ORIGINAL PAPER

VARIATIONAL ITERATION METHOD: A COMPUTATIONAL TOOL FOR SOLVING COUPLED SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *In this research work, a computational method which is based on the Variational Iteration Method (VIM) is used for the numerical solution of the coupled system of nonlinear partial differential equations. Numerical examples are presented to show the convergence and accuracy of the method.*

Keywords: *variational iteration method, system of nonlinear partial differential equations, Lagrange multiplier.*

1. INTRODUCTION

The wide applicability of system of partial differential equations has made its study attracted much attention in a variety of applied sciences. Researchers [1-10] have used different numerical methods to solve forms of partial differential equations. These systems were formally modeled to investigate wave propagation and to control the shallow water waves [6-10].

Inokuti et al [3] proposed a Lagrange multiplier method to solve nonlinear problems in quantum mechanics in 1978. He [4] later modified it into a variational iteration method to solve a large class of linear and nonlinear differential equations.

In this work, the variational iteration method introduced by He [4] will be used to solve coupled nonlinear partial differential equations numerically.

2. VARIATIONAL ITERATION METHOD

According to the variational iteration method [4], we consider the differential equation:

$$L(u) + N(u) = g(t) \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is an inhomogeneous term. A correction functional can be constructed as follows:

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$$u_{u+1}(t) = u_n(t) + \int_u^t \lambda \{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\} d\xi \tag{2}$$

where λ is a general Lagrangian multiplier which can be identified optimally via variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

3. APPLICATIONS

In this section, the VIM will be used to solve systems of nonlinear partial differential equations.

Example 3.1. Consider the coupled system of nonlinear homogeneous partial differential equations [1] of the form:

$$\begin{aligned} V_t - V_{xx} - 2VV_x + (VW)_x &= 0 \\ W_t - W_{xx} - 2WW_x + (VW)_x &= 0 \end{aligned} \tag{3}$$

Therefore, according VIM, the correction functional can be derived as follows:

$$V_{n+1}(x,t) = V_n(x,t) + \int_0^t \lambda_1(x,\xi) \left[\begin{array}{l} \frac{\partial \tilde{V}_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{V}_n(x,\xi)}{\partial x^2} - 2V_n(x,\xi)\frac{\partial}{\partial x}\tilde{V}_n(x,\xi) \\ + \frac{\partial}{\partial x}\left(\tilde{V}_n(x,\xi)\tilde{W}_n(x,\xi)\right) \end{array} \right] d\xi$$

$$W_{n+1}(x,t) = W_n(x,t) + \int_0^t \lambda_2(x,\xi) \left[\begin{array}{l} \frac{\partial \tilde{W}_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{W}_n(x,\xi)}{\partial x^2} - 2W_n(x,\xi)\frac{\partial}{\partial x}\tilde{W}_n(x,\xi) \\ + \frac{\partial}{\partial x}\left(\tilde{V}_n(x,\xi)\tilde{W}_n(x,\xi)\right) \end{array} \right] d\xi \tag{4}$$

Where $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ are general Lagrange multipliers, $\tilde{V}_n(x,\xi)$ and $\tilde{W}_n(x,\xi)$ denote restricted variations i.e. $\delta\tilde{V}_n(x,\xi) = \delta\tilde{W}_n(x,\xi) = 0$

Making the above correction functional stationary, then

$$\partial V_{n+1}(x,t) = V_n(x,t) + \partial\int_0^t \lambda_1(x,\xi) \left[\begin{array}{l} \frac{\partial \tilde{V}_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{V}_n(x,\xi)}{\partial x^2} - 2V_n(x,\xi)\frac{\partial}{\partial x}\tilde{V}_n(x,\xi) \\ + \frac{\partial}{\partial x}\left(\tilde{V}_n(x,\xi)\tilde{W}_n(x,\xi)\right) \end{array} \right] d\xi \tag{5}$$

and

$$\partial W_{n+1}(x,t)=W_n(x,t)+\partial\int_0^t \lambda_2(x,\xi)\left[\begin{array}{l}\dfrac{\partial \tilde{W}_n(x,\xi)}{\partial \xi}-\dfrac{\partial^2 \tilde{W}_n(x,\xi)}{\partial x^2}-2W_n(x,\xi)\dfrac{\partial}{\partial x}\tilde{W}_n(x,\xi)\\ +\dfrac{\partial}{\partial x}\left(\tilde{V}_n(x,\xi)\tilde{W}_n(x,\xi)\right)\end{array}\right]d\xi \qquad (6)$$

therefore

$$\begin{aligned}&\lambda_1^{'}(x,\xi)=0\\ &1+\lambda_1(x,\xi)\big|_{\xi=t}=0\\ &\lambda_2^{'}(x,\xi)=0\\ &1+\lambda_2(x,\xi)\big|_{\xi=t}=0\end{aligned} \qquad (7)$$

The Lagrange multipliers can be identified as:

$$\lambda_1(x,\xi)=\lambda_1(x,\xi)=-1 \qquad (8)$$

Substituting equation (8) into equation (4) results in the following iteration formula.

$$\begin{aligned}V_{n+1}(x,t)&=V_n(x,t)-\int_0^t\left[\begin{array}{l}\dfrac{\partial V_n(x,\xi)}{\partial \xi}-\dfrac{\partial^2 V_n(x,\xi)}{\partial x^2}-2V_n(x,\xi)\dfrac{\partial}{\partial x}V_n(x,\xi)\\ +\dfrac{\partial}{\partial x}(V_n(x,\xi)W_n(x,\xi))\end{array}\right]d\xi\\ W_{n+1}(x,t)&=W_n(x,t)-\int_0^t\left[\begin{array}{l}\dfrac{\partial W_n(x,\xi)}{\partial \xi}-\dfrac{\partial^2 W_n(x,\xi)}{\partial x^2}-2W_n(x,\xi)\dfrac{\partial}{\partial x}W_n(x,\xi)\\ +\dfrac{\partial}{\partial x}(V_n(x,\xi)W_n(x,\xi))\end{array}\right]d\xi\end{aligned} \qquad (9)$$

with the initial conditions $V_0(x,0)=W_0(x,0)=\sin x$. The following terms can be obtained

$$\begin{aligned}V_1(x,t)&=\cos x-t\cos x\\ W_1(x,t)&=\cos x-t\cos x\end{aligned} \qquad (10)$$

$$\begin{aligned}V_2(x,t)&=\cos x-t\cos x+\frac{t^2}{2}\cos x=\cos x(1-t+)\frac{t^2}{2}\\ W_2(x,t)&=\cos x-t\cos x+\frac{t^2}{2}\cos x=\cos x(1-t+)\frac{t^2}{2}.\end{aligned} \qquad (11)$$

$$V_3(x,t) = \cos x\left(1 - t + \right)\frac{t^2}{2!} - \frac{t^3}{3!} + ...)$$
$$W_3(x,t) = \cos x\left(1 - t + \right)\frac{t^2}{2!} - \frac{t^3}{3!} + ...) \quad (12)$$

Hence, in a closed form, the exact solutions of the coupled systems are given by

$$V(x,t) = e^{-t}\sin x$$
$$W(x,t) = e^{-t}\sin x \quad (13)$$

Example 3.2. Consider the system of homogenous and inhomogeneous nonlinear partial differential equations of the form:

$$U_t + U_x V_x = 2$$
$$V_t + U_x V_x = 0 \quad (14)$$

with initial conditions

$$U(x,0) = x, V(x,0) = x \quad (15)$$

According to VIM, the iteration formula for equations (14) is

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda_1(x,\xi)\left[\frac{\partial U_n(x,\xi)}{\partial \xi} + \frac{\partial U_n(x,\xi)}{\partial x}\frac{\partial V_n(x,\xi)}{\partial x} - 2\right]d\xi$$

$$V_{n+1}(x,t) = V_n(x,t) + \int_0^t \lambda_2(x,\xi)\left[\frac{\partial V_n(x,\xi)}{\partial \xi} + \frac{\partial U_n(x,\xi)}{\partial x}\frac{\partial V_n(x,\xi)}{\partial x}\right]d\xi \quad (16)$$

where

$$\lambda_1(x,\xi) = \lambda_2(x\xi) = -1 \quad (17)$$

The following terms can be obtained from equations (16)

$$U_1(x,t) = x + t$$
$$V_1(x,t) = x - t \quad (18)$$

$$U_2(x,t) = x + t$$
$$V_2(x,t) = x - t \quad (19)$$

Therefore, the exact solutions are :

$$U(x,t) = x + t$$
$$V(x,t) = x - t \quad (20)$$

Example 3.3. Consider the inhomogeneous system of partial differential equations.

$$\begin{aligned} &U_t + 2VU_x - U = 2 \\ &V_t - 3UV_x + V = 3 \end{aligned} \tag{21}$$

with initial conditions $U(x,0) = e^x,\ V(x,0) = e^{-x}$

The correction functional of equation (21) is:

$$\begin{aligned} U_{n+1}(x,t) &= U_n(x,t) + \int_0^t \lambda_1(x,\xi)\left[\frac{\partial \tilde{U}(x,\xi)}{\partial \xi} + 2V_n(x,\xi)\frac{\partial \tilde{U}_n(x,\xi)}{\partial x} - U_n(x,\xi) - 2\right] d\xi \\ V_{n+1}(x,t) &= V_n(x,t) + \int_0^t \lambda_2(x,\xi)\left[\frac{\partial \tilde{V}(x,\xi)}{\partial \xi} - 3U_n(x,\xi)\frac{\partial \tilde{V}_n(x,\xi)}{\partial x} + V_n(x,\xi) - 3\right] d\xi \end{aligned} \tag{22}$$

Making the above correction functional stationary, then, the stationary conditions become:

$$\begin{aligned} &\lambda'_1(x,\xi) = 0 \\ &1 + \lambda_1(x,\xi)\big|_{\xi=t} = 0 \\ &\lambda'_2(x,\xi) = 0 \\ &1 + \lambda_2(x,\xi)\big|_{\xi=t} = 0 \end{aligned} \tag{23}$$

which gives:

$$\lambda_1(x,\xi) = \lambda_1(x,\xi) = -1 \tag{24}$$

This gives the iteration formula as follows:

$$\begin{aligned} U_{n+1}(x,t) &= U_n(x,t) - \int_0^t \left[\frac{\partial U_n(x,\xi)}{\partial \xi} + 2V_n(x,\xi)\frac{\partial U_n(x,\xi)}{\partial x} - \tilde{U}_n(x,\xi) - 2\right] d\xi \\ V_{n+1}(x,t) &= V_n(x,t) - \int_0^t \left[\frac{\partial V_n(x,\xi)}{\partial \xi} - 3U_n(x,\xi)\frac{\partial V_n(x,\xi)}{\partial x} + \tilde{V}_n(x,\xi) - 3\right] d\xi \end{aligned} \tag{25}$$

Therefore, the following successive terms were obtained.

$$\begin{aligned} U(x,0) &= e^x \\ V(x,0) &= e^{-x} \end{aligned} \tag{26}$$

$$U_1(x,t) = e^x + e^x t = e^x(1+t)$$
$$V_1(x,t) = e^{-x} - e^{-x}t = e^{-x}(1-t) \quad (27)$$

$$U_2(x,t) = e^x + e^x t + \frac{e^x t^2}{2!} + noise \quad terms$$
$$V_2(x,t) = e^{-x} - e^{-x}t + \frac{e^{-x} t^2}{2!} + noise \quad terms \quad (28)$$

By cancelling the noise terms between $U_2, U_3, \dots$ and between $V_2, V_3, \dots,$ the exact solution

$$U(x,t) = e^{x+t}$$
$$V(x,t) = e^{-(x+t)} \quad (29)$$

can be obtained.

CONCLUSION

In this paper, the variational iteration method has been used to solve coupled system of nonlinear partial differential equations, homogeneous, inhomogeneous and both directly without any restrictive assumptions or linearization.

This method is very effective and accelerate the convergent of solution to exact for the examples presented. The method also can be applied to system that comprises of more than two linear and nonlinear partial differential equations of higher order.

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