrt{-\frac{\lambda^2\sigma+\mu^2}{\lambda}},\ \ V=-\lambda,\ \ \sigma=A^2-B^2. \tag{38}$$

Now, the traveling wave solution of Eq. (34) becomes:

$$\begin{aligned}u(x,z,t)=&-\frac{(-\lambda)^{3/2}\left(A\cosh\left(\sqrt{-\lambda}\xi\right)+B\sinh\left(\sqrt{-\lambda}\xi\right)\right)}{A\sinh\left(\sqrt{-\lambda}\xi\right)\lambda+B\cosh\left(\sqrt{-\lambda}\xi\right)\lambda+\mu}+\sqrt{-\frac{\lambda^2\sigma+\mu^2}{\lambda}}\\&\times\left(\frac{\lambda}{A\sinh\left(\sqrt{-\lambda}\xi\right)\lambda+B\cosh\left(\sqrt{-\lambda}\xi\right)\lambda+\mu}\right)+\alpha_0,\end{aligned} \tag{39}$$

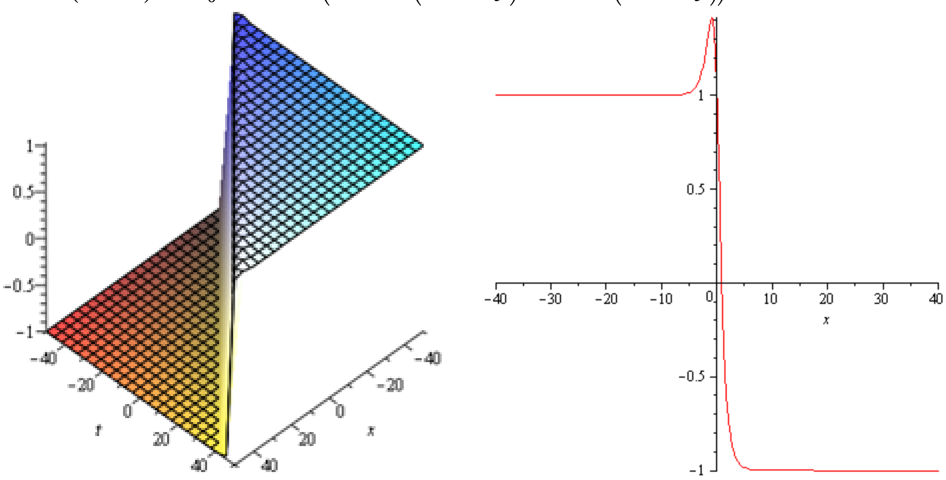
where

$$\xi=x+y+\lambda t.$$

In particular, if we take $A=0,B\neq 0$ and $\mu=0$ or $A\neq 0,B=0$ and $\mu=0$ in (39), we have the following solution

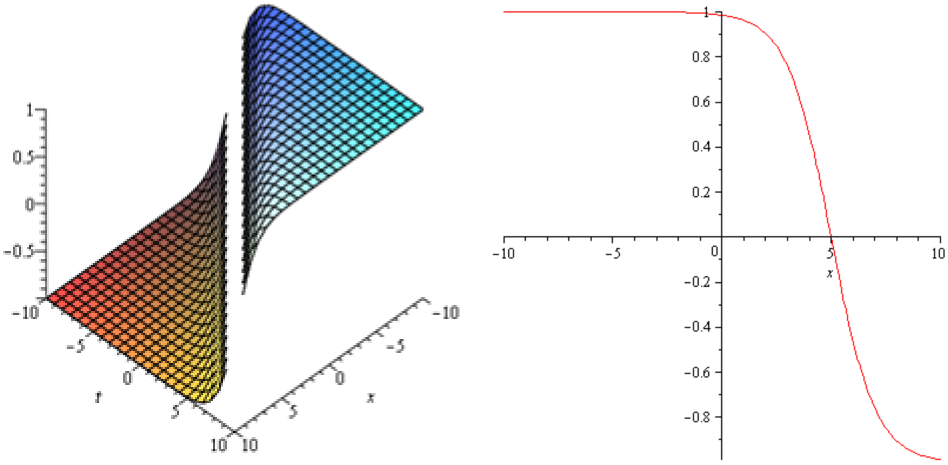
$$u(x,z,t)=\alpha_0+k\sqrt{\lambda}\left(-i\tanh\left(\sqrt{-\lambda}\xi\right)+\operatorname{sec}h\left(\sqrt{-\lambda}\xi\right)\right), \tag{40}$$

$$u(x,z,t)=\alpha_0+\sqrt{-\lambda}\left(-\coth\left(\sqrt{-\lambda}\xi\right)+\operatorname{csc}h\left(\sqrt{-\lambda}\xi\right)\right). \tag{41}$$



(a) **(b)**

Figure 5. (a) 3D and (b) 2D symbolize the kink solutions of Eq. (40) for different values of parameters.



(a) **(b)**

Figure 6. (a) 3D and (b) 2D represent the singular kink solutions of Eq. (41) for different values of parameters.

Case 2: If $\lambda > 0$, substituting (37) into (36) and by means of equations (7) and (9), the left-hand side of (36) becomes a polynomial in φ and ψ. Similar to case 1, after solving the system of algebraic equations, we obtain

$$\alpha_0 = \alpha_0, \alpha_1 = 1,\ \beta_1 = \sqrt{\frac{\lambda^2\varepsilon - \mu^2}{\lambda}},\ V = -\lambda,\ \ \varepsilon = A^2 + B^2. \tag{42}$$

Now, we get the traveling wave solution of Eq. (10) as follows:

$$u(x,y,t) = \frac{(\lambda)^{3/2}\left(A\cos\left(\sqrt{\lambda}\xi\right) - B\sin\left(\sqrt{\lambda}\xi\right)\right)}{A\sin\left(\sqrt{\lambda}\xi\right)\lambda + B\cos\left(\sqrt{\lambda}\xi\right)\lambda + \mu} + \sqrt{\frac{\lambda^2\varepsilon - \mu^2}{\lambda}} \times\left(\frac{\lambda}{A\sin\left(\sqrt{\lambda}\xi\right)\lambda + B\cos\left(\sqrt{\lambda}\xi\right)\lambda + \mu}\right) + \alpha_0, \tag{43}$$

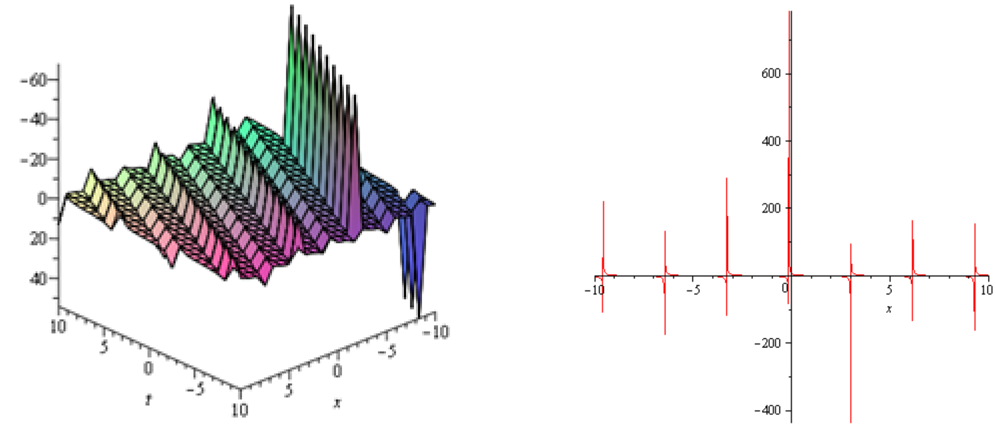
where

$$\xi = x + y + \lambda t.$$

In meticulous, if we set $A = 0, B \neq 0$ and $\mu = 0$ or $A \neq 0, B = 0$ and $\mu = 0$ in (43), we have the following solitary solutions

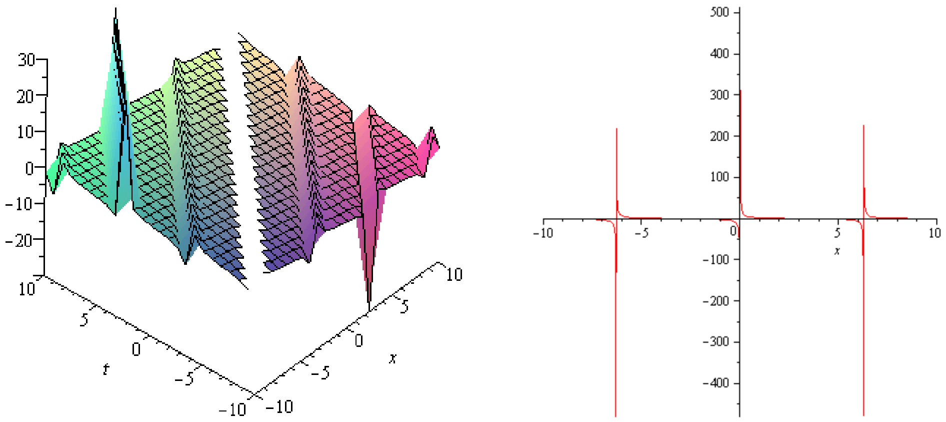
$$u(x,y,t) = \alpha_0 + \sqrt{\lambda}\left(-\tan\left(\sqrt{\lambda}\xi\right) + \sec\left(\sqrt{\lambda}\xi\right)\right), \tag{44}$$

$$u(x,z,t) = \alpha_0 + \sqrt{\lambda}\left(\cot\left(\sqrt{\lambda}\xi\right) + \csc\left(\sqrt{\lambda}\xi\right)\right). \tag{45}$$



(a) (b)

Figure 7. (a) 3D and (b) 2D characterize the periodic traveling wave solutions of Eq. (44) for different values of parameters.



(a) (b)

Figure 8. (a) 3D and (b) 2D show the singular periodic solutions of Eq. (45) for different values of parameters.

Remark: If we take $\lambda = \frac{c-1}{c}$ in equations (16), (24) and (25), our results are identical to the results (5.19), (5.20) and (5.21) obtained by [38].

5. CONCLUSIONS

In this article, we have applied the $(G'/G, 1/G)$- expansion method to obtain solitary solutions of modified Benjamin-Bona-Mahony (mBBM) equation and the (2+1) - dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. For different values of parameters A, B and λ, we obtained the solitary wave solutions. The method used in this article is more effective and general than the basic (G'/G)- expansion method. The main advantage of this method over other methods is that, it possesses all types of the solutions; i.e. hyperbolic function solution, trigonometric function solution and rational solution. It is decisive to mention out that three of our obtained solutions are identical with the existing solutions. From this analysis we can conclude that the proposed method is quite resourceful and practically well suited to be used in finding exact solutions of NLEEs. Numerical results and graphical presentation reveal the complete reliability of the method.

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