ORIGINAL PAPER

A COMPARISON BETWEEN TWO KINDS OF SUBSPACES TOPOLOGIES

MUSTAFA H. HADI[1]

Manuscript received: 15.04.2017; Accepted paper: 21.08.2017;
Published online: 30.09.2017.

Abstract. *In [3] the author introduced such definition (Definition 1.6.1) p.25, for some weak $\omega -$open sets in the subspace topology. And prove some properties and theorems for them. In this paper we shall improve the definition mentioned above and answer the question: Are these properties and theorems satisfied for this improving definition?*
Keywords: *$\omega -$open set, Separation axioms, $\omega - regular$, $\omega - normal$.*
Mathematics Subject Classification (2010): *54B05, 54C08, 54D10.*

1. INTRODUCTION

In 1982 the $\omega -$closed set was first introduced by Hdeib, H. Z. in [4], and he defined it as: A is $\omega -$closed set if it contains all its condensation points and the $\omega -$open set is the complement of the $\omega -$closed set. The union of all $\omega -$open sets contained in A is the $\omega -$interior of A and will denoted by $\mathrm{int}_\omega(A)$. In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha - \omega -$open, pre $-\omega -$open, $b - \omega -$open and $\beta - \omega -$open sets which are weaker than $\omega -$open set. Let us introduce these notions in the following definition:

Definition 1.1. [2, 5] A subset A of a space X is called
1. $\alpha - \omega -$open if $A \subseteq \mathrm{int}_\omega\left(\mathrm{cl}\left(\mathrm{int}_\omega(A)\right)\right)$, and the complement of the $\alpha - \omega -$open set is called $\alpha - \omega -$closed set.
2. pre $-\omega -$open if $A \subseteq \mathrm{int}_\omega(\mathrm{cl}(A))$, and the complement of the pre $- \omega -$open set is called pre $- \omega -$closed set.
3. b $-\omega -$open if $A \subseteq \mathrm{int}_\omega(\mathrm{cl}(A)) \cup \mathrm{cl}(\mathrm{int}_\omega(A))$, and the complement of the $b - \omega -$open set is called $b - \omega -$closed set.
4. $\beta - \omega -$open if $A \subseteq \mathrm{cl}\left(\mathrm{int}_\omega(\mathrm{cl}(A))\right)$, and the complement of the $\beta - \omega -$open set is called $\beta - \omega -$closed set.

2. WEAK $\omega -$ OPEN SETS IN THE SUBSPACE TOPOLOGY

In this article we introduce our definition of the weak $\omega -$open sets in the subspace topology and it's related results. Before that, we remember you that you can review the definition of subspace topology in [1] or [6].

[1] University of Babylon, College of Education for Pure Science, Department of Mathematics, Babylon/ Hillah, Iraq. E-mail: mustafahh1984@yahoo.com.

Now we can introduce our definition for weak $\omega-$open sets above in the subspace topology:

Definition 2.1. Let Y be a subset of the topological space (X,T) , then $A \subset Y$ is $\omega-$open (resp. $\alpha-\omega-$open , pre $-\omega-$open , b $-\omega-$open and $\beta-\omega-$open) in Y if A is $\omega-$open (resp. $\alpha-\omega-$open , pre $-\omega-$open , $\beta-\omega-$open , b $-\omega-$open) with respect to the subspace topology (Y,T_Y). We say that $A \subset Y$ is $\omega-$closed (resp. $\alpha-\omega-$closed , pre $-\omega-$closed , b $-\omega-$open and $\beta-\omega-$open) in Y if A is $\omega-$closed (resp. $\alpha-\omega-$closed , pre $-\omega-$closed , b $-\omega-$open and $\beta-\omega-$open) with respect to the subspace topology (Y,T_Y).

Lemma 2.2. [6]. Let Y be a subspace of a topological space X. If $A \subset Y$ is open (closed) in X, then A is also open (closed) in Y.

Theorem 2.3. Let Y be a subspace of a topological space X. And let $A \subset Y$. If A is $\omega-$open set in X , then A is $\omega-$open in Y.
Proof: Let $x \in A$, then since A is $\omega-$open in X, the there exists an open set U in X with $x \in U \cap Y$. Then we have $(U \cap Y)\backslash A=(U\backslash A) \cap (Y\backslash A)$ is countable.

Corollary 2.4. Let Y be a subspace of a topological space X. And let $A \subset Y$. If A is $\omega-$closed set in X , then A is $\omega-$closed in Y.
Proof: Let A be an $\omega-$closed subset of the topological space X. Then A^c is $\omega-$open set in X . By Theorem 2.3 we get A^c is $\omega-$open in Y. Hence A is $\omega-$closed in Y.

Theorem 2.5. If Y is a subspace of the topological space X, and $A \subset Y$, then $cl_{\omega X}(A) \subseteq cl_{\omega Y}(A)$.
Proof: $cl_{\omega X}(A) = \cap \{K: K\ \omega-\text{closed in } X, A \subset K\}$
$\subseteq \cap \{K: K\ \omega-\text{closed in } Y, A \subset K\}$
$= cl_{\omega Y}(A)$.

Theorem 2.6. If Y be a subspace of a topological space X, and $A \subset Y$, then $int_{\omega X}(A) \subseteq int_{\omega Y}(A)$.
Proof: $int_{\omega X}(A) = \cup \{G \subset X: G \text{ is } \omega-\text{open in } X , G \subseteq A \subseteq Y\}$. Then since A subset of Y and $G \subseteq A$, so $G \subseteq Y$. And since any $\omega-$open set in X is also an $\omega-$open set in Y for any subset of Y. This completes the proof.

Theorem 2.7. Let Y be a subspace of a topological space X, and $A \subset Y$. If A is $\alpha-\omega-$open (resp. pre $-\omega-$open, b $-\omega-$open and $\beta-\omega-$open) in X, then A is $\alpha-\omega-$open (resp. pre $-\omega-$open, b $-\omega-$open , and $\beta-\omega-$open) in Y.
Proof: Let A be a subset of Y and A is an $\alpha-\omega-$open set in X, so
$A \subseteq int_{\omega X}\left(cl_X\left(int_{\omega X}(A)\right)\right)$
$\subseteq int_{\omega Y}\left(cl_Y\left(int_{\omega Y}(A)\right)\right)$, Therefore, A is $\alpha-\omega-$open in Y.
By the same way we can prove the other cases.

Theorem 2.8. Let Y be a subspace of a topological space X, and $A \subset Y$. If A is $\alpha-\omega-$closed (resp. pre $-\omega-$closed, b $-\omega-$closed and $\beta-\omega-$closed) in X, then A is $\alpha-\omega-$closed (resp. pre $-\omega-$closed, b $-\omega-$closed , and $\beta-\omega-$closed) in Y.
Proof: The proof is similar to that of Corollary 2.4 .

3. SOME SEPARATION AXIOMS AND SUBSPACE TOPOLOGY

In this article let us study the characterization of some separation axioms introduced in [3] and our new definition of subspace topology. First let us recall the definition of weak T_0 spaces introduced in [3]:

Definition 3.1. [3] Let X be a topological space. If for each $x \neq y \in X$, either there exists a set U, such that $x \in U$, $y \notin U$,or there exists a set U such that $x \notin U, y \in U$. Then X called
1. $\omega - T_0$ space, whenever U is ω-open set in X.
2. $\alpha - \omega - T_0$ space, whenever U is $\alpha - \omega -$open set in X.
3. pre$-\omega - T_0$ space, whenever U is pre $-\omega -$open set in X.
4. $b - \omega - T_0$ space, whenever U is $b - \omega -$open set in X.
5. $\beta - \omega - T_0$ space, whenever U is $\beta - \omega -$open set in X.

To prove our first result we need to recall the following result from [5]:

Lemma 3.2. [5] The intersection of an $\alpha - \omega -$open (resp. pre $- \omega -$open, $b - \omega -$open and $\beta - \omega -$open) subset of any topological space and an open subset is $\alpha - \omega -$open (resp. pre $- \omega -$open, $b - \omega -$open and $\beta - \omega -$open) set.

Theorem 3.3. Let (X, T) be a topological space, and A is an open subspace of X . If X is $\omega - T_0$, (resp. $\alpha - \omega - T_0$, , pre $-\omega - T_0$, $b -\omega - T_0$, $\beta - \omega - T_0$. Then so is A.

Proof: Let X be an $\alpha - \omega - T_0$ space. Let $x \neq y$ in $A \subset X$. Since X is $\alpha - \omega - T_o$, so there is a set U in X, such that U is $\alpha - \omega -$ open set containing x, but not containing y . Then by Lemma 3.2 and Theorem 2.7, $U \cap A$ is an $\alpha - \omega -$open set in A. Thus A is an $\alpha - \omega - T_0$ space .

Similarly by using the other cases of Lemma 3.2, and Theorem 2.7, and Theorem 2.3, one can prove the other cases.

Secondly let us recall definition of weak T_1 spaces:

Definition 3.4. [3] Let X be a topological space. For each $x \neq y \in X$, there exists a set U, such that $x \in U, y \notin U$, and there exists a set V such that $y \in V, x \notin V$, then X is called
1. $\omega - T_1$ space if U is open and V is $\omega -$open sets in X.
2. $\alpha - \omega - T_1$ space if U is open and V is $\alpha - \omega -$open sets in X.
3. $\omega^\star - T_1$ space [4] if U and V are $\omega -$open sets in X.
4. $\alpha - \omega^\star - T_1$ space if U is ω-open and V is $\alpha - \omega -$open sets in X.
5. $\alpha - \omega^{\star\star} - T_1$ space if U and V are $\alpha - \omega -$open sets in X.
6. $pre -\omega - T_1$ space if U is open and V is $pre - \omega -$open sets in X.
7. $pre -\omega^\star - T_1$ space if U is ω-open and V is $pre -\omega -$open sets in X.
8. $\alpha - pre -\omega - T_1$ space if U is $\alpha - \omega -$ open and V is $pre -\omega -$open sets in X.
9. $pre -\omega^{\star\star} - T_1$ space if U and V are $pre -\omega -$open sets in X.
10. $b - \omega - T_1$ space if U is open and V is $b - \omega -$open sets in X.
11. $b - \omega^\star - T_1$ space if U is $\omega -$open and V is $b - \omega -$open sets in X.
12. $\alpha - b - \omega - T_1$ space if U is $\alpha - \omega -$open and V is $b - \omega -$open sets in X.
13. $pre - b - \omega - T_1$ space if U is $pre -\omega -$open and V is $b - \omega -$open sets in X.
14. $b - \omega^{\star\star} - T_1$ space if U and V are $b - \omega -$open sets in X.
15. $\beta - \omega - T_1$ space if U is open and V is $\beta - \omega -$open sets in X.
16. $\beta - \omega^\star - T_1$ space if U is $\omega -$open and V is $\beta - \omega -$open sets in X.
17. $\alpha - \beta-\omega - T_1$ space if U is $\alpha - \omega -$open and V is $\beta - \omega -$open sets in X.
18. $pre - \beta - \omega - T_1$ space if U is $pre -\omega -$open and V is $\beta - \omega -$open sets in X.

19. $\beta - \omega^{\star\star} - T_1$ space if U and V are $\beta - \omega$ −open sets in X.
20. $b - \beta - \omega - T_1$ space if U is $b - \omega$ −open and V is $\beta - \omega$ −open sets in X.

Then we can prove the following result:

Theorem 3.5. Let (X, T) be a topological space, and A is an open subspace of . If X is $\omega - T_1$ (resp. $\alpha - \omega - T_1$, $\alpha - \omega^{\star} - T_1$,$\omega^{\star} - T_1$, $\alpha - \omega^{\star\star} - T_1$, $pre - \omega - T_1$, $pre - \omega^{\star} - T_1$, $\alpha - pre - \omega - T_1$, $pre - \omega^{\star\star} - T_1$, $b - \omega - T_1$, $b - \omega^{\star} - T_1$, $\alpha - b - \omega - T_1$, $pre - b - \omega - T_1$, $b - \omega^{\star\star} - T_1$, $\beta - \omega - T_1$, $\beta - \omega^{\star} - T_1$, $\alpha - \beta - \omega - T_1$, $pre - \beta - \omega - T_1$, $\beta - \omega^{\star\star} - T_1$, and $b - \beta - \omega - T_1$). Then so is A.

Proof: Let X be an $\alpha - \omega - T_1$ space and let $x \neq y$ in $A \subset X$. Since X is $\alpha - \omega - T_1$, so there are two sets U and V in X, such that U is open set containing x, and V is $\alpha - \omega$ −open set containing y . Then $U \cap A$ is open in A. and by Lemma 3.2, and Theorem 2.7, $V \cap A$ is an $\alpha - \omega$ −open set in A. Thus A is an $\alpha - \omega - T_1$ space.

Similarly by using the other cases of Lemma 3.2, and Theorem 2.7, and Theorem 2.3, one can prove the other cases.

Third let us recall definition of weak T_2 spaces:

Definition 3.6. [3] Let X be a topological space. And for each $x \neq y \in X$, there exist two disjoint sets U and V with $x \in U$, and $y \in V$, then X is called:
1. $\omega - T_2$ space if U is open and V is ω −open sets in X.
2. $\alpha - \omega - T_2$ space if U is open and V is $\alpha - \omega$ −open sets in X.
3. $\omega^{\star} - T_2$ space if U and V are ω −open sets in X.
4. $\alpha - \omega^{\star} - T_2$ space if U is ω-open and V is $\alpha - \omega$ −open sets in X.
5. $\alpha - \omega^{\star\star} - T_2$ space if U and V are $\alpha - \omega$ −open sets in X.
6. $pre - \omega - T_2$ space if U is open and V is $pre - \omega$ −open sets in X.
7. $pre - \omega^{\star} - T_2$ space if U is ω −open and V is $pre - \omega$ −open sets in X.
8. $\alpha - pre - \omega - T_2$ space if U is α −open and V is $pre - \omega$ −open sets in X.
9. $pre - \omega^{\star\star} - T_2$ space if U and V are $pre - \omega$ −open sets in X.
10. $b - \omega - T_2$ space if U is open and V is $b - \omega$ −open sets in X.
11. $b - \omega^{\star} - T_2$ space if U is ω −open and V is $b - \omega$ −open sets in X.
12. $\alpha - b - \omega - T_2$ space if U is $\alpha - \omega$ −open and V is $b - \omega$ −open sets in X.
13. $pre - b - \omega - T_2$ space if U is $pre - \omega$ −open and V is $b - \omega$ −open sets in X.
14. $b - \omega^{\star\star} - T_2$ space if U and V are $b - \omega$ −open sets in X.
15. $\beta - \omega - T_2$ space if U is open and V is $\beta - \omega$ −open sets in X.
16. $\beta - \omega^{\star} - T_2$ space if U is ω −open and V is $\beta - \omega$ −open sets in X.
17. $\alpha - \beta - \omega - T_2$ space if U is $\alpha - \omega$ −open and V is $\beta - \omega$ −open sets in X.
18. $pre - \beta - \omega - T_2$ space if U is $pre - \omega$ −open and V is $\beta - \omega$ −open sets in X.
19. $\beta - \omega^{\star\star} - T_2$ space if U and V are $\beta - \omega$ −open sets in X.
20. $b - \beta - \omega - T_2$ space if U is $b - \omega$ −open and V is $\beta - \omega$ −open sets in X.

And our theorem concerning the subspace topology is:

Theorem 3.7. Let (X, T) be a topological space, and A is an open subspace of X. If X is $\omega - T_2$ (resp. $\alpha - \omega - T_2$, $\omega^{\star} - T_2$, $\alpha - \omega^{\star} - T_2$, $\alpha - \omega^{\star\star} - T_2$, $pre - \omega - T_2$, $pre - \omega^{\star} - T_2$, $\alpha - pre - \omega - T_2$, $pre - \omega^{\star\star} - T_2$, $b - \omega - T_2$, $b - \omega^{\star} - T_2$, $\alpha - b - \omega - T_2$, $pre - b - \omega - T_2$, $b - \omega^{\star\star} - T_2$, $\beta - \omega - T_2$, $\beta - \omega^{\star} - T_2$, $\alpha - \beta - \omega - T_2$, $pre - \beta - \omega - T_2$, $\beta - \omega^{\star\star} - T_2$, $b - \beta - \omega - T_2$). Then so is A.

Proof: Let X be an $\alpha - \omega - T_2$ space. Let $x \neq y$ in $A \subset X$. Since X is $\alpha - \omega - T_2$, so there are two disjoint sets U and V in X, such that U is open set containing x, and V is $\alpha - \omega$ −open set containing y. Then $U \cap A = G$, is open in A. and by Lemma 3.2, and Theorem 2.7

$V \cap A = H$ is an $\alpha - \omega -$open set in A, and $G \cap H = (U \cap A) \cap (V \cap A) = U \cap V \cap A = \emptyset \cap A = \emptyset$. Thus A is an $\alpha - \omega - T_2$ space. Similarly by using the other cases of Lemma 3.2, and Theorem 2.7, and Theorem 2.3, one can prove the other cases .

Now let us introduce the following definition of weak regularity:

Definition 3.8. Let X be a topological space. If for a given $x \in X$, and a set $F \subset X$, with $x \notin F$, there exist two disjoint sets U and V with $x \in U$, and $F \subset V$ then X is called:
1. $\omega - regular$ space if F is closed, U is $\omega -$open, and V is open sets in X.
2. $\alpha - \omega - regular$ space if F is closed, U is $\alpha - \omega -$open, and V is open sets in X.
3. $\omega^\star - regular$ space if F is closed, U and V are $\omega -$open sets in X.
4. $\alpha - \omega^\star - regullar$ space if F is $\omega -$ closed, U is $\alpha - \omega -$open, and, V is $\omega -$open sets in X.
5. $\alpha - \omega^{\star\star} - regular$ space if F is $\alpha - \omega -$closed U and V are $\alpha - \omega$-open sets in X.
6. $pre - \omega - regular$ space if F is $pre - \omega -$closed, U is open and V is $pre - \omega -$open sets in X.
7. $pre - \omega^\star - regular$ space if F is $pre - \omega -$closed, U is $\omega -$open and V is $pre - \omega -$open sets in X.
8. $\alpha - pre - \omega - regular$ space if F is $pre - \omega -$closed, U is $\alpha -$open and V is $pre - \omega -$open sets in X.
9. $pre - \omega^{\star\star} - regular$ space if F is $pre - \omega -$closed, and U and V are $pre - \omega -$open sets in X.
10. $b - \omega - regular$ space if F is $b - \omega -$closed, U is open and V is $b - \omega -$open sets in X.
11. $b - \omega^\star - regular$ space if F is $b - \omega -$closed U is $\omega -$open and V is $b - \omega -$open sets in X.
12. $\alpha - b - \omega - regular$ space if F is $b - \omega -$closed, U is $\alpha - \omega -$open and V is $b - \omega -$open sets in X.
13. $pre - b - \omega - regular$ space if F is $b - \omega -$closed, U is $pre - \omega -$open and V is $b - \omega -$open sets in X.
14. $b - \omega^{\star\star} - regular$ space if F is $b - \omega -$closed, and U and V are $b - \omega -$open sets in X.
15. $\beta - \omega - regular$ space if F is $\beta - \omega -$closed, U is open and V is $\beta - \omega -$open sets in X.
16. $\beta - \omega^\star - regular$ space if F is $\beta - \omega -$closed, U is $\omega -$open and V is $\beta - \omega -$open sets in X.
17. $\alpha - \beta - \omega - regular$ space if F is $\beta - \omega -$closed U is $\alpha - \omega -$open and V is $\beta - \omega -$open sets in X.
18. $pre - \beta - \omega - regular$ space if F is $\beta - \omega -$closed U is pre $-\omega -$open and V is $\beta - \omega -$open sets in X.
19. $\beta - \omega^{\star\star} - regular$ space if F is $\beta - \omega -$closed, and U and V are $\beta - \omega -$open sets in X.
20. $b - \beta - \omega - regular$ space if F is $\beta - \omega -$closed, U is $b - \omega -$open and V is $\beta - \omega -$open sets in X.

For these weak spaces let us introduce the following result:

Theorem 3.9. For any topological space.
1. An $\omega - T_0$ space is $pre - \omega^\star - T_2$ (resp. $\omega^\star - T_2, \alpha - \omega^\star - T_2, b - \omega^\star - T_2, \quad \beta - \omega - T_2, \quad \beta - \omega^\star - T_2 \quad \beta - \omega^\star - T_2, \; b - \omega - T_2, \alpha - \omega - T_2, pre - \omega - T_2, and \; \omega - T_2$) if it is $pre \; - \omega^\star - regular$ (resp. $\omega^\star - regular$, $\alpha - \omega^\star - regular$, $b - \omega^\star -$

$regular$, $\beta-\omega-regular$, $\beta-\omega^{\star}-regular$, $b-\omega-regular$, $\alpha-\omega-regular$, $pre-\omega-regular$ and $\omega-regular$).

2. A T_0 space is $\alpha-\omega-T_2$ (resp. $\omega-T_2, pre-\omega-T_2, b-\omega-T_2$,and $\beta-\omega-T_2$ if it is $\alpha-\omega-regular$ (resp. ω $regular$, pre - ω $regular$, b - ω - $regular$ and $\beta-\omega-regular$).

3. An $\alpha-\omega-T_0$ space is $\alpha-pre-\omega-T_2$ (resp. $\alpha-\omega^{\star}-T_2$, $\alpha-b-\omega-T_2, \alpha-\beta-\omega-T_2$, and $\alpha-\omega-T_2$) if it is $\alpha-pre-\omega-regular$ (resp $\alpha-\omega^{\star}-regular$, $\alpha-b-\omega-regular$, $\alpha-\beta-\omega-regular$ and $\alpha-\omega-regular$).

4. A $pre-\omega-T_0$ space is $pre-\omega^{\star\star}-T_2$ (resp. $pre-b-\omega-T_2$, pre$-\beta-\omega-T_2, pre-\omega-T_2, pre-\omega^{\star}-T_2$, and $\alpha-pre-\omega-T_2$) if it is pre $-\omega^{\star\star}-regular$ (resp. pre $-b-\omega-regular$, $pre-\beta-\omega-regular$, $pre-\omega-regular$, $pre-\omega^{\star}-regular$ and $\alpha-pre-\omega-regular$).

5. A $b-\omega-T_0$ space is $b-\omega^{\star\star}-T_2$, (resp. $b-\omega-T_2$, $b-\omega^{\star}-T_2$, $\alpha-b-\omega-T_2$, $pre-b-\omega-T_2$ and $b-\beta-\omega-T_2$) if it is $b-\omega^{\star\star}-T_3$, (resp. $b-\omega-regular$, $b-\omega^{\star}-regular$, $\alpha-b-\omega-regular$, $pre-b$ ω $regular$ and $b-\beta-\omega-regular$).

6. A $\beta-\omega-T_0$ space is $\beta-\omega-T_2$(resp. $\beta-\omega^{\star}-T_2$ $\beta-\omega^{\star\star}-T_2$, $b-\beta-\omega-T_2$, $pre-\beta-\omega-T_2$, and $\alpha-\beta-\omega-T_2$) if it is $\beta-\omega-regular$ (resp. $\beta-\omega^{\star}-regular$, $\beta-\omega^{\star\star}$ $regular$, $b-\beta-\omega-regular$, $pre-\beta-\omega-regular$, $\alpha-\beta-\omega-regular$ and $\beta-\omega^{\star\star}-regular$).

Proof of (1): Let us prove one case and the other are the same. Let X be $\omega-T_0$ space and $\alpha-\omega^{\star}-regular$, let $x, y \in X$, such that $x \neq y$. There is an ω-open set U_x, such that $x \in U_x, y \notin U_x$. Then $X\backslash U_x$, is $\omega-$closed set and $x \notin X\backslash U_x$. Since X is $\alpha-\omega^{\star}-regular$, there are disjoint sets V_1, and V_2, such that V_1 is ω-open and V_2 is $\alpha-\omega-$open, with $x \in V_2$, and $X\backslash U_x \subset V_1$, so $x \in V_2$ and $y \in V_1$. Hence X is $\alpha-\omega^{\star}-T_2$.
Similarly for (2), (3), (4), (5), and (6). .

We can extend Theorem 3.9 by other weak T_2s spaces that possess weak $\omega-$open set weaker than that of the weak T_0s spaces.

And the following theorem is related to the subspace topology:

Theorem 3.10. Let (X,T) be a topological space, and A is an open subspace of X . If X is $\omega-regular$ (resp. $\alpha-\omega-regular$, $pre-\omega-regular$, $b-\omega-regular$ and $\beta-\omega-regular$) . Then so is A.

Proof: Let X be $\alpha-\omega-regular$ space. We must prove A is $\alpha-\omega-regular$. Let $F \subset A$ be a closed set in A, $x \notin F$and let $\overline{F} = \cap\, G$, G is a closed set in X containing F. Then $\cap\, \acute{G} = \cap\,(G \cap A) = F$, where $\acute{G}$ is a closed set in A containing F, and F the closure of F in A. Since X is $\alpha-\omega-regular$ space so there are two disjoint sets U and V such that U is open in X and V is an $\alpha-\omega-$open in X, with $x \in V$, and $\overline{F} \subset U$. Then by Lemma 3.2 and Theorem 2.7 $V \cap A$ is an $\alpha-\omega-$open set containing x. And $U \cap A$ is an open set containing the closed set $F = \overline{F} \cap A$. Also we have $(V \cap A) \cap (U \cap A) = \emptyset$

Similarly by using the other cases of Lemma 3.2, and Theorem 2.7, and Theorem 2.3, one can prove the other cases .

If we consider A is closed and open subspace of X, we can get A is $\omega-T_3$ (resp. $\alpha-\omega-T_3$, $\omega^{\star}-T_3$, $\alpha-\omega^{\star\star}-T_3$, $pre-\omega-T_3$, $pre-\omega^{\star}-T_3$, $\alpha-pre-\omega-T_3$, $pre-\omega^{\star\star}-T_3$, $b-\omega-T_3$, $b-\omega^{\star}-T_3$, $pre-b-\omega-T_3$, $\alpha-b-\omega-T_3$, $b-\omega^{\star\star}-T_3$, $\beta-\omega-T_3$, $\beta-\omega^{\star}-T_3$, $\alpha-\beta-\omega-T_3$, $pre-\beta-\omega-T_3$, $\beta-\omega^{\star\star}-T_{3i}$, and $b-\beta-\omega-T_3$) , whenever X is it also.

Since the intersection of two $\alpha-\omega-$closed sets is also $\alpha-\omega-$closed set, and the same for the other types of the weak closed sets.

Therefore we can write another type of Theorem 3.10 as:

Theorem 3.11. Let (X,T) be a topological space, and A is open and closed subspace of X. If X is $\omega-T_3$ (resp. $\alpha-\omega-T_3$, $\omega^{\star}-T_3$, $\alpha-\omega^{\star\star}-T_3$, $\alpha-\omega^{\star}-T_3$, $pre-\omega-T_3$, $pre-\omega^{\star}-T_3$, $\alpha-pre-\omega-T_3$, $pre-\omega^{\star\star}-T_3$, $b-\omega-T_3$, $b-\omega^{\star}-T_3$, $pre-b-\omega-T_3$, $\alpha-b-\omega-T_3$, $b-\omega^{\star\star}-T_3$, $\beta-\omega-T_3$, $\beta-\omega^{\star}-T_3$, $\alpha-\beta-\omega-T_3$, $pre-\beta-\omega-T_3$, $\beta-\omega^{\star\star}-T_3$, and $b-\beta-\omega-T_3$) . Then so is A.
Proof: Similar to the proof of the theorem above with simple modification.

After weak regularity let us recall the weak normality defined in [3] and prove a related theorem for subspace topology.

Definition 3.12. Let X be a topological space. If for every pair of disjoint subsets G and H of X, there exist pair of subsets U and V of X, such that $G\subset U$ and $H\subset V$ and $U\cap V=\emptyset$, then X is called

1. $\omega-normal$ space if G is closed, H is $\omega-$closed, U is open and V is $\omega-$open sets in X.
2. $\alpha-\omega-normal$ space if G is closed and H is $\alpha-\omega-$closed, U is open and V is $\alpha-\omega-$open sets in X.
3. $\omega^{\star}-normal$ space if G and H are $\omega-$closed sets and U and V are $\omega-$open sets in X.
4. $\alpha-\omega^{\star}-normal$ space if G is ω-closed and H is $\alpha-\omega-$closed, U is $\omega-$open and V is $\alpha-\omega-$open sets in X.
5. $\alpha-\omega^{\star\star}-normal$ space if G and H are $\alpha-\omega-$closed, U and V are $\alpha-\omega-$open sets in X.
6. $pre-\omega-normal$ space if G is closed and H is $pre-\omega-$closed, U is open and V is $pre-\omega-$open sets in X.
7. $pre-\omega^{\star}-normal$ space if G is $\omega-$closed and H is $pre-\omega-$closed, U is $\omega-$open and V is $pre-\omega-$open sets in X.
8. $\alpha-pre-\omega-normal$ space if , G is $\alpha-\omega-$ closed and H is $pre-\omega-$closed, U is $\alpha-\omega-$ open and V is $pre-\omega-$open sets in X.
9. $pre-\omega^{\star\star}-normal$ space if , G and H are $pre-\omega-$closed, U and V are $pre-\omega-$open sets in X.
10. $b-\omega-normal$ space if G is closed and H is $b-\omega-$closed, U is open and V is $b-\omega-$open sets in X.
11. $b-\omega^{\star}-normal$ space if G is $\omega-$closed and H is $b-\omega-$closed, U is $\omega-$open and V is $b-\omega-$open sets in X.
12. $\alpha-b-\omega-normal$ space if G is $\alpha-\omega-$closed and H is $b-\omega-$closed U is $\alpha-\omega-$open and V is $b-\omega-$open sets in X.
13. $pre-b-\omega-normal$ space if G is $pre-\omega-$closed and H is $b-\omega-$closed, U is $pre-\omega-$open and V is $b-\omega-$open sets in X.
14. $b-\omega^{\star\star}-normal$ space if G and H are $b-\omega-$closed, sets U and V are $b-\omega-$open sets in X.
15. $\beta-\omega-normal$ space if G is closed and H is $\beta-\omega-$closed, U is open and V is $\beta-\omega-$open sets in X.
16. $\beta-\omega^{\star}-normal$ space if G is $\omega-$closed and H is $\beta-\omega-$closed, U is ω-open and V is $\beta-\omega-$open sets in X.
17. $\alpha-\beta-\omega-normal$ space if G is $\alpha-\omega-$closed and H is $\beta-\omega-$closed, U is $\alpha-\omega-$open and V is $\beta-\omega-$open sets in X.
18. $pre-\beta-\omega-normal$ space if G is $pre-\omega-$closed and H is $\beta-\omega-$closed, U is $pre-\omega-$open and V is $\beta-\omega-$open sets in X.

19. $\beta-\omega^{\star\star}-normal$ space if G and H are $\beta-\omega$ −closed, U and V are $\beta-\omega$ −open sets in X.
20. $b-\beta-\omega-normal$ space if G is $b-\omega$ −closed and H is $\beta-\omega$ −closed, U is $b-\omega$ −open and V is $\beta-\omega$ −open sets in X.

Theorem 3.13. Let (X,T) be a topological space, and A is open and closed subspace of X. If X is $\omega-normal$ (resp. $\alpha-\omega-normal$, $\omega^{\star}-normal$, $\alpha-\omega^{\star\star}-normal$, $\alpha-\omega^{\star}-normal$, $pre-\omega-normal$, $pre-\omega^{\star}-normal$, $\alpha-pre-\omega-normal$, $pre-\omega^{\star\star}-\text{normal}$, $b-\omega-normal$, $b-\omega^{\star}-normal$, $pre-b-\omega-normal$, $\alpha-b-\omega-normal$, $b-\omega^{\star\star}-normal$, $\beta-\omega-normal$, $\beta-\omega^{\star}-normal$, $\alpha-\beta-\omega-\text{normal}$, $\text{pre}-\beta-\omega-\text{normal}$, $\beta-\omega^{\star\star}-\text{normal}$, and $\text{b}-\beta-\omega-\text{normal}$) . Then so is A.
Proof: Let F and H are disjoint sets in X, such that F is closed set and H is ω −closed set in A. Let $\overline{\text{F}}=\cap\ \text{G}$, G is closed set in X , and containing F . $\text{F}=\cap\ (\text{G}\cap\text{A})=\cap\ \acute{\text{G}}$, where $\acute{\text{G}}$ is closed set in A and containing F. Let $\overline{\text{H}}=\cap\ \text{W}$, W is ω −closed set in X , and containing H . $\text{H}=\cap\ (\text{W}\cap\text{A})=\cap\ \acute{\text{W}}$, where $\acute{\text{W}}$ is ω −closed set in A and containing H. Since X is $\omega-$ normal space so there are two disjoint sets U open and V is ω −open in X, such that $\text{F}\subset\text{U}$ and $\text{H}\subset\text{V}$. We have $\text{F}\subset\text{U}\cap\text{A}=\text{B}$, also $\text{H}\subset\text{V}\cap\text{A}=\text{C}$, and $\text{B}\cap\text{C}=\emptyset$. Lemma 3.2, and Theorem 2.7 implies B is open in A and C is ω −open in A .

Similarly by using the other cases of Lemma 3.2, and Theorem 2.7, and Theorem 2.3, one can prove the other cases.

4. CONCLUSIONS

In this paper we try to define new definitions for the weak openness and closeness in the subspace topology different from that in [3]. The new is more complicated than the other in [3]. In this article we make a comparison between the two definitions in terms of some results that we proved in [3], and the inheritance of the weak separation axioms.

We conclude that the same results (with more complicated different proof) still the same as that for the definition in [3] such as: If $\text{Y}\subseteq\text{X}$, then any (ω −open, $\alpha-\omega$ −open, $\text{pre}-\omega$ −open, $\text{b}-\omega$ −open, $\beta-\omega$ −open, ω −closed, $\alpha-\omega$ −closed, $\text{pre}-\omega$ −closed, $\text{b}-\omega$ −closed, and $\beta-\omega$ −closed) subsets of X is the same in Y. $\text{cl}_{\omega\text{X}}(\text{A})\subseteq\text{cl}_{\omega\text{Y}}(\text{A})$, and $\text{int}_{\omega\text{X}}(\text{A})\subseteq\text{int}_{\omega\text{Y}}(\text{A})$.

While some of the results are still satisfied under conditions. We show how a subset Y of a topological space X inherits the weak separation axioms from the space X under conditions as in Theorem 3.3, Theorem 3.5, Theorem 3.7, Theorem 3.11, and Theorem 3.13.

REFERENCES

[1] Adams, C., Franzosa, R., *Introduction to topology pure and applied*, India, 2009.
[2] Al Swidi, L.A., Mustafa, H.H., *Eur.J.Sci.Res.*, **57**(4), 577, 2011.
[3] Hadi, M.H., *Weak forms of ω-open sets and decomposition of separation axioms* - M.Sc. Thesis, Babylon University, 2011.
[4] Hdeib, H.Z., *Rev. Colomb. Mat.*, **16**(3-4), 65, 1982.
[5] Noiri, T., Al-Omari, A., Noorani, M.S.M., *E.J.P.A.M.*, **2**(1), 73, 2009.
[6] Sharma, J.N., *Topology*, Krishna Prakashan Mandir, Meerut, 1977.