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SOLVING SYSTEM OF DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER BY HOMOTOPY ANALYSIS METHOD

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Abstract. *This paper applies homotopy analysis method (HAM) to obtain analytical solutions of system of fractional order DEs, which arises very frequently in mechanical engineering, control theory, solid mechanics and applied sciences. Analytical results reveal the complete compatibility of proposed algorithm for such problems. Some mathematical problems are presented to show the efficiency and simplicity of the method.*
Keywords: *Homotopy Analysis Method, Fractional Calculus, Differential Equations.*

1. INTRODUCTION

Differential equations arise in almost all fields of the applied and engineering sciences [1-18]. Several numerical and analytical techniques including Homotopy analysis method (HAM), Perturbation methods, Modified adomian's decomposition method (MADM), Finite difference method, Spline method, Variational iteration method (VIM) have been developed to solve such problems, see [1-18] and the references therein. Recently, many researchers have started working on a very special type of differential equations which are called fractional differential equations [12-18] and are extremely important in number of physical problems related to applied and engineering sciences. It is to be highlighted that system of fractional ODEs, which arises very frequently in mechanical engineering, control theory, solid mechanics, mathematical modeling and applied sciences have inspired and motivated by the ongoing research in present time. We apply a very efficient and reliable technique which is called homotopy analysis method (HAM) [1-10, 15, 16] to obtain analytical solutions of system of fractional ODEs. Obtained results are very encouraging.

2. HOMOTOPY ANALYSIS METHOD (HAM)

We consider the following equation

$$\widetilde{N}[u(\tau)] = 0, \tag{1}$$

where $\widetilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao [6-10] constructed zero-order deformation equation

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$$(1-p)\mathcal{L}[\emptyset(\tau;p)-u_0(\tau)] = p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{2}$$

where $\mathcal{L}$ is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathcal{L}[\emptyset(\tau;0)-u_0(\tau)] = 0 \implies \emptyset(\tau;0) = u_0(\tau), \tag{3}$$

$$\hbar\widetilde{N}[\emptyset(\tau;1)] = 0 \implies \emptyset(\tau;1) = u(\tau), \tag{4}$$

The solution $\emptyset(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\emptyset(\tau;p)$ in Taylor series about the embedding parameter

$$\emptyset(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \emptyset(\tau;p)}{\partial p^m}\bigg|_{p=0} \tag{6}$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1, one has

$$\emptyset(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \dots\dots\dots, u_n(\tau)\}$$

If we differentiate the zeroth-order deformation equation i.e. Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p=0$, we obtain the following mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - X_m u_{m-1}(\tau)] = \hbar\Re_m(\vec{u}_{m-1}), \tag{8}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\widetilde{N}[\emptyset(\tau;p)]}{\partial p^{m-1}}\bigg|_{p=0} \tag{9}$$

and

$$X_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \tag{10}$$

If we multiply with $\mathcal{L}^{-1}$ each side of Eq. (8), we will obtain the following mth order deformation equation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar\,\Re_m\,(\vec{u}_{m-1})$$

3. SOLUTION PROCEDURE

In this section, we solve system of fractional order linear and nonlinear boundary value problems to illustrate the implementation of the homotopy analysis method.

Problem 1. Consider a fractional homogeneous 2-by-2 stiff system of linear ODEs:

$$x_1^{\alpha} = k(-1-\varepsilon)x_1 + k(1-\varepsilon)x_2, \qquad 0 < \alpha \le 1,$$

$$x_2^{\alpha} = k(1-\varepsilon)x_1 + k(-1-\varepsilon)x_2, \qquad 0 < \alpha \le 1,$$

where k, ε are constants, with the initial condition

$$x_1(0) = 1, \; x_2(0) = 3,$$

Now we apply the HAM to solve system of fractional order linear ODE. The solutions of $x_1(t)$ and $x_2(t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2,\dots\dots\dots..\},$$

In the following forms

$$x_1(t) = \sum_{n=o}^{+\infty} a_n t^n \,.\; x_2(t) = \sum_{n=o}^{+\infty} b_n t^n,$$

where a_n and b_n are coefficients. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$x_{1,0}(t) = 1\,, \quad x_{2,0}(t) = 3,$$

As the initial approximations of $x_1(t)$ and $x_2(t)$, to choose the auxiliary linear operator

$$L[\emptyset_i(t;q)] = \frac{\partial^{\alpha}\emptyset_i(t;q)}{\partial t^{\alpha}}, \qquad i = 1,2$$

with the property

$$L[C_i] = 0,$$

where C_i $(i = 1,2)$ are integral constants. Furthermore, we define a system of nonlinear operators as

$$N_1[\emptyset_i(t;q)] = \frac{\partial^{\alpha}\emptyset_1(t;q)}{\partial t^{\alpha}} - k(-1-\varepsilon)\emptyset_1(t;q) - k(1-\varepsilon)\emptyset_2(t;q),$$

$$N_2[\emptyset_i(t;q)] = \frac{\partial^{\alpha}\emptyset_2(t;q)}{\partial t^{\alpha}} - k(-1-\varepsilon)\emptyset_1(t;q) - k(1-\varepsilon)\emptyset_2(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}\left[\emptyset_i(t;q) - x_{i,0}(t)\right] = q\hbar N_i[\emptyset_i(t;q)], \;\; i = 1,2$$

Obviously, when q=0 and q=1,

$$\emptyset_i(t;0) = x_{i,0}(t), \quad \emptyset_i(t;1) = x_i(t)$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1,2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset_i(t;q) = x_{i,0}(t) + \sum_{m=1}^{+\infty} x_{i,m}(t)q^m,$$

where

$$x_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \emptyset_i(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{x}_{i,n} = \left\{x_{i,0}(t), x_{i,1}(t), \dots\dots\dots., x_{i,n}(t)\right\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!

We gain the mth order deformation equations

$$L\left[x_{i,m}(t) - \chi_m x_{i,m-1}(t)\right] = \hbar R_{i,m}\left(\bar{x}_{i,m-1}\right),$$

Subject to initial condition $x_{1,m}(0) = 0$, $x_{2,m}(0) = 0$,

$$R_{1,m}\left(\bar{x}_{i,m-1}\right) = x_{1,m-1}^{\alpha}(t) - k(-1-\varepsilon)x_{1,m-1}(t) - k(1-\varepsilon)x_{2,m-1}(t),$$

$$R_{2,m}\left(\bar{x}_{i,m-1}\right) = x_{2,m-1}^{\alpha}(t) - k(1-\varepsilon)x_{1,m-1}(t) - k(-1-\varepsilon)x_{2,m-1}(t),$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$x_{i,m}(t) = \chi_m x_{i,m-1}(t) + h_i j_t^{\alpha}\left[R_{i,m}\left(\bar{x}_{i,m-1}\right)\right].$$

We now successfully obtain

$$x_{1,1}(t) = \hbar(-2k + 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$x_{1,2}(t) = (-2\hbar k + 4\hbar k\varepsilon - 2\hbar^2 k + 4\hbar^2 k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (8\hbar^2 k^2\varepsilon^2 - 4\hbar^2 k^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$\vdots$

$$x_{2,1}(t) = \hbar(2k + 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$x_{2,2}(t) = (2\hbar k + 4\hbar k\varepsilon + 2\hbar^2 k + 4\hbar^2 k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (4\hbar^2 k^2 + 8\hbar^2 k^2\varepsilon^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

Then the solution expression can be written in the form

$$x_1(t) = \sum_{n=o}^{+\infty} a_n t^n\,, \quad x_2(t) = \sum_{n=o}^{+\infty} b_n t^n,$$

From this the first three terms of the series solution when $\hbar = -1$ are

$$x_{1,1}(t) = (2k - 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$x_{1,2}(t) = (8k^2\varepsilon^2 - 4k^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$\vdots$$
$$x_{2,1}(t) = (-2k - 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$x_{2,2}(t) = (4k^2 + 8k^2\varepsilon^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

Thus the analytic solution is

$$x_1(t) = \sum_{m=0}^{\infty} x_{1,m}(t),$$

$$x_1(t) = (2k - 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (8k^2\varepsilon^2 - 4k^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cdots,$$

and

$$x_2(t) = \sum_{m=0}^{\infty} x_{2,m}(t),$$

$$x_2(t) = (-2k - 4k\varepsilon)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (4k^2 + 8k^2\varepsilon^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

Problem 2. Consider the following fractional stiff system of non linear ODEs:

$$x_1^{\alpha} = -1002x_1 + 1000{x_2}^2,$$

$$x_2^{\alpha} = x_1 - x_2 - {x_2}^2, \qquad 0 < \alpha \le 1,$$

with the initial condition

$$x_1(0) = 1, \;\; x_2(0) = 1,$$

Now we apply the HAM to solve fractional system.

Initial approximations are,

$$x_{1,0}(t) = 1\,, \quad x_{2,0}(t) = 1,$$

the auxiliary linear operators are

$$L[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_i(t;q)}{\partial t^\alpha}, \qquad i = 1,2,$$

with the property,

$$L[C_i] = 0,$$

where C_i $(i = 1,2)$ are integral constants. Furthermore, we define a system of nonlinear operators as

$$N_1[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_1(t;q)}{\partial t^\alpha} + 1002\emptyset_1(t;q) - 1000{\emptyset_2}^2(t;q),$$

$$N_2[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_2(t;q)}{\partial t^\alpha} - \emptyset_1(t;q) + \emptyset_2(t;q) + {\emptyset_2}^2(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L\big[\emptyset_i(t;q) - x_{i,0}(t)\big] = q\hbar N_i[\emptyset_i(t;q)], \;\; i = 1,2,$$

Obviously, when q=0 and q=1,

$$\emptyset_i(t;0) = x_{i,0}(t), \quad \emptyset_i(t;1) = x_i(t),$$

Therefore as the embedding parameter q increases from 0 to 1, varies from the initial guess to the solution for $i = 1,2$. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has

$$\emptyset_i(t;q) = x_{i,0}(t) + \sum_{m=1}^{+\infty} x_{i,m}(t) q^m,$$

where

$$x_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \emptyset_i(t;q)}{\partial q^m}\Big|_{q=0}$$

define the vector

$$\bar{x}_{i,n} = \big\{x_{i,0}(t), x_{i,1}(t), \dots \dots \dots., x_{i,n}(t)\big\}.$$

Differentiating the zero-order deformation equation, m-times with respect to q, and finally dividing by m!

We gain the mth order deformation equations

$$L\left[x_{i,m}(t) - \chi_m x_{i,m-1}(t)\right] = \hbar R_{i,m}\left(\bar{x}_{i,m-1}\right), \quad i = 1,2$$

Subject to initial condition $x_{1,m}(0) = 0$, $x_{2,m}(0) = 0$,

$$R_{1,m}\left(\bar{x}_{i,m-1}\right) = x_{1,m-1}^{\alpha}(t) + 1002 x_{1,m-1}(t) - 1000 \sum_{j=0}^{m-1} x_{2,j}(t) x_{2,m-1-j}(t),$$

$$R_{2,m}\left(\bar{x}_{i,m-1}\right) = x_{2,m-1}^{\alpha}(t) - x_{1,m-1}(t) + x_{2,m-1}(t) + \sum_{j=0}^{m-1} x_{2,j}(t) x_{2,m-1-j}(t),$$

Now the solution of the mth-order deformation equation (12) for m ≥1 becomes

$$x_{i,m}(t) = \chi_m x_{i,m-1}(t) + \hbar j_t^{\alpha}\left[R_{i,m}\left(\bar{x}_{i,m-1}\right)\right] \qquad i = 1,2$$

We now successfully obtain

$$x_{1,1}(t) = 2\hbar \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$x_{1,2}(t) = (2\hbar + 2h^2)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\hbar^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

⋮

$$x_{2,1}(t) = \hbar \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$x_{2,2}(t) = (\hbar + \hbar^2)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \hbar^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

⋮

Then the solution expression can be written in the form

$$x_1(t) = \sum_{n=o}^{+\infty} a_n t^n, \quad x_2(t) = \sum_{n=o}^{+\infty} b_n t^n.$$

From this the first two terms of the series solution when $\hbar$ = -1 are

$$x_{1,1}(t) = -2\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$x_{1,2}(t) = 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

⋮

$$x_{2,1}(t) = -\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$x_{2,2}(t) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$\vdots$

Then the solution expression can be written in the form

$$x_1(t) = \sum_{m=0}^{\infty} x_{1,m}(t)$$
$$x_1(t) = -2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$
$$x_2(t) = \sum_{m=0}^{\infty} x_{2,m}(t),$$
$$x_2(t) = -\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

Problem 3. Consider the following fractional non linear matrix Ricatti differential equation

$$x^{\alpha} = -y^2 + Q, \qquad y(0) = 0, \qquad\qquad 0 < \alpha \le 1$$

where

$$Q = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 100\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix},$$

We can write fractional non linear matrix Ricatti differential equation in system of equations as:

$$x^{\alpha} = -x^2 - yz + \frac{101}{2},$$
$$y^{\alpha} = -xy - yw - \frac{99}{2},$$
$$z^{\alpha} = -xy - yw - \frac{99}{2},$$
$$w^{\alpha} = -x^2 - yz + \frac{101}{2},$$

Now we apply the HAM to solve the above equations.
Initial approximations are

$$x_0(t) = \frac{101t}{2}, \quad y_0(t) = -\frac{99t}{2}, \quad z_0(t) = -\frac{99t}{2}, \quad w_0(t) = \frac{101t}{2},$$

the auxiliary linear operators are

$$L[\emptyset_i(t;q)] = \frac{\partial^{\alpha}\emptyset_i(t;q)}{\partial t^{\alpha}}, \qquad i = 1,2,3,4,$$

with the property

$$L[C_i] = 0,$$

$$N_1[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_1(t;q)}{\partial t^\alpha} + \emptyset_1^2(t;q) + \emptyset_2(t;q)\,\emptyset_3(t;q) - \frac{101}{2},$$

$$N_2[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_2(t;q)}{\partial t^\alpha} + \emptyset_1(t;q)\emptyset_2(t;q) + \emptyset_2(t;q)\emptyset_4(t;q) + \frac{99}{2},$$

$$N_3[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_3(t;q)}{\partial t^\alpha} + \emptyset_1(t;q)\emptyset_2(t;q) + \emptyset_2(t;q)\emptyset_4(t;q) + \frac{99}{2},$$

$$N_4[\emptyset_i(t;q)] = \frac{\partial^\alpha \emptyset_4(t;q)}{\partial t^\alpha} + \emptyset_1^2(t;q) + \emptyset_2(t;q)\,\emptyset_3(t;q) - \frac{101}{2},$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)L\left[\emptyset_i(t;q) - x_{i,0}(t)\right] = q\hbar N_i[\emptyset_i(t;q)], \quad i = 1,2,$$

Obviously, when q=0 and q=1,

$$\emptyset_i(t;0) = x_{i,0}(t), \quad \emptyset_i(t;1) = x_{i,1}(t)$$

Therefore as the embedding parameter q increases from 0 to 1, $\emptyset_i(t;q)$ varies from the initial guess $x_{i,0}(t)$, to the solution $x_i(t)$ for i=1,2,3,4. Expanding $\emptyset_i(t;q)$ in Taylor series with respect to q, one has

$$\emptyset_i(t;q) = x_{i,0}(t) + \sum_{m=1}^{+\infty} x_{i,m}(t)q^m,$$

where

$$x_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \emptyset_i(t;q)}{\partial q^m}\Big|_{q=0}.$$

define the vector

$$\bar{x}_{i,n} = \left\{x_{i,0}(t), x_{i,1}(t), \dots\dots\dots, x_{i,n}(t)\right\}.$$

Differentiating the zero-order deformation equation, m times with respect to q, and finally dividing by m!

We gain the m-th order deformation equations

$$L\left[x_{i,m}(t) - \chi_m x_{i,m-1}(t)\right] = \hbar R_{i,m}\left(\bar{x}_{i,m-1}\right),$$

Subject to initial condition $x_m(0) = 0$, $y_m(0) = 0$, $z_m(0) = 0$, $w_m(0) = 0$,

$$R_m(\bar{x}_{m-1}) = x_{m-1}^{\alpha}(t) + \sum_{j=0}^{m-1} x_j(t)x_{m-1-j}(t) + \sum_{j=0}^{m-1} y_j(t)z_{m-1-j}(t) - \frac{101}{2},$$

$$R_m(\bar{y}_{m-1}) = y_{m-1}^{\alpha}(t) + \sum_{j=0}^{m-1} x_j(t)y_{m-1-j}(t) + \sum_{j=0}^{m-1} y_j(t)w_{m-1-j}(t) + \frac{99}{2},$$

$$R_m(\bar{z}_{m-1}) = z_{m-1}^{\alpha}(t) + \sum_{j=0}^{m-1} x_j(t)y_{m-1-j}(t) + \sum_{j=0}^{m-1} y_j(t)w_{m-1-j}(t) + \frac{99}{2},$$

$$R_m(\bar{w}_{m-1}) = w_{m-1}^{\alpha}(t) + \sum_{j=0}^{m-1} x_j(t)x_{m-1-j}(t) + \sum_{j=0}^{m-1} y_j(t)z_{m-1-j}(t) - \frac{101}{2},$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$x_m(t) = \chi_m x_{m-1}(t) + \hbar j_t^{\alpha}[R_m(\bar{x}_{m-1})],$$

We now successfully obtain

$$x_1(t) = \hbar\left(\frac{101t}{2} + \frac{10001t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{2\Gamma(\alpha+1)}\right),$$

$$x_2(t) = \hbar\left(\frac{101t}{2} + \frac{10001t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{\Gamma(\alpha+1)}\right) + \hbar^2\left(\frac{101t}{2} + \frac{30003t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{2\Gamma(\alpha+1)} + \frac{2000002\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} - \frac{10001\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right),$$

⋮

$$y_1(t) = \hbar\left(-\frac{99t}{2} - \frac{9999\Gamma(3)t^{\alpha+2}}{2\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)}\right),$$

$$y_2(t) = h\left(-\frac{99t}{2} - \frac{9999t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)}\right) + h^2\left(-\frac{99t}{2} - \frac{29997t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)} - \frac{1999998\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{9999\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right),$$

⋮

$$z_1(t) = \hbar\left(-\frac{99t}{2} - \frac{9999\Gamma(3)t^{\alpha+2}}{2\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)}\right),$$

$$z_2(t) = h\left(-\frac{99t}{2} - \frac{9999t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)}\right) + h^2\left(-\frac{99t}{2} - \frac{29997t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{99t^{\alpha}}{2\Gamma(\alpha+1)} - \frac{1999998\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{9999\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right),$$

$\vdots$

$$w_1(t) = \hbar\left(\frac{101t}{2} + \frac{10001\Gamma(3)t^{\alpha+2}}{2\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{2\Gamma(\alpha+1)}\right),$$

$$w_2(t) = \hbar\left(\frac{101t}{2} + \frac{10001t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{\Gamma(\alpha+1)}\right) + \hbar^2\left(\frac{101t}{2} + \frac{30003t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{101t^{\alpha}}{2\Gamma(\alpha+1)} + \frac{2000002\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} - \frac{10001\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right),$$

$\vdots$

Then the solution expression can be written in the closed form as

$$v_i(t) = \sum_{m=0}^{\infty} a_{i,m}(\hbar)t^{2m+1}, \qquad i = 1,2,3,4,$$

4. CONCLUSION

In this study, we used homotopy analysis method (HAM) to obtain analytical solution of system of fractional order differential equations. The applied algorithm is very beneficial to validate the results attained by the exact solution. The intimacy among the outcomes reveals that it is a powerful tool for solving system of fractional order DEs. Series solution results reveal the complete compatibility of proposed algorithm for such problems. It is also observed that the results obtained by this method are in complete agreement with other exiting results in literature. The attained results through under discussion procedure approves that this technique is very inspiring and trustworthy for handling different classes of non-linear EEs.

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