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ON FIRST ORDER FUZZY DIFFERENTIAL SUPERORDINATION

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Abstract. *The concept of fuzzy differential superordination was introduced by Waggas Galib Atshan and Khudair O. Hussain in [5] as a dual concept of fuzzy differential subordination [3]. Let Ω be a set in $\mathbb{C}^2$ and $\varphi: \mathbb{C}^2 \times U \to \mathbb{C}$ be an analytic function in Ω and let the function p be an analytic in the unit open disk U such that $\varphi(p(z), zp'(z))$ is univalent in U and satisfies $F_{h(U)}(h(z)) \leq F_{\varphi(\mathbb{C}^2 \times U)}(\varphi(p(z), zp'(z)))$. In this article, we have identified the conditions on functions θ, w, μ and h so that $F_{h(U)}(h(z)) \leq F_{\varphi(\mathbb{C}^2 \times U)}(\varphi(p(z), zp'(z)))$, implies $F_{q(U)}(q(z)) \leq F_{p(U)}(p(z))$.*

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1. INTRODUCTION AND PRELIMINARIES

Let $H(U)$ denote the class of holomorphic functions in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ and let $H[a, n] = \{f \in H(U): f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in U\}$ for $a \in \mathbb{C}, n \in \mathbb{N}^*$.
Let $f \in H(U)$ is called to be convex (univalent) if

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \quad z \in U.$$

A function $f \in H(U)$ is called to be starlike (univalent) if

$$Re\left(\frac{zf'(z)}{f(z)} + 1\right) > 0, \quad z \in U$$

If $f, g \in H(U)$ the function f is said to be fuzzy subordinate to g, written $f \prec_F g$ or $f(z) \prec_F g(z)$ if $f(0) = g(0)$ and $F_{f(U)} f(z) \prec_F F_{g(U)} g(z)$. See [2]

Let $\varphi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be holomorphic function in U and $q \in H[a, n]$. In [5] the authors determined conditions on φ such that

$$F_{h(U)}(h(z)) \leq F_{\varphi(\mathbb{C}^3 \times U)}\left(\varphi(p(z), zp'(z), z^2 p''(z))\right)$$

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implies

$$F_{q(U)}q(z) \leq F_{p(U)}p(z), \qquad z \in U. \qquad (1.1)$$

for all p functions that satisfies (1.1). Furthermore, they determined sufficient conditions as well the function q is the largest function with this property called the fuzzy best subordinat of this fuzzy differential superordination.

The main objective of this paper is the case when

$$\varphi\big(p(z), zp'(z)\big) = \theta\left(p(z) + w\big(p(z)\big)\mu\big(zp'(z)\big)\right).$$

We identified the conditions on functions θ, w, μ and h so that this fuzzy subordination $q(z) \prec_F p(z)$ and we find its fuzzy best subordinat q.

In order to prove our main results we will need to use the next definition.

Definition (1.1): [1] we denote by Q the set of functions q that are analytic and injective on $\overline{U}/E(q)$, where

$$E(q) = \left\{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. The set $E(q)$ is called exemption set.

Definition (1.2): [4] A function $L(z,t), z \in U, t \geq 0,$ is a fuzzy subordination chain if $L(.,t)$ is analytic and univalent in U. For all $t \geq 0$, $L(z,t)$ is continuously differentiable on $[0,\infty)$ for all $z \in U$, and

$$F_{L[U\times[0,\infty)]}\big(L(z,t_1)\big) \leq F_{L[U\times[0,\infty)]}\big(L(z,t_2)\big), t_1 \leq t_2.$$

2. MAIN RESULTS

Theorem (2.1): Let q be a convex univalent function in U, let $\theta, w \in H(E)$ where $q(U) \subset E$ and let $\mu \in H(\mathbb{C})$. Suppose that

$$Re\left(\frac{\theta'\big(q(z)\big) + w'\left(q(z)\mu\big(tzq'(z)\big)\right)}{\theta(q(z))\mu'(tzq'(z))}\right) > 0, \forall z \in U, \text{and } \forall t \geq 0. \qquad (2.1)$$

If $p \in H[q(0),1] \cap Q$, with $p(U) \subset E$ and $\theta(p(z)) + w\big(p(z)\big)\mu\big(zp'(z)\big)$ is univalent in the unit disk U, then

$$F_{h(U)}\left(h(z) = \theta(q(z)) + \mu\big(zq'(z)\big)\right) \leq F_{\varphi(\mathbb{C}^2\times U)}\left(\theta(p(z)) + \mu\big(zp'(z)\big)\right)$$

Implies

$$q(z) \prec_F p(z)\,, \;\; i.e. F_{q(U)}q(z) \leq F_{p(U)}p(z)$$

and q is the fuzzy best subordinate.

Proof: Let $\varphi\big(p(z), zp'(z)\big) = \theta(p(z)) + w\big(p(z)\big)\mu\big(zp'(z)\big)$, using the hypothesis we get

$$F_{h(U)}\big(h(z)\big) \le F_{\varphi(\mathbb{C}^2\times U)}\Big(\varphi\big(p(z), zp'(z)\big)\Big)$$

and $\varphi\big(p(z), zp'(z)\big)$ is univalent in U, if we set

$$L(z,t) = \theta(q(z)) + w(q(z))\mu\big(tzq'(z)\big)$$
$$= a_1(t)z + \cdots,$$

Then

$$\frac{\partial L(0,t)}{\partial z} = w\big(q(0)\big)\mu'(0)q'(0)\left[t + \frac{\theta'\big(q(0)\big) + w'\big(q(0)\big)\mu(0)}{w\big(q(0)\big)\mu'(0)}\right].$$

Since q univalent function we have $q'(0) \neq 0$ and by using (2.1) for $z = 0$. We get that

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} \neq 0, \forall t \ge 0 \text{ and } \lim_{t\to\infty}|a_1(t)| = \infty$$

Using simple calculations, we obtain

$$Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) = Re\left(\frac{\theta'\big(q(z)\big) + w'\big(q(z)\big)\mu\big(tzq'(z)\big)}{w\big(q(z)\big)\mu'\big(tzq'(z)\big)} + t\left(1 + \frac{zq''(z)}{q'(z)}\right)\right)$$

According to (2.1) and Using the properties of the function q where is convex univalent function in U we obtain

$$Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0, z \in U, t \ge 0$$

and by [Lemma (C), 4] we conclude that L is a fuzzy subordination chain. Now, applying [Theorem(2.15), 5] we obtain our result.
Taking $w(\delta) = 1$ in the last theorem we get the next corollary:

Corollary (2.2): Let q be convex univalent function in the open unit disk U and $\theta \in H(E)$ where $q(U) \subset E$, suppose that

$$Re\left(\frac{\theta'\big(q(z)\big)}{\mu'\big(tzq'(z)\big)}\right) > 0, \forall z \in U, \forall t \ge 0,$$

if $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq E$ and $\theta(p(z)) + \mu\big(zp'(z)\big)$ is univalent in the unit disk U, then

$$F_{h(U)}\Big(h(z) = \theta(q(z)) + \mu\big(zq'(z)\big)\Big) \le F_{\varphi(\mathbb{C}^2\times U)}\Big(\theta(p(z)) + \mu\big(zp'(z)\big)\Big)$$

implies

$$F_{q(U)}q(z) \leq F_{p(U)}p(z),$$

and q is the fuzzy best subordinate.

For the particular case when $\mu(\delta) = \delta$ using a similar proof as in Theorem (2.1) we obatain:

Corollary (2.3): Let q be univalent function in the unit disk U and let $\theta, w \in H(E)$ where $q(U) \subset E$ is a domain. Suppose that

1. $Re\left(\frac{\theta'(q(z))}{w(q(z))}\right) > 0, \forall z \in U$

2. $\vartheta(z) = zq'(z)w\big(q(z)\big)$ is starlike univalent function in U, if $p \in H[q(0), 1] \cap Q$ with $p(U) \subset E$ and $\theta(p(z)) + zp'(z)w\big(p(z)\big)$ is univalent in U. Then

$$F_{h(U)}\big(h(z) = \theta(q(z)) + zq'(z)w\big(q(z)\big)\ \big) \leq F_{\varphi(\mathbb{C}^2\times U)}\big(\theta(p(z)) + zp'(z)w\big(p(z)\big)\ \big)$$

implies

$$q(z) \prec_F p(z)$$

and q is the fuzzy best subordinat.

For case $w(\delta) = 1$ Using the properties of the function it$Q(z)$, where $\vartheta(z) = zq'(z)$, is starlike univalent in U if and only if q is convex univalent in U, Corollary (2.3) becomes:

Corollary (2.4): Let q be convex univalent function in U and let $\theta \in H(E)$, where $q(U) \subset E$. Suppose that

$$Re\left(\theta'\big(q(z)\big)\right) > 0, \forall\ \ z \in U, \tag{2.2}$$

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subset E$ and $\theta\big(p(z)\big) + zp'(z)$ is univalent in U, then

$$F_{h(U)}\big(h(z) = \theta(q(z)) + zq'(z)\big) \leq F_{\varphi(\mathbb{C}^2\times U)}\left(\theta\big(p(z)\big) + zp'(z)\right)$$

Implies

$$F_{q(U)}q(z) \leq F_{p(U)}p(z)$$

and q is the fuzzy best subordinat.

Now we will give some special cases of the results above.

Example (2.5): Let q be a convex function in the open unit disk U with

$$|Im\ q(z)| < \frac{\pi}{2},\ \ z \in U. \tag{2.3}$$

If we let the function $p \in H[q(0), 1] \cap Q$ and $e^{p(z)} + zp'(z)$ is univalent in U, then

$$F_{h(U)}\left(h(z) = e^{q(z)} + zq'(z)\right) \leq F_{\varphi(\mathbb{C}^2\times U)}\left(e^{p(z)} + zp'(z)\right), \quad z \in U.$$

Implies

$$F_{q(U)}\big(q(z)\big) \leq F_{p(U)}\big(p(z)\big) \,,$$

and q is the fuzzy best subordinant.

Proof: By using Corollary (2.4) let $\theta(\delta) = e^{\delta}$, then condition (2.2) becomes

$$Re\left(\theta'\big(q(z)\big)\right) = e^{Re(q(z))}\cos\left(Im\,\big(q(z)\big)\right) > 0, z \in U,$$

that is equivalent with (2.3).

Remark (2.1): If we put $q(z) = \gamma z,\ \ |\gamma| \leq \frac{\pi}{2}$, in example (2.5) we obtain the next result.

If $p \in H[0,1] \cap Q$ such that $e^{p(z)} + zp'(z)$ is univalent in the open unit disk U, and $|\gamma| \leq \frac{\pi}{2}$, then

$$F_{h(U)}(h(z) = e^{\gamma z} + \gamma z\,) \leq F_{\varphi(\mathbb{C}^2\times U)}\left(e^{p(z)} + zp'(z)\right)$$
$$F_{q(U)}(q(z) = \gamma z) \leq F_{p(U)}\big(p(z)\big)$$

and γz is the fuzzy best subordinant.

Example (2.6): Let q be a convex function in U and let

$$Re\big(q(z)\big) > \alpha,\ \ z \in U \qquad (2.4)$$

If $p \in H[q(0), 1] \cap Q$ and $\frac{p^2(z)}{2} - \alpha p(z) + zp'(z)$ is univalent in the unit disk U, then

$$F_{h(U)}\left(h(z) = \frac{q^2(z)}{2} - \alpha q(z) + zq'(z)\right) \leq F_{\varphi(\mathbb{C}^2\times U)}\left(\frac{p^2(z)}{2} - \alpha p(z) + zp'(z)\right), z \in U.$$

implies

$$F_{q(U)}\big(q(z)\big) \leq F_{p(U)}\big(p(z)\big),$$

and q is the fuzzy best subordinant.

Proof: By using Corollary (2.4) the case $\theta(\delta) = \frac{\delta^2}{2} - \alpha\delta$, then we can see that (2.2) is equivalent with (2.4).

Remark (2.2): The function $q(z) = \frac{1-z}{1+z}$ is convex in the unit open disk U and

$$Re\big(q(z)\big) > 0\,,\ \ z \in U$$

To prove that if we consider in Example (2.6) $(z) = \frac{1-z}{1+z}$, then we get :

If $p \in H[1,1] \cap Q$ such that $\frac{p^2(z)}{2} + zp'(z)$ is univalent in , then

$$F_{h(U)}\left(h(z) = \frac{(1-z)^2 - 4z}{2(1+z)^2}\right) \leq F_{\varphi(\mathbb{C}^2 \times U)}\left(\frac{p^2(z)}{2} + zp'(z)\right)$$

implies

$$\frac{1-z}{1+z} \prec_F p(z).$$

and $\frac{1-z}{1+z}$ is the fuzzy best subordinant.

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