ORIGINAL PAPER

A NOTE ON THE SINGULARITY OF LCM MATRICES ON GCD - CLOSED SETS WITH 9 ELEMENTS

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Abstract. *In the present paper, we obtain some classes of gcd - closed sets with 9 elements on which LCM matrices are singular by adding an element to a gcd - closed set with 8 elements on which the LCM matrix is also singular. Finally, examining these examples we present two new conjectures.*

Keywords: *LCM matrix, Bourque - Ligh conjecture, gcd - closed set, meet semilattice, Möbius function.*

1. INTRODUCTION

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n distinct positive integers. The set S is said to be *gcd - closed* if $(x_i, x_j) \in S$ for all $1 \leq i, j \leq n$. The matrix having least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j - entry is called the *least common multiple (LCM) matrix*, denoted by $[S]$. In 1992, Beslin and Ligh [2] gave the following formula for calculating the determinant of an LCM matrix defined on a gcd - closed set $S = \{x_1, x_2, \dots, x_n\}$ is equal to

$$\prod_{i=1}^{n} {x_i}^2 \,.\, \alpha_i,$$

where

$$\alpha_i = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ x_t < x_i}} g(d)$$

with the arithmetical function g defined by $g(m) = \frac{1}{m}\sum_{d|m} d\mu(d)$, and μ is the Möbius function. In the same paper, they conjectured that the LCM matrix on any gcd - closed set is invertible. But, finding a counterexample with a gcd - closed set having 9 elements, this was proven to be false by Haukkanen et al. [4]. In 1999, Hong [5] proved that the conjecture holds for any gcd - closed set which has less than 8 elements and gave a counterexample which presents a gcd - closed set with n elements for every $n \geq 8$.

Although the Bourque - Ligh conjecture is not true in general, there are great interests on the invertibility of LCM matrices on gcd - closed sets [1, 3, 9]. Hong raised many open

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problems and conjectures on this subject. In 2004 Hong [6] conjectured that if ε is a nonzero real number and $S = \{x_1, x_2, \dots, x_n\}$ is an odd – gcd - closed set or an odd – lcm - closed set then the matrix $\left([x_i, x_j]^{\varepsilon}\right)$ on S is nonsingular. These conjectures were proven to be false for $\varepsilon = 1$ by Haukkanen et al. [3]. They stated that more counterexamples can be found by using the method presented in the proof of Theorem 5.1 in the same paper.

Let x be a positive integer and let $S = \{x_1, x_2, \dots, x_n\}$. If $\alpha_n(x_1, \dots, x_n) = 0$, where $1 \leq x_1 < \cdots < x_n = x$, then x is said to be a singular number. Moreover, x is said to be a primitive singular number, if x is a singular number and for any $x'|x$, $1 \leq x' \leq x$, x' is a nonsingular number. In 2005, Hong [7] conjectured that there does not exist an odd primitive singular number. Giving a counterexample to this conjecture, Haukkanen et al. [3] showed that there exists such a number. There are two conjectures about the number of primitive singular numbers which are currently proven or disproven. The former is that there are infinitely many primitive singular numbers and the latter is that there are infinitely many even primitive singular numbers. In this frame, Hong raised many other conjectures on the invertibility of the LCM matrix $\left(f[x_i, x_j]\right)$ associated with a multiplicative arithmetic function f on gcd - closed and lcm - closed sets (see [6] and [8]). Currently, almost all of them are open.

Beside their results on the invertibility of LCM matrices and their relatives on gcd - closed sets, and counterexamples to some conjectures above, Haukkanen, Mattila and Mäntysalo [3] mainly showed that if S is a GCD closed set with 8 elements and the LCM matrix $[S]$ is singular, then the semilattice $(S, |)$ is isomorphic to $\mathbf{2^3}$. In this paper, following the paper of Haukkanen et. al. [3], we investigate the structure of gcd - closed sets with 9 elements and the singularity or nonsingularity of lcm matrices. We try to classify gcd - closed sets with 9 elements on which the LCM matrix is singular. Our method is to add an element to a gcd - closed set with 8 elements on which the LCM matrix is singular and to determine that the LCM matrix on this new set is whether singular or not. For these sets, we determine all positions of the element added such that the LCM matrix on the new gcd - closed set is singular.

2. PRELIMINARIES

Let $(P, \preccurlyeq)$ be a meet semilattice, f be a function $P \to \mathbb{C}$ and let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of P with distinct elements arranged so that $x_i \preccurlyeq x_j \Longrightarrow i \leq j$. Then the join matrix of the set S with respect to the function f has $f\left(x_i \vee x_j\right)$ as its i, j - entry. We denote this join matrix by $[S]_f$. The meet matrix $(S)_f$ is defined similarly.

If we take $(P, \preccurlyeq) = (\mathbb{Z}^+, |)$ and $f = N^{\alpha}$, where $N^{\alpha}(m) = m^{\alpha}$ for all $m \in \mathbb{Z}^+$, then the matrix $(S)_f$ and $[S]_f$ become the power – GCD and the power – LCM matrices with $(x_i, x_j)^{\alpha}$ and $[x_i, x_j]^{\alpha}$ as their i, j - entries, respectively. In the case $\alpha = 1$ we obtain the usual GCD matrix and LCM matrix and denote them by (S) and $[S]$, respectively.

In this paper, our main goal is to investigate the singularity and nonsingularity of the LCM matrices defined on particular gcd - closed sets with 9 elements. Indeed, in the light of recent papers, with respect to divisibility, the semilattice structure of the set on which the LCM matrix is defined and hence its Möbius function are very important for the study of singularity and nonsingularity of these matrices. If μ is the Möbius function on P, then we have $\mu(x, x) = 1$ for all $x \in P$ and $\mu(x, y) = -\sum_{x \preccurlyeq z \prec y} \mu(x, z)$ for all $x \prec y$ in P. Now, we follow the specialization on the Möbius function on a subset S of P given in [3]. Throughout this paper, let $S = \{x_1, x_2, \dots, x_n\}$ be a gcd - closed set of distinct positive integers. Denoting

$S_i = \{x_1, x_2, \dots, x_i\}$, we obtain a chain of gcd - closed sets $S_1 \subset S_2 \subset \cdots \subset S_n = S$. It is clear that every set S_i is also lower closed in $(S, |)$. From this observation we can use the Möbius function μ_S of the set S, which can be given recursively as

$$\mu_s(x_i, x_i) = 1, \quad \mu_s(x_i, x_j) = -\sum_{\substack{x_i|x_k|x_j \\ x_k \neq x_i}} \mu_s(x_k, x_j) = -\sum_{\substack{x_i|x_k|x_j \\ x_k \neq x_j}} \mu_s(x_i, x_k).$$

Since $[S]_{N^\alpha} = diag(x_1{}^\alpha, x_2{}^\alpha, \dots, x_n{}^\alpha)(S)_{\frac{1}{N^\alpha}} diag(x_1{}^\alpha, x_2{}^\alpha, \dots, x_n{}^\alpha)$, in particular, we can conclude that the usual LCM matrix $[S]$ is singular if and only if $(S)_{\frac{1}{N}}$ is singular. Moreover, defining the function on the gcd - closed set S as

$$\psi_{S,\frac{1}{N}}(x_i) = \sum_{x_k|x_i} \frac{\mu_S(x_k, x_i)}{x_k}, \tag{1}$$

one can get

$$det(S)_{\frac{1}{N}} = \psi_{S,\frac{1}{N}}(x_1).\psi_{S,\frac{1}{N}}(x_2).\dots.\psi_{S,\frac{1}{N}}(x_n). \tag{2}$$

Finally, we obtain the following result as a particular case of Propositon 2.1 in [3].

Proposition 2.1. The LCM matrix $[S]$ and the GCD matrix $(S)_{\frac{1}{N}}$ are both invertible if and only if $\psi_{S,\frac{1}{N}}(x_i) \neq 0$ for all $i = 1,2, \dots, n$ [3].

The first general result on the Bourque - Ligh conjecture is presented by Hong in 1999.

Theorem 2.2. Let n be a positive integers.
(i) If $n \leq 7$, then the Bourque - Ligh conjecture is true.
(ii) If $n \geq 8$ then there exists a gcd - closed set $S = \{x_1, x_2, \dots, x_n\}$ of n distinct positive integers such that $\alpha_n(x_1, x_2, \dots, x_n) = 0$. Therefore, the Bourque - Ligh conjecture is not true [5].

It has been known that the smallest gcd - closed set S for which the LCM matrix $[S]$ is singular has 8 elements since Hong's proof [5]. However, the uniqueness of the structure of such a set was proven at the first time by Haukkanen et. al. [3]. Indeed, they proved the following result which triggered the main idea of the present paper.

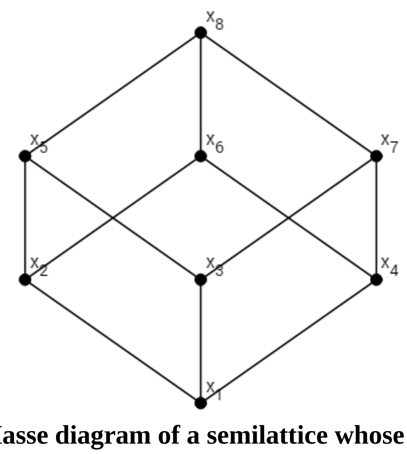


Figure 1. The Hasse diagram of a semilattice whose shape is a cube.

Theorem 2.3. If S is a GCD closed set with 8 elements and the LCM matrix $[S] = \big[[x_i, x_j]\big]$ is singular, then the semilattice $(S, |)$ always belongs the class presented in Fig. 1 [3].

By Hong's theorem [5] and Theorem 2.3, if the LCM matrix on $S_8 = \{x_1, x_2, \dots, x_8\}$ is singular then it is clear that

$$\frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} + \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} - \frac{1}{x_1} = 0 \qquad (3)$$

In the case that the LCM matrix $[S_8]$ is singular, recently Altınışık and Yıldız proved the following theorem for the structure of the set $S_8 = \{x_1, x_2, \dots, x_8\}$.

Theorem 2.4. Let $S_8 = \{x_1, x_2, \dots, x_8\}$ be a gcd - closed set of distinct positive integers such that $[x_2, x_3]\alpha = x_5$, $[x_2, x_4]\beta = x_6$, $[x_3, x_4]\gamma = x_7$, $[x_2, x_3, x_4]A = x_8$ and $[x_5, x_6, x_7]B = x_8$ for positive integers $\alpha, \beta, \gamma, A, B$. If the LCM matrix $[S_8] = \big[[x_i, x_j]\big]$ is singular, then $A = \alpha\beta\gamma$, $B = 1$ and $\alpha, \beta, \gamma > 1$ [1].

3. THE MAIN RESULTS

Throughout this section, let $S_8 = \{x_1, x_2, \dots, x_8\}$ be a gcd - closed set of positive integers with $x_1 \le x_2 \le \cdots \le x_8$ and let the LCM matrix $[S_8]$ be singular. Let a be a positive integer such that $a \notin S_8$ and $S = S_8 \cup \{a\}$ is gcd - closed. In this section, firstly we proved that there cannot exist the following classes of meet semilattices $(S, |)$ on which LCM matrices are nonsingular such that the LCM matrix $[S_8]$ is singular.

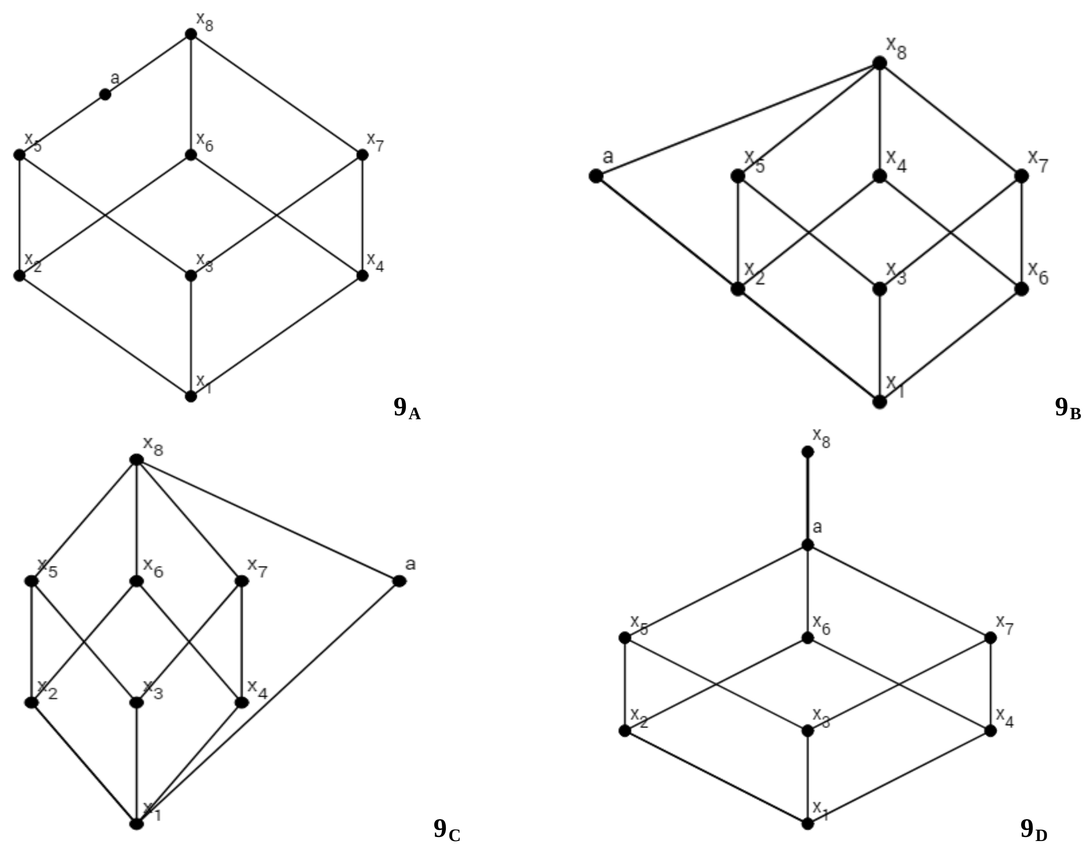
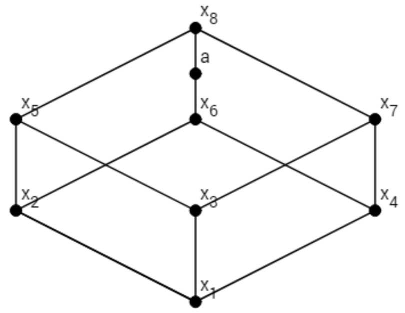


Figure 2. The Hasse diagrams of semilattices $S_8 \cup \{a\}$ in Theorem 3.1.

In Fig. 2, all possible diagrams of nonisomorphic meet semilattices are listed. For example, the following diagram in Fig. 3 is not listed in Fig. 2 since the meet semilattice belongs to $9_A{}'$ which is isomorphic to the meet semilattice belongs to 9_A.



$\mathbf{9_A{}'}$

Figure 3. The Hasse diagram of a semilattice which is isomorphic to the semilattice in 9_A.

Theorem 3.1. Let $S_8 = \{x_1, x_2, \dots, x_8\}$ be a gcd - closed set of positive integers with $x_1 \le x_2 \le \cdots \le x_8$ and let the LCM matrix $[S_8]$ be singular. Also let a be a positive integer such that $a \notin S_8$ and $S = S_8 \cup \{a\}$ is gcd - closed. Then, S cannot belong to one of the classes $9_A, 9_B, 9_C, 9_D$.

Proof of Theorem 3.1. Let S be in 9_A. Then $x_5|a$ and $a|x_8$. It is clear that $a = kx_5$ for $k > 1$. Thus,

$$[x_5, x_6, x_7] \le [a, x_6, x_7] \le [x_8, x_6, x_7] = x_8.$$

Since we have $[x_5, x_6, x_7] = x_8$ by Theorem 2.4, $[a, x_6, x_7] = x_8$ and hence $[x_5, x_6, x_7] = [a, x_6, x_7]$. Then, we have $(kx_5, x_6)(kx_5, x_7) = kx_2x_3$. Suppose $(k, x_6) = 1$ and $(k, x_7) = 1$. Then $(x_5, x_6)(x_5, x_7) = kx_2x_3$. Since $(x_5, x_6) = x_2$ and $(x_5, x_7) = x_3$, it is a contradiction. Thus, we must have $(k, x_6) > 1$ or $(k, x_7) > 1$. Without loss of generality we can assume that $(k, x_6) > 1$. Then, $(a, x_6) = (kx_5, x_6) = t(x_5, x_6) = tx_2 > x_2$, which contradicts to the fact that $S = S_8 \cup \{a\}$ is gcd - closed.

Let S be in 9_B. Then $x_2|a$ and $a|x_8$ and hence

$$[x_5, x_6, x_7] \le [a, x_5, x_6, x_7] \le x_8.$$

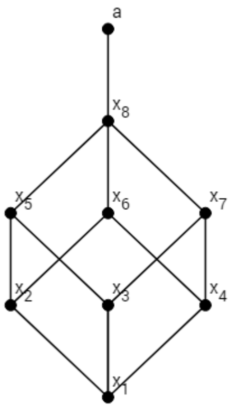
By Theorem 2.4, we have $[a, x_5, x_6, x_7] = x_8$. Thus $[x_5, x_6, x_7] = [a, x_5, x_6, x_7]$. Since $[x_5, x_6, x_7] = \frac{x_5x_6x_7x_1}{x_2x_3x_4}$ and $[a, x_5, x_6, x_7] = \frac{ax_6x_7x_1}{x_2x_2x_3x_4}$, we obtain $a = x_2$. It is a contradiction and hence S cannot be in 9_B.

Let S be in 9_C. Then $x_1|a$ and $a|x_8$. Similarly, by Theorem 2.4, it is clear that $[a, x_5, x_6, x_7] = [x_5, x_6, x_7]$. Also, it is obvious that $[a, x_5, x_6, x_7] = \frac{ax_5x_6x_7}{x_2x_3x_4}$. Thus, we obtain $a = x_1$, which is a contradiction and hence S cannot be in 9_C.

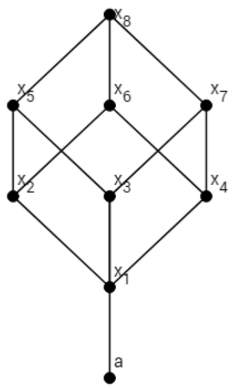
Finally, by Theorem 2.4, we know that $[x_5, x_6, x_7] = x_8$. Therefore, there does not exist such a a as in 9_D.

The proof is complete.

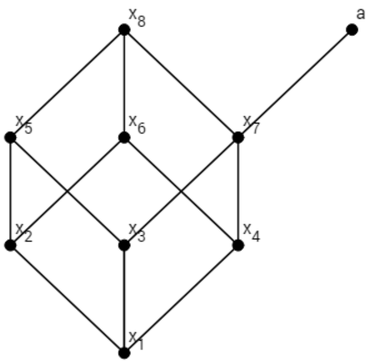
In this section, secondly we present some classes of meet semilattices $(S, |)$ on which LCM matrices $[S_8]$ and $[S]$ are singular under the same assumptions on the sets S_8 and S.



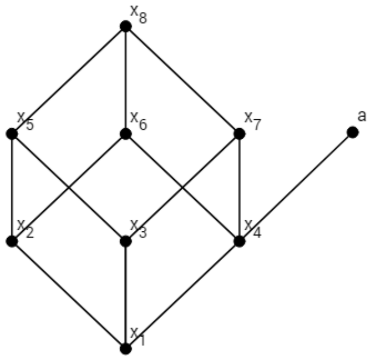
$\mathbf{9_E}$



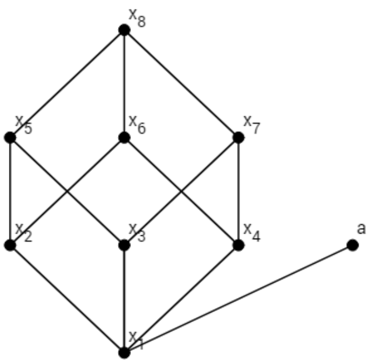
$\mathbf{9_F}$



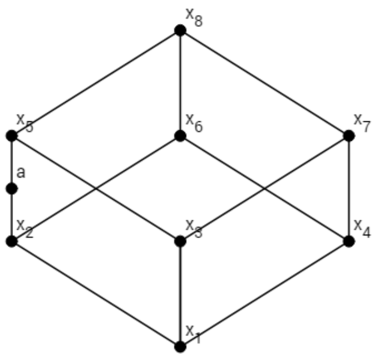
$\mathbf{9_G}$



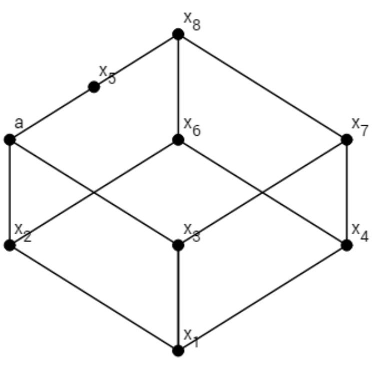
$\mathbf{9_H}$



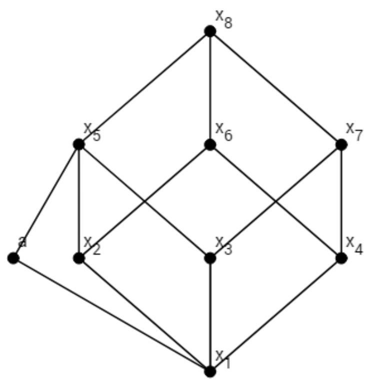
$\mathbf{9_I}$



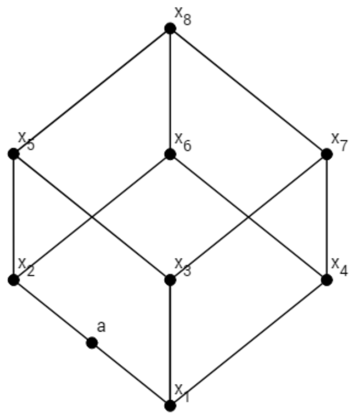
$\mathbf{9_J}$



$\mathbf{9_K}$



$\mathbf{9_L}$



$\mathbf{9_M}$

Figure 4. The Hasse diagrams of semilattices $S_8 \cup \{a\}$ in Theorem 3.2.

Theorem 3.2. Let $S_8 = \{x_1, x_2, \dots, x_8\}$ be a gcd - closed set of positive integers with $x_1 \le x_2 \le \cdots \le x_8$ and let the LCM matrix $[S_8]$ be singular. Also let a be a positive integer such that $a \notin S_8$ and $S = S_8 \cup \{a\}$ is gcd - closed. If the semilattice of $(S, |)$ belongs to one of classes 9_E, 9_F, 9_G, 9_H, 9_I, 9_J, 9_L, 9_K and 9_M, then the LCM matrix [S] is singular.

Proof of Theorem 3.2. Suppose that $S_8 = \{x_1, x_2, \dots, x_8\}$ is a gcd - closed set of positive integers with $x_1 \le x_2 \le \cdots \le x_8$ and the LCM matrix $[S_8]$ is singular. Then, we have the equality in (3), again.

Let $(S, |) \in 9_E$. Since $\psi_{S,\frac{1}{N}}(x_8) = \psi_{S_8,\frac{1}{N}}(x_8)$, by (3) we have $\psi_{S,\frac{1}{N}}(x_8) = 0$. Thus, by (2), we obtain $\det[S] = 0$. This is the proof for (a).

Let $(S, |) \in 9_F$, 9_G, 9_H or 9_I. By Lemma 3.1 in [3], it is clear that $\mu(a, x_8) = 0$ in each case. Then, we have $\psi_{S,\frac{1}{N}}(x_8) = \psi_{S_8,\frac{1}{N}}(x_8)$, and hence, by (3), we obtain $\psi_{S,\frac{1}{N}}(x_8) = 0$. Thus, by (2), we obtain $\det[S] = 0$. This completes the proof of the cases (b), (c), (d) and (e).

Let $(S, |) \in 9_J$, 9_K, 9_L or 9_M. In each case, since $\mu(a, x_8) = 0$, we have $\psi_{S,\frac{1}{N}}(x_8) = \psi_{S_8,\frac{1}{N}}(x_8)$. Thus, by (2), we obtain $\det[S] = 0$. This completes the proof of the cases (f), (g), (h) and (i).

4. COMMENTS ON EXAMPLES AND NEW CONJECTURES

In the literature, there exist a couple of examples to disprove the Bourque - Ligh conjecture and other conjectures on the singularity of LCM matrices on particular gcd - closed sets. Among these examples, there are 4 gcd - closed sets with 9 elements and 2 gcd - closed sets with 8 elements. Using these gcd - closed sets with 8 elements and Theorem 3.2 we can obtain infinitely many gcd - closed sets with 9 elements on which LCM matrices are singular.

Example 4.1. Let $S_8 = \{1, 2, 3, 5, 42, 110, 2295, 353430\}$. It is clear that the Hasse diyagram of the semilattice $(S, |)$ is a cube and the LCM matrix on S_8 is singular [7]. By Theorem 3.2 and the classes from 9_E to 9_L we have the following gcd - closed sets $S = S_8 \cup \{a\}$ given in Table 4.1.

Table 4.1. Gcd - closed sets belonging to the classes from 9_E to 9_M such that $[S_8]$ and $[S]$ are singular and $S_8 = \{1, 2, 3, 5, 42, 110, 2295, 353430\}$.

The class	Gcd-closed sets belonging to the class ($k \geq 2$ is an integer)
9_E	$S = S_8 \cup \{a = 353430k\}$
9_F	Since $x_1 = 1$ in S_8, we cannot derive an example belonging to 9_F directly.
9_G	$S = S_8 \cup \{a = 42k\}$ such that $3 \nmid k$, $5 \nmid k$, $11 \nmid k$, $17 \nmid k$ $S = S_8 \cup \{a = 110k\}$ such that $3 \nmid k$, $7 \nmid k$, $17 \nmid k$ $S = S_8 \cup \{a = 2295k\}$ such that $2 \nmid k$, $7 \nmid k$, $11 \nmid k$
9_H	$S = S_8 \cup \{a = 5k\}$ such that $2 \nmid k$, $3 \nmid k$, $7 \nmid k$, $11 \nmid k$, $17 \nmid k$ $S = S_8 \cup \{a = 3k\}$ such that $2 \nmid k$, $3 \nmid k$, $5 \nmid k$, $7 \nmid k$, $11 \nmid k$ $17 \nmid k$ $S = S_8 \cup \{a = 2k\}$ such that $3 \nmid k$, $5 \nmid k$, $7 \nmid k$, $11 \nmid k$, $17 \nmid k$
9_I	$S = S_8 \cup \{a\}$ such that $2 \nmid a$ $3 \nmid a$, $5 \nmid a$, $7 \nmid a$, $11 \nmid a$, $17 \nmid a$
9_J	$S = S_8 \cup \{a = 14\}$, $S = S_8 \cup \{a = 21\}$, $S = S_8 \cup \{a = 22\}$, $S = S_8 \cup \{a = 55\}$, $S = S_8 \cup \{a = 85\}$ $S = S_8 \cup \{a = 3^k 17^t\}$ such that $k \in \{1, 2, 3\}$, $t \in \{0, 1\}$ and $(k, t) \neq (1, 0)$
9_K	$S = S_8 \cup \{a = 6\}$, $S = S_8 \cup \{a = 10\}$ $S = S_8 \cup \{a = 5.3^k 17^t\}$ such that $k \in \{1, 2, 3\}$, $t \in \{0, 1\}$ and $(k, t) \neq (3, 1)$
9_L	$S = S_8 \cup \{a = 7\}$, $S = S_8 \cup \{a = 11\}$, $S = S_8 \cup \{a = 17\}$
9_M	We do not have any example for this class.

Example 4.2. Let $n \geq 8$ and $a > 1$ be any integer. Now let $x_i = a^{i-1}$ for $1 \leq i \leq n-7$ and $b = a^{n-8}$, $x_{n-6} = 2b$, $x_{n-5} = 3b$, $x_{n-4} = 5b$, $x_{n-3} = 36b$, $x_{n-2} = 230b$, $x_{n-1} = 825b$, $x_n = 227700b$. Then, for each $n \geq 8$, the LCM matrix $[S_n]$ on $S_n = \{x_1, x_2, \dots, x_n\}$ is singular [5]. Indeed, from the proof of Theorem 2.2 (see, Theorem in [5]) the LCM matrix $[S_n]$ on $S = \{1, 2, 3, 5, 36, 230, 825, 227700\}$ is singular. By Theorem 3.2 and the classes from 9_E to 9_L we have the following gcd - closed sets $S = S_8 \cup \{a\}$ given in Table 4.2.

Table 4.2. Gcd - closed sets belonging to the classes from 9_E to 9_M such that $[S_8]$ and $[S]$ are singular and $S = \{1, 2, 3, 5, 36, 230, 825, 227700\}$.

The class	Gcd - closed sets belonging to the class ($k \geq 2$ is an integer)
9_E	$S = S_8 \cup \{a = 227700k\}$
9_F	Since $x_1 = 1$ in S_8, we cannot derive an example belonging to 9_F directly.
9_G	$S = S_8 \cup \{a = 36k\}$ such that $5 \nmid k$, $11 \nmid k$, $23 \nmid k$ $S = S_8 \cup \{a = 230k\}$ such that $2 \nmid k$, $3 \nmid k$, $5 \nmid k$, $11 \nmid k$ $S = S_8 \cup \{a = 825k\}$ such that $2 \nmid k$, $3 \nmid k$, $23 \nmid k$
9_H	$S = S_8 \cup \{a = 2k\}$, $S = S_8 \cup \{a = 3k\}$ such that $2 \nmid k$, $3 \nmid k$, $5 \nmid k$, $11 \nmid k$, $23 \nmid k$ $S = S_8 \cup \{a = 5k\}$,
9_I	$S = S_8 \cup \{a\}$ such that $2 \nmid a$ $3 \nmid a$, $5 \nmid a$, $11 \nmid a$, $23 \nmid a$
9_J	$S = S_8 \cup \{a = 4\}$, $S = S_8 \cup \{a = 9\}$, $S = S_8 \cup \{a = 33\}$, $S = S_8 \cup \{a = 46\}$, $S = S_8 \cup \{a = 115\}$ $S = S_8 \cup \{a = 5^k 11^t\}$ such that $k \in \{1, 2\}$, $t \in \{0, 1\}$ and $(k, t) \neq (1, 0)$
9_K	$S = S_8 \cup \{a = 10\}$ $S = S_8 \cup \{a = 2^k 3^t\}$ such that $k \in \{1, 2\}$, $t \in \{1, 2\}$ and $(k, t) \neq (2, 2)$ $S = S_8 \cup \{a = 3.5^k 11^t\}$ such that $k \in \{1, 2\}$, $t \in \{0, 1\}$ and $(k, t) \neq (2, 1)$
9_L	$S = S_8 \cup \{a = 11\}$, $S = S_8 \cup \{a = 23\}$
9_M	We do not have any example for this class.

After above examples, it should be noted that we cannot produce an example for the class 9_M although the results in the literature and Theorem 3.2 support the existence of such an example. Therefore, in the light of our calculations, we are in a position to raise the following conjecture.

Conjecture 4.1. There cannot exist a gcd - closed set $S = S_8 \cup \{a\}$ belonging to the class 9_M such that LCM matrices $[S_8]$ and $[S]$ are singular.

Example 4.3. Hong [6, 7] presented the gcd - closed sets

$$S = \{1, 2, 5, 8, 10, 13, 26, 65, 520\},$$

$$S = \{1, 2, 3, 5, 6, 10, 15, 27, 270\}$$

on which LCM matrices are singular. In addition to these, Haukkanen, Wang and Sillanpää [4], and Haukkanen et. al. [3] found the following gcd - closed sets

$$S = \{1, 2, 3, 4, 5, 6, 10, 45, 180\}$$

$$S = \{1, 3, 5, 7, 195, 291, 1407, 4025, 1020180525\}$$

on which LCM matrices are singular, again. Indeed, they all are in the class 9_M.

Beside above examples, from the proof of Theorem 2.2 (see, Theorem in [5]), we obtain another gcd - closed set, namely

$$S = \{1, a, 2a, 3a, 5a, 36a, 230a, 825a, 227700a\}$$

for every integer $a > 1$ which belongs to the class 9_F. In the light of these examples, we raise the following conjecture.

Conjecture 4.2. Let S be a gcd - closed set with 9 elements. If $\det[S] = 0$ then the semilattice $(S, |)$ has a sublattice whose Hasse diyagram is a cube.

Finally, we should note that we obtain some classes of gcd - closed sets with 9 elements, however, the problem of determining all structures of gcd - closed sets with 9 elements on which the LCM matrix is singular is still open.

REFERENCES

[1] Altınışık, E., Yıldız, M., *The Bourque-Ligh Conjecture on LCM Matrices*, International Conference on Mathematics and Mathematics Education, Elazığ - Turkey, 2016.
[2] Bourque, K., Ligh, S., *Linear Algebra Appl*., **174**, 65, 1992.
[3] Haukkanen, P., Mattila, M., Mäntysalo, J., *J Comb. Theory Series A*, **135**, 181, 2015.
[4] Haukkanen, P., Wang, J., Sillanpää, J., *Linear Algebra Appl*., **258**, 251, 1997.
[5] Hong, S., *J. Algebra*, **218**, 216, 1999.
[6] Hong, S., *J. Algebra*, **281**, 1, 2004.
[7] Hong, S., *J. Number Theory*, **113**, 1, 2005.
[8] Hong, S., *Linear Algebra Appl*., **416**(1), 124, 2006.
[9] Korkee I., Mattila M., Haukkanen P., *A lattice-theoretic approach to the Bourque-Ligh conjecture*, arXiv: 1403.5428.