ORIGINAL PAPER

ON AN EFFICIENT TECHNIQUE FOR SOLVING (1+1)-DIMENSIONAL BENJAMIN-BONA MAHONY EQUATION

QAZI MAHMOOD UL-HASSAN[1], MUHAMMAD ASHRAF[1], MADIHA AFZAL[2], KAMRAN AYUB[3], MUHAMMAD YAQUB KHAN[3]

Manuscript received: 07.02.2017; Accepted paper: 12.06.2017;
Published online: 30.09.2017.

Abstract. *In this work, improved modified Exp-function method is used to obtain the exact solution for (1+1)-dimensional Benjamin-Bona Mahony equation of fractional order. This differential equation arises in modeling of long surface gravity waves of small amplitude. It is appropriate to use a generalized fractional complex transformation to convert this equation to ordinary differential equation which further resulted into number of exact solutions.*

Keywords: *(1+1)-dimensional Benjamin-Bona Mahony equation, improved modified Exp-function method, fractional calculus, Maple 18.*

Mathematics Subject Classification: *35Q35; 35C08; 35B35.*

1. INTRODUCTION

The one of the most convenient classes of fractional differential equation is the class of fractional calculus which viewed as generalized differential equations [1]. In that sense, we can extend much of the theory and, hence, applications of differential equation smoothly to fractional differential equations without changing flavor and spirit of the realm of differential equation. The works on fractional calculus (that is, the theory of integrals and derivatives of any arbitrary real or complex order) were started over 300 years ago. Since then, many researchers have been done great job to this field. Recently, it has turned out those differential equations involving derivatives of non-integer [2]. For example, with the help of fractional derivatives [3], the nonlinear oscillation of earthquakes can be modeled. There has been some attempt to solve linear problems with multiple fractional derivatives (the so -called multi-term equations) [3-5]. There is not too much work on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivatives. We use a generalized fractional complex transform [6-10] for converting fractional order differential equation to ordinary differential equation. Finally, we apply a novel technique [11-12] called improved modified Exp-function method to obtain its exact solutions, and to obtain generalized solitary solutions and periodic solutions. Mohyud-Din [13-16] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [17] tried this novel approach for Fisher's equation; Wu et. al. [18-19] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [20] for MBBN equations, Zhang [21] for high-dimensional nonlinear evolution equations; Zhu [22-23] for the Hybrid-Lattice system and discrete mKdV lattice; Kudryashov [24] for exact soliton solutions of the

[1] University of Wah, Department of Mathematics, Wah, Pakistan. E-mail: qazimahmood@uow.edu.pk.
[2] Allama Iqbal Open University, Department of Mathematics, Islamabad, Pakistan.
[3] Riphah International University, Department of Mathematics, Islamabad, Pakistan.

generalized evolution equation of wave dynamics; Momani [25] for an explicit and numerical solutions of the fractional KdV equation. Most scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly [29-34]. This method has been effectively and accurately shown to solve a large class of nonlinear problems. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can be easily extended to other kinds of nonlinear evolution equations. In engineering and science, scientific phenomena give a variety of solutions that are characterized by distinct features. Traveling waves appear in many distinct physical structures in solitary wave theory [26-27] such as solitons, kinks, peakons, cuspons, and compactons and many others. Solitons are localized traveling waves which are asymptotically zero at large distances. In other words, solitons are localized wave packets with exponential wings or tails. Solitons are generated from arobust balance between nonlinearity and dispersion. Solitons exhibit properties typically associated with particles. Kink waves [27-28] are solitons that rise or descend from one asymptotic state to another, and hence another type of traveling waves as in the case of the Burgers hierarchy. Peakons, that are peaked solitary wave solutions, are another type of travelling waves as in the case of Camassa-Holm equation. For peakons, the traveling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in 1st derivative of the solution. Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. The compactons, which are solitons with compact spatial support such that each compacton is a soliton confined to a finite core or a soliton without exponential tails or wings. Other types of travelling waves arise in science such as negatons, positons and complexitons. In this research, we use the improved modified Exp-funion method along with generalized fractional complex transform to obtain new solitary waves solutions for the under study differential equation.

2. PRELIMINARIES AND NOTATION

Some important results of fractional calculus are discussed in under study section. The fractional integral and derivatives defined on $[a,b]$ are given below:

Definition 1. A real valued function $h(x), x > 0$ in the space $E_v, \nu \in R$ is said to be in the space $E_v^n \nu$, if $h^n \in E_v, n \in N$. There exists a real number ($s > \nu$) such that $h(x) = x^s h_1(x)$, where $h_1(x) = E(0,\infty)$.

Definition 2. Riemann-Liouville integral operator of the order $\beta \geq 0$ can be defined as

$$J^{\beta}(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1} h(t)dt, \beta > 0, x > 0, J^0(x) = h(x).$$

where $h \in E_v, \nu \geq -1$.

Some essential properties of the operator J^{β} are as

$$J^{\beta} J^{\gamma} h(x) = J^{\beta+\gamma} h(x) \quad (1)$$

$$J^{\beta} J^{\gamma} h(x) = J^{\gamma} J^{\beta} h(x) \quad (2)$$

$$J^{\beta} x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha} \quad (3)$$

For $h \in E_v, v \geq -1, \beta, \gamma \geq 0$ and $\alpha \geq -1$.

There arise some demerits of Riemann-Liouville derivative when we apply it to model real world problems in the form of fractional differential equations. There is a need to overcome this deficiency. For this M. Caputo introduce modified version of fractional differential operator which we used in our article.

Definition 3. Caputo time fractional derivative operator is defined below, for the smallest integer n that exceeds.

$$D_t^{\beta} h(x) = \begin{cases} \dfrac{\partial^{\beta} w(x,t)}{\partial t^{\beta}} = \dfrac{1}{\Gamma(n-\beta)} \displaystyle\int_0^x (x-t)^{n-\beta-1} h(t)dt, -1 << n, n \in N, \\ \dfrac{\partial^{\beta} w(x,t)}{\partial t^{\beta}}, \beta = n. \end{cases}$$

Chain rule

In this segment we used a complex fractional transformation to convert differential equation of fractional order into classical differential equation. We apply the following chain rule

$$\frac{\partial^{\beta} w}{\partial t^{\beta}} = \frac{\partial w}{\partial q} \cdot \frac{\partial^{\beta} q}{\partial t^{\beta}} \tag{4}$$

It is a Jumarie's modification of Riemann-Liouville derivative. There are few results, which are very important and useful

$$D_t^{\beta} w = \sigma_t' \frac{dw}{d\xi} D_t^{\beta} \xi$$

$$D_x^{\beta} w = \sigma_x' \frac{dw}{d\xi} D_t^{\beta} \xi$$

The value of σ_s is determined by assuming a special case given below

$$q = t^{\beta} \text{ and } w = q^n$$

We have

$$\frac{\partial^{\beta} w}{\partial t^{\beta}} = \frac{\Gamma(1+n\beta) t^{n\beta-\beta}}{\Gamma(1+n\beta-\beta)} = \sigma . \frac{\partial w}{\partial q} = \sigma n t^{n\beta-\beta}$$

Thus we can calculate σ_q as

$$\sigma_q = \frac{\Gamma(1+n\beta)}{\Gamma(1+n\beta-\beta)}$$

Other fractional indexes $\left(\sigma_x', \sigma_y', \sigma_z'\right)$ can determine in similar way.

3. ANALYSIS OF TECHNIQUE

We suppose the nonlinear FPDE of the form

$$Q(s, s_t, s_x, s_{xx}, \ldots, D_t^{\beta} s, D_x^{\beta} s, D_{xx}^{\beta} s, \ldots..) = 0, 0 < \beta \leq 1. \tag{5}$$

where $D_t^{\beta} s, D_x^{\beta} s, D_{xx}^{\beta} s$ are the modified Riemann-Liouville derivative of s w.r.t t, x, xx respectively.

Invoking the transformation

$$s(x,t)=s(\xi),\xi=\frac{lx^{\gamma}}{\Gamma(1+\gamma)}+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}+\frac{Mx^{\alpha}}{\Gamma(1+\alpha)} \tag{6}$$

Here l,ψ and M are all constants.

We can write equation (5) again in the form of following nonlinear ODE

$$P(s,s',s'',s''',...)=0. \tag{7}$$

where prime signify the drivative of s with respect to ξ.

If possible, integrate equation (7) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero.

According to improved modified Exp-function method, the solution will be

$$s(\xi)=\frac{\sum_{n=1}^{2K}a_n\exp[n\xi]}{\sum_{n=1}^{2K}b_n\exp[n\xi]}+\frac{\sum_{n=1}^{2K}a_{-n}\exp[-n\xi]}{\sum_{n=1}^{2K}b_{-n}\exp[-n\xi]} \tag{8}$$

where K is positive integers which are unknown to be determine, a_n and b_n are unknown constants.

To determine the value of K, we balance the linear term of highest order of equation (7) with the highest order nonlinear term.

Substituting equation (8) in to the equation (7), equating the coefficients of each power of $\exp(n\xi)$ to zero gives the system of algebraic equations for a_n and b_n, then solve the system with the help of Maple 18 to determine these constants.

4. SOLUTION PROCEDURE

Consider $(1+1)-$ Dimensional Benjamin-Bona Mahony equation of fractional order,

$$D_t^{\beta}+s_x-\gamma s^2 s_x+s_{xxx}=0 \tag{9}$$

where γ is nonzero positive constant. This equation was first derived to describe an approximation for surface long waves in nonlinear media. It can also be characterize the hydromagnetic waves in cold plasma, a caustic wave in inharmonic crystals and a costicgravity waves in compressible fluids.

Using transformation

$$\xi=px+qy+rz+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}+\xi_0,\ p,q,r,\psi,\xi_0 \text{ are all constants with } p,\psi\neq 0. \tag{10}$$

We can rewrite equation (9) in the following nonlinear ODE

$$\psi s+ps-p\gamma s^2 s'+p^3 s'''=0. \tag{11}$$

Integrate once time, we get

$$(\psi+p)s-\frac{p\gamma}{3}s^3+p^3 s''=0. \tag{12}$$

Balancing the s'' and s^3 by using homogenous principal

$K+2=3K$

$K=1$

then the trail solution is

$$s(\xi)=\frac{c_1e^{\xi}+c_2e^{2\xi}}{d_1e^{\xi}+d_2e^{2\xi}}+\frac{c_{-1}e^{-\xi}+c_{-2}e^{-2\xi}}{d_{-1}e^{-\xi}+d_{-2}e^{-2\xi}} \tag{13}$$

Substituting equation (13) in to equation (12) we have

$$\frac{-1}{3N}\left[l_3\exp(3\xi)+l_2\exp(2\xi)+l_1\exp(\xi)+l_0+l_{-1}\exp(-\xi)+l_{-2}\exp(-2\xi)+l_{-3}\exp(-3\xi)\right]=0$$

$$N=(d_1+d_2\exp(\xi))^3(d_{-1}+d_{-2}\exp(-\xi))^3$$

where $l_i\,(i=-3,-2,..........,2,3)$ are constants obtained by Maple 18.

Equating the coefficients of $\exp(n\xi)$ to be zero, we obtain

$$\left(l_{-3}=0,l_{-2}=0,l_{-1}=0,l_0=0,l_1=0,l_2=0,l_3=0\right) \tag{14}$$

we have following solution sets satisfy the given equation

1st Solution set

$$\left\{p=p,\psi=\psi,c_2=\frac{-c_{-1}d_2}{d_{-1}},c_{-1}=c_{-1},c_1=\frac{-c_{-1}d_1}{d_{-1}},c_{-2}=0,d_{-2}=0,d_{-1}=d_{-1},d_1=d_1,d_2=d_2\right\}$$

We, therefore, obtained the following generalized solitary solution

$$s(x,t)=\frac{\frac{-c_{-1}d_1}{d_{-1}}e^{px+\frac{\psi t^\beta}{\Gamma(1+\beta)}}-\frac{-c_{-1}d_2}{d_{-1}}e^{px+\frac{2\psi t^\beta}{\Gamma(1+\beta)}}+c_{-1}e^{px-\frac{\psi t^\beta}{\Gamma(1+\beta)}}}{d_1e^{px+\frac{\psi t^\beta}{\Gamma(1+\beta)}}+d_2e^{px+\frac{2\psi t^\beta}{\Gamma(1+\beta)}}+d_{-1}e^{px-\frac{\psi t^\beta}{\Gamma(1+\beta)}}}$$

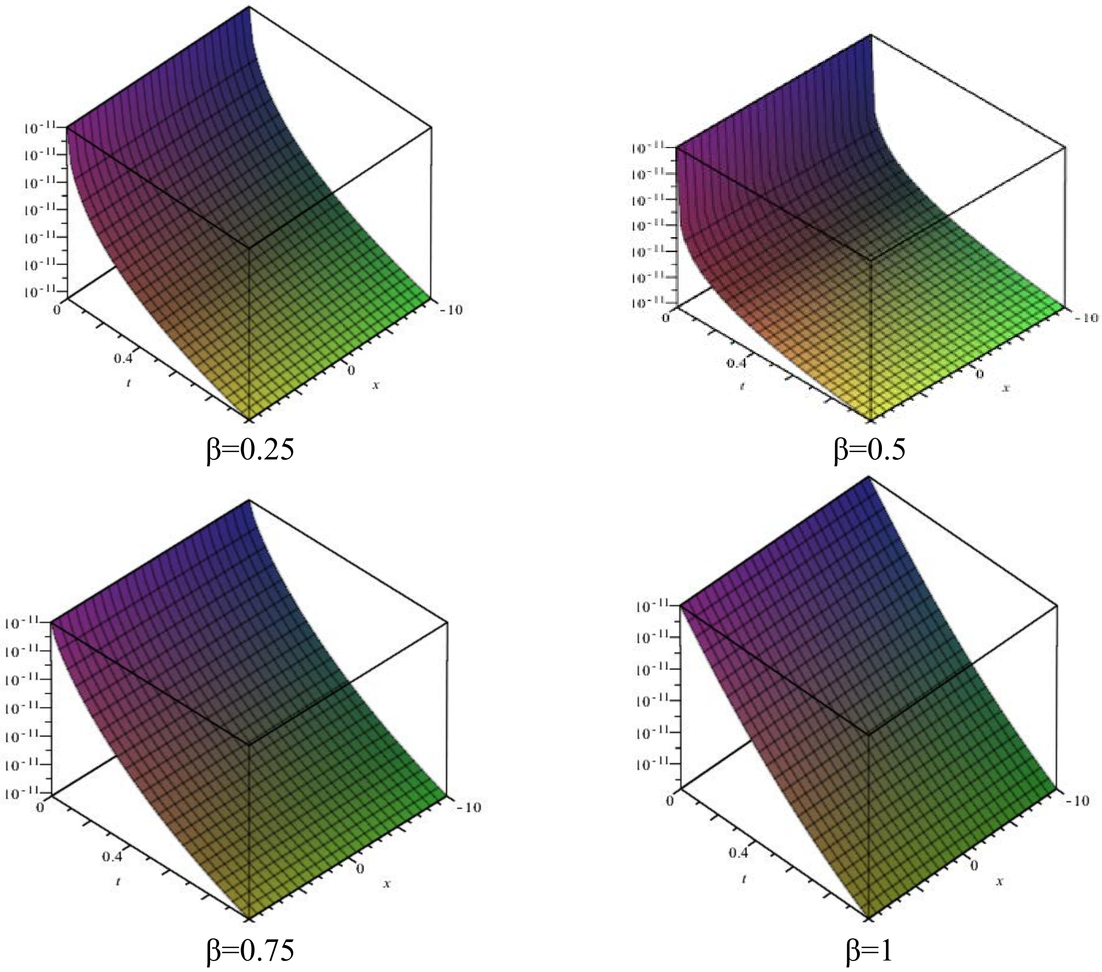


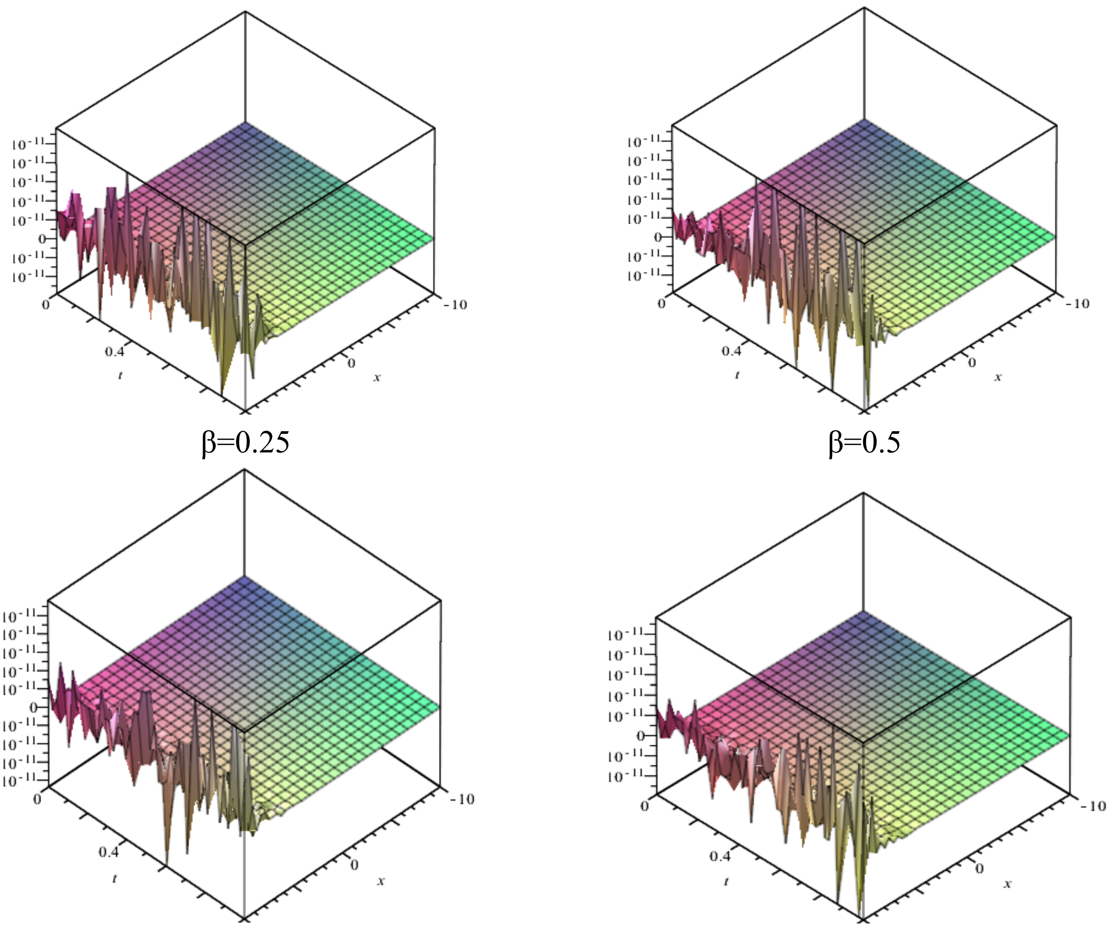
Figure 1

2nd Solution set

$$\left\{p = p, \psi = \psi, c_2 = c_2, c_{-1} = c_{-1}, c_1 = \frac{c_2 c_{-2}}{c_{-1}}, c_{-2} = c_{-2}, d_{-2} = d_{-2}, d_{-1} = 0, d_1 = \frac{-c_2 d_{-2}}{c_{-1}}, d_2 = 0\right\}$$

We, therefore, obtained the following generalized solitary solution

$$s(x,t) = \frac{\dfrac{c_{-2}c_2}{c_{-1}} e^{px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}} + c_2 e^{2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}} + c_{-1} e^{-px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}} + c_{-2} e^{-2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}}{\dfrac{-d_{-2}c_2}{c_{-1}} e^{px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}} + d_{-2} e^{-2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}}$$



β=0.25 β=0.5

β=0.75 β=1

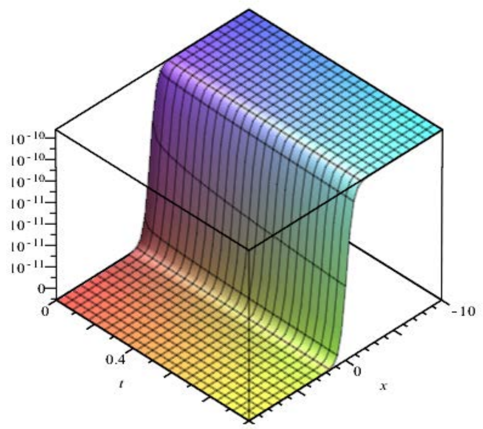
Figure 2

3rd Solution set

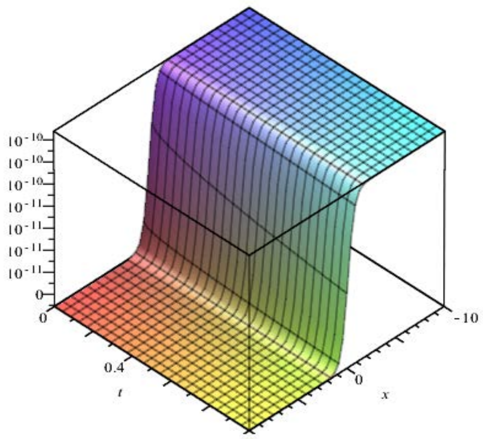
$$\left\{\begin{aligned} & p = p, \psi = \frac{p\left(\gamma c_{-2}^2 d_2^2 + 2\gamma c_{-2} c_2 d_{-2} d_2 + \gamma c_2^2 d_{-2}^2 - 3 d_{-2}^2 d_2^2\right)}{3 d_{-2}^2 d_1^2}, c_2 = c_2, c_{-1} = 0, \\ & c_1 = \frac{c_2 d_1}{d_2}, c_{-2} = c_{-2}, d_{-2} = d_{-2}, d_{-1} = 0, d_1 = d_1, d_2 = d_2 \end{aligned}\right\}$$

We, therefore, obtained the following generalized solitary solution

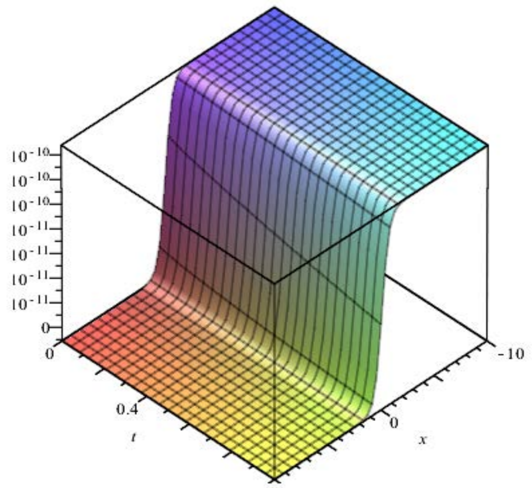
$$s(x,t)=\frac{\dfrac{c_2 d_1}{d_2}e^{px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}+c_2 e^{2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}+c_{-2}e^{-2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}}{d_1 e^{px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}+d_2 e^{2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}+d_{-2}e^{-2px+\frac{\psi t^{\beta}}{\Gamma(1+\beta)}}}$$



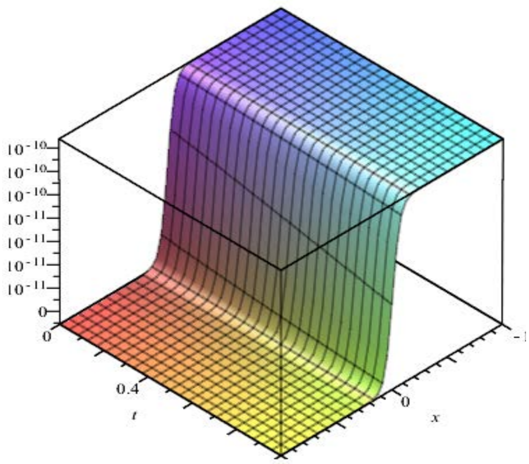
β=0.25



β=0.5



β=0.75



β=1

Figure 3

5. CONCLUSION

This article is devoted to attain, test and analyze the novel soliton wave solutions and physical properties of nonlinear partial differential equation well known as nonlinear (1+1)-dimensional Benjamin-Bona Mahony equation of fractional order. For this, fractional order nonlinear evolution equation is considered as test case and we apply improved modified Exp-function method. We attain desired soliton solutions of various types for different values of parameters. It is guaranteed that the accuracy of the attain results by backward substitution into the original equation with Maple 18. The scheming procedure of this method is simple, straight and productive. We observed that the under study technique is more reliable and have minimum computational task, so widely applicable. In precise we can say that this method is quite competent and much operative for evaluating exact solution of NLEEs. The validity of given algorithm is totally hold up with the help of the computational work, the graphical representations and successive results. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate the solitary solutions.

REFERENCES

[1] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[2] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Vol. 204, Elsevier, Amsterdam, The Netherlands, 2006.
[3] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.
[4] Podlubny, I., *Fractional Differential Equations*, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.
[5] Adomian, G., *J. Math. Anal. Appl.*, **135**(2), 501, 1988.
[6] Shawagfeh N.T., *Appl. Math. Comput.*, **131**(2-3), 517, 2002.
[7] Ray, S.S., Bera, R.K., *Appl. Math. Comput.*, **167**(1), 561, 2005.
[8] He, J.H., *Bull. Sci. Technol.*, **15**(2), 86, 1999.
[9] Yildirim, A., Mohyud-Din, S.T., Sarıaydın, S., *J. of King Saud Uni – Sci*, **23**, 205, 2011.
[10] He, J.H., Wu, X.H., *Chaos Solitons & Fractals*, **30**(3), 700, 2006.
[11] He, J.H., *Int. J. Mod. Phys. B*, **22**(21), 3487, 2008.
[12] He, J.H., Abdou, M.A., *Chaos, Solitons & Fractals*, **34**, 1421, 2007.
[13] Mohyud-Din, S.T., Noor, M.A., Waheed, A., *J. Appl. Math. Computg.*, **30**, 439, 2009.
[14] Mohyud-Din, S.T., Noor, M.A., Noor, K.I., *Mathematical Problems in Engineering*, Article ID 234849, 2009.
[15] Noor, M.A., Mohyud-Din, S.T., Waheed, A., *J. Appl. Math. Computg.*, **29**, 1, 2008.
[16] Noor, M.A., Mohyud-Din, S.T., Waheed, A., *Acta Applicand.Mathem.*, **104**, 131, 2008.
[17] Ozis, T., Koroglu, C., *Phys Lett. A*, **372**, 3836, 2008.
[18] Wu, X.H., He, J.H., *Chaos, Solitons & Fractals*, **30**(3), 700, 2006.
[19] Wu, X.H., He, J.H., *Comput. Math. Appl.*, **54**, 966, 2007.
[20] Yusufoglu, E., *Phys. Lett. A.*, **372**, 442, 2008.
[21] Zhang, S., *Chaos, Solitons & Fractals*, **365**, 448, 2007.
[22] Zhu, S.D., *Inter. J. Nonlin. Sci. Num. Simulation*, **8**, 461, 2007.
[23] Zhu, S.D., *Inter. J. Nonlin. Sci. Num. Simulation*, **8**, 465, 2007.
[24] Kudryashov, N.A., *J. Appl Math and Mech*, 52(3), 361, 1988.
[25] Momani, S., *Math. Comput. Simul.*, **70**(2), 110, 2005.
[26] He, J.H., Li, Z.B., *Math and Comput Appl*, **15**(5), 970, 2010.
[27] Li, Z.B., *J. of Nonlinear Sc and Numl Simul*, **11**, 335, 2010.
[28] He J.H., Li Z.B., *Thermal Sc.*, **16**(2), 331, 2006.
[29] Jumarie, G., *Comput. Math. Appl.*, **51**(9-10), 1367, 2006.
[30] Wazwaz, A.M., *Applied Mathematics and Computation*, **204**, 942, 2008.
[31] Momani, S., *Chaos, Solitons & Fractals*, **28**, 930, 2006.
[32] Ebaid, A., *J. Math. Anal. Appl.*, **392**, 1, 2012.
[33] Abdou, M.A., Soliman, A.A., Basyony, S.T., *Phys. Lett. A*, **369**, 469, 2007.
[34] Bin, Z., *Commun. Theor. Phys.*, **58**, 623, 2012.