ORIGINAL PAPER

ON K-JACOBSTHAL AND K-JACOBSTHAL- LUCAS QUATERNIONS

DURSUN TASCI[1]

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Abstract: *In this paper, we generalize Jacobsthal and Jacobsthal- Lucas quaternions to k- Jacobsthal and k- Jacobsthal-Lucas quaternions. We give Binet- like formulas, generating functions and matrix representation of k- Jacobsthal quaternions and k- Jacobsthal-Lucas quaternions.*

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1. INTRODUCTION

A quaternion is defined by

$$q = a + i_1 b + i_2 c + i_3 d,$$

where a,b,c,and d are real numbers and i_1, i_2, i_3 are the standart orthonormal basis in $\mathbb{R}^3$. Let $q_1 = a_1 + i_1 b_1 + i_2 c_1 + i_3 d_1$ and $q_2 = a_2 + i_1 b_2 + i_2 c_2 + i_3 d_2$ be any two quaternions.

Then addition, equality and multiplication by scalar of two quaternions are defined by

$$q_1 + q_2 = (a_1 + a_2) + i_1(b_1 + b_2) + i_2(c_1 + c_2) + i_3(d_1 + d_2),$$

$$q_1 = q_2 \ \ only\ if \ \ a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$$

and the multiplication by scaler is defined by

$$\alpha q_1 = \alpha a_1 + i_1 \alpha b_1 + i_2 \alpha c_1 + i_3 \alpha d_1$$

for α ϵ IR. We note that the quaternion multiplication is defined using the rules

$${i_1}^2 = {i_2}^2 = {i_3}^2 = i_1 i_2 i_3 = -1.$$

For any positive real number k, the k-Jacobsthal sequence say $\{J_{k,n}\}_{n\in N}$ is defined recurrently by

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}\,; for\ n \geq 1 \qquad (1.1)$$

[1] Gazi University, Faculty of Sciences, Departments of Mathematics, 06500 Ankara, Turkey.
E-mail: dtasci@gazi.edu.tr.

with initial conditions $J_{k,0} = 0$, $J_{k,1} = 1$. The sequence of k-Jacobsthal-Lucas numbers $\{j_{k,n}\}_{n\in N}$ is defined by

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}\,; for\ n \geq 1 \tag{1.2}$$

where initial conditions are $j_{k,0} = 2, j_{k,1} = k$ for any positive real number k.

In Eq(1.1) and Eq(1.2), if k = 1, we get the sequence of Jacobsthal numbers and the sequence of Jacobsthal- Lucas numbers defined by Horadam[5,6], respectively. In [1], Cerda-Moralles introduced third order Jacobsthal quaternions. Also in [9], Tasci D. and Yalcin N.F. studied Fibonacci p-quaternions. Fibonacci quaternions, complex Fibonacci quaternions, Jacobsthal quaternions, Pell and Pell- Lucas quaternions can be found [2,3,4,7].

In this paper, we generalize Jacobsthal and Jacobsthal –Lucas quaternions[8] to k-Jacobsthal and k- Jacobsthal –Lucas quaternions, respectively. Moreover we give their properties also using matrix representations.

2. k- JACOBSTHAL QUATERNIONS

Firstly we give the definition of k-Jacobsthal quaternions.

Definition 2.1. The k- Jacobsthal quaternions are defined by

$$QJ_{k,n} = J_{k,n} + i_1 J_{k,n+1} + i_2 J_{k,n+2} + i_3 J_{k,n+3}$$

where $J_{k,n}$ is the *n*th k- Jacobsthal number.

We note that for $n \geq 0$

$$QJ_{k,n+2} = kQJ_{k,n+1} + 2QJ_{k,n} \tag{2.1}$$

can be written.

Theorem 2.2. The Binet-like formula for the k- Jacobsthal quaternions is

$$QJ_{k,n+2} = \frac{\propto r_1^n - \beta r_2^n}{r_1 - r_2} \tag{2.2}$$

where $\propto = 1 + i_1 r_1 + i_2 {r_1}^2 + i_3 {r_1}^3, \beta = 1 + i_1 r_2 + i_2 {r_2}^2 + i_3 {r_2}^3$ and r_1, r_2 are roots of the equation $r^2 - kr - 2 = 0$, i.e

$$r_1 = \frac{k+\sqrt{k^2+8}}{2}, r_2 = \frac{k-\sqrt{k^2+8}}{2}.$$

Proof: As well known the Binet formula of k- Jacobsthal sequence is

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

On the other hand using Definition 2.1, the proof is easily seen.

Theorem 2.3. The generating function of the k- Jacobsthal quaternions is

$$\sum_{n=0}^{\infty} QJ_{k,n}x^n = \frac{QJ_{k,0}+(QJ_{k,1}-kQJ_{k,0})x}{1-kx-2x^2}$$

where

$$QJ_{k,0} = i_1 + i_2 k + i_3(k^2+2)$$

and

$$QJ_{k,1} = 1 + i_1 k + i_2(k^2+2) + i_3(k^3+4k).$$

Proof: Let

$$g(x) = \sum_{n=0}^{\infty} QJ_{k,n}x^n = QJ_{k,0} + QJ_{k,1}x + QJ_{k,2}x^2 + \cdots + QJ_{k,n}x^n + \cdots$$

be generating function of the k- Jacobsthal quaternions. Since

$$kxg(x) = kQJ_{k,0}x + kQJ_{k,1}x^2 + kQJ_{k,2}x^3 + \cdots + kQJ_{k,n-1}x^n + \cdots$$

and

$$2x^2g(x) = 2QJ_{k,0}x^2 + 2QJ_{k,1}x^3 + 2QJ_{k,2}x^4 + \cdots + 2QJ_{k,n-2}x^n + \cdots$$

we have

$$(1-kx-2x^2)g(x) = QJ_{k,0} + (QJ_{k,1}-kQJ_{k,0})x + (QJ_{k,2}-kQJ_{k,1}-2QJ_{k,0})x^2 + \cdots + (QJ_{k,n}-kQJ_{k,n-1}-2QJ_{k,n-2})x^n + \cdots.$$

Using (2.1), we obtain

$$(1-kx-2x^2)g(x) = QJ_{k,0} + (QJ_{k,1}-kQJ_{k,0})x$$

or

$$g(x) = \frac{QJ_{k,0}+(QJ_{k,1}-kQJ_{k,0})x}{(1-kx-2x^2)}.$$

So the proof is complete.

Theorem 2.4.(Catalan Identity)

$$QJ_{k,n-r}QJ_{k,n+r} - {QJ_{k,n}}^2 = -(\propto\beta)(-2)^{n-r}{J_{k,r}}^2 \qquad (2.3)$$

where $n, r \in Z^+, n > r$ and $J_{k,r}$ is the *r* th k- Jacobsthal number and

$$\propto = 1 + i_1 r_1 + i_2 {r_1}^2 + i_3 {r_1}^3,$$

$$\beta = 1 + i_1 r_2 + i_2 {r_2}^2 + i_3 {r_2}^3.$$

Proof: Using the Binet- like formula for the k- Jacobsthal formula and taking into account that $r_1 r_2 = -2$, we write

$$QJ_{k,n-r}QJ_{k,n+r} - {QJ_{k,n}}^2 = \left(\frac{\propto r_1^{n-r}-\beta r_2^{n-r}}{r_1-r_2}\right)\left(\frac{\propto r_1^{n+r}-\beta r_2^{n+r}}{r_1-r_2}\right) - \left(\frac{\propto r_1^{n}-\beta r_2^{n}}{r_1-r_2}\right)^2$$
$$= \frac{1}{(r_1-r_2)^2}\left[-(\propto\beta)(-2)^n\left(\frac{r_1^{2r}+r_2^{2r}}{(r_1r_2)^r}-2\right)\right]$$

$$= \frac{1}{(r_1 - r_2)^2}\left[-(\propto \beta)(-2)^n \left(\frac{r_1^r - r_2^r}{(-2)^r}\right)^2\right]$$
$$= -(\propto \beta)(-2)^{n-r}\left(\frac{r_1^r - r_2^r}{r_1 - r_2}\right)^2$$
$$= -(\propto \beta)(-2)^{n-r} {J_{k,r}}^2.$$

So the theorem is proved.

Corollary 2.5 (Cassini's Identity)

$$QJ_{k,n-1}QJ_{k,n+1} - {QJ_{k,n}}^2 = -(\propto \beta)(-2)^{n-1}$$

Proof: Note that for r = 1, equality (2.3) gives Cassini's identity. Moreover we remark that ${J_{k,1}}^2 = 1$.

Theorem 2.6 (D'ocagne's Identity) If $m > n$, then

$$QJ_{k,m}QJ_{k,n+1} - QJ_{k,m+1}QJ_{k,n} = -(\propto \beta)(-2)^n J_{k,m-n},$$

where $\propto = 1 + i_1 r_1 + i_2 {r_1}^2 + i_3 {r_1}^3, \beta = 1 + i_1 r_2 + i_2 {r_2}^2 + i_3 {r_2}^3$.

Proof: Using Binet- like formula and $r_1 r_2 = -2$,

$$QJ_{k,m}QJ_{k,n+1} - QJ_{k,m+1}QJ_{k,n} = \left(\frac{\propto r_1^m - \beta r_2^m}{r_1 - r_2}\right)\left(\frac{\propto r_1^{n+1} - \beta r_2^{n+1}}{r_1 - r_2}\right)$$
$$-\left(\frac{\propto r_1^{m+1} - \beta r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{\propto r_1^n - \beta r_2^n}{r_1 - r_2}\right)$$
$$= (\propto \beta)\left(\frac{(r_1 r_2)^n}{(r_1 - r_2)^2}\right) \text{x}$$
$$(-{r_1}^{m-n} r_2 - r_1 {r_2}^{m-n} + {r_1}^{m-n} r_1 - r_2 {r_2}^{m-n})$$
$$= (\propto \beta)\frac{(-2)^n}{(r_1 - r_2)^2}[{r_1}^{m-n}(r_1 - r_2) + {r_2}^{m-n}(r_1 - r_2)]$$
$$= (\propto \beta)\frac{(-2)^n}{(r_1 - r_2)^2}(r_1 - r_2)[{r_1}^{m-n} - {r_2}^{m-n}]$$
$$= (\propto \beta)(-2)^n\left(\frac{{r_1}^{m-n} - {r_2}^{m-n}}{(r_1 - r_2)}\right)$$
$$= (\propto \beta)(-2)^n J_{k,m-n}.$$

Theorem 2.7 For the integer $n \geq 1$

$$\begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QJ_{k,2} & QJ_{k,1} \\ QJ_{k,1} & QJ_{k,0} \end{bmatrix} = \begin{bmatrix} QJ_{k,n+2} & QJ_{k,n+1} \\ QJ_{k,n+1} & QJ_{k,n} \end{bmatrix}. \quad (2.4)$$

Proof: (By induction on n) If $n = 1$ then the result is obvious. We assume that it is true n-1 i.e.,

$$\begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} QJ_{k,2} & QJ_{k,1} \\ QJ_{k,1} & QJ_{k,0} \end{bmatrix} = \begin{bmatrix} QJ_{k,n+1} & QJ_{k,n} \\ QJ_{k,n} & QJ_{k,n-1} \end{bmatrix}.$$

By simple calculation using induction's hypothesis we write

$$\begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QJ_{k,2} & QJ_{k,1} \\ QJ_{k,1} & QJ_{k,0} \end{bmatrix} = \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} QJ_{k,2} & QJ_{k,1} \\ QJ_{k,1} & QJ_{k,0} \end{bmatrix}$$
$$= \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} QJ_{k,n+1} & QJ_{k,n} \\ QJ_{k,n} & QJ_{k,n-1} \end{bmatrix}$$
$$= \begin{bmatrix} kQJ_{k,n+1} + 2QJ_{k,n} & kQJ_{k,n} + 2QJ_{k,n-1} \\ QJ_{k,n+1} & QJ_{k,n} \end{bmatrix}$$
$$= \begin{bmatrix} QJ_{k,n+2} & QJ_{k,n+1} \\ QJ_{k,n+1} & QJ_{k,n} \end{bmatrix}$$

which ends the proof.

Corollary 2.8.

$$QJ_{k,n+2}QJ_{k,n} - {QJ_{k,n+1}}^2 = (-2)^n\left(QJ_{k,2}QJ_{k,0} - {QJ_{k,1}}^2\right).$$

Proof: If we get the determinant of equality (2.4) then the proof is easily seen.

3. k- JACOBSTHAL-LUCAS QUATERNIONS

Definition 3.1. The k- Jacobsthal-Lucas quaternions are defined by

$$Qj_{k,n} = j_{k,n} + i_1 j_{k,n+1} + i_2 j_{k,n+2} + i_3 j_{k,n+3}$$

where $j_{k,n}$ is the *n* th k- Jacobsthal-Lucas number.

We give Binet-like formula of k- Jacobsthal- Lucas quaternions.

Theorem 3.2.(Binet-like formula for k- Jacobsthal- Lucas quaternions)

$$Qj_{k,n} = \propto r_1^n + \beta r_2^n \tag{3.1}$$

where $r_1 = \frac{k+\sqrt{k^2+8}}{2}, r_2 = \frac{k-\sqrt{k^2+8}}{2}$ are roots of the equation x^2-kx-2 $= 0$, and

$$\propto = 1 + i_1 r_1 + i_2 {r_1}^2 + i_3 {r_1}^3, \beta = 1 + i_1 r_2 + i_2 {r_2}^2 + i_3 {r_2}^3.$$

Proof: We know that the Binet- like formula of k- Jacobsthal-Lucas numbers

$$j_{k,n} = r_1^n + r_2^n$$

where $r_1 = \frac{k+\sqrt{k^2+8}}{2}, r_2 = \frac{k-\sqrt{k^2+8}}{2}$.

On the other hand using the equality

$$Qj_{k,n} = j_{k,n} + i_1 j_{k,n+1} + i_2 j_{k,n+2} + i_3 j_{k,n+3}$$

we obtain

$$Qj_{k,n} = \propto r_1^n + \beta r_2^n.$$

Theorem 3.3 The generating function of k- Jacobsthal-Lucas quaternions is

$$\sum_{n=0}^{\infty} Qj_{k,n}x^n = \frac{Qj_{k,0} + (Qj_{k,1} - kQj_{k,0})x}{1 - kx - 2x^2},$$

where

$$Qj_{k,0} = 2 + i_1 k + i_2(k^2 + 4) + i_3(k^3 + 6k)$$

and

$$Qj_{k,1} = k + i_1(k^2 + 4) + i_2(k^3 + 6k) + i_3(k^4 + 8k^2 + 8).$$

Proof: Let

$$f(x) = \sum_{n=0}^{\infty} Qj_{k,n}x^n$$

be generating function of k-Jacobsthal –Lucas quaternions. Then we have

$$kxf(x) = kQj_{k,0}x + kQj_{k,1}x^2 + kQj_{k,2}x^3 + \cdots + kQj_{k,n-1}x^n + \cdots$$

and

$$2x^2 f(x) = 2Qj_{k,0}x^2 + 2Qj_{k,1}x^3 + 2Qj_{k,2}x^4 + \cdots + 2Qj_{k,n-2}x^n + \cdots$$

So we obtain

$$(1 - kx - 2x^2)f(x) = Qj_{k,0} + (Qj_{k,1} - kQj_{k,0})x + (Qj_{k,2} - kQj_{k,1} - 2Qj_{k,0})x^2 + \cdots + (Qj_{k,n} - kQj_{k,n-1} - 2Qj_{k,n-2})x^n + \cdots$$

We note that k-Jacobsthal – Lucas quaternions can be written as

$$QJ_{k,n} = kQj_{k,n-1} + 2Qj_{k,n-2}$$

Thus we have

$$f(x) = \frac{Qj_{k,0} + (Qj_{k,1} - kQj_{k,0})x}{1 - kx - 2x^2}.$$

Theorem 3.4 (D'ocagne's identity for k-Jacobsthal –Lucas quaternions) For $m > n$

$$Qj_{k,m}Qj_{k,n+1} - Qj_{k,m+1}Qj_{k,n} = (\propto \beta)(-2)^n\sqrt{k^2+8}\left[j_{k,m-n} - 2^{n-m+1}(k + \sqrt{k^2+8})^{m-n}\right].$$

Proof. For $m > n$ using Binet- like formula and $r_1 r_2 = -2$, we have

$$\begin{aligned}
Qj_{k,m}Qj_{k,n+1} - Qj_{k,m+1}Qj_{k,n} &= (\propto r_1^m - \beta r_2^m)(\propto r_1^{n+1} + \beta r_2^{n+1}) \\
&\quad -(\propto r_1^{m+1} + \beta r_2^{m+1})(\propto r_1^n + \beta r_2^n) \\
&= (\propto \beta)(r_1 r_2)^n({r_1}^{m-n}r_2 + r_1{r_2}^{m-n} - {r_1}^{m-n}r_1 - r_2{r_2}^{m-n}) \\
&= (-2)^n(\propto \beta)[{r_1}^{m-n}(r_2 - r_1) + {r_2}^{m-n}(r_1 - r_2)] \\
&= (-2)^n(\propto \beta)(r_1 - r_2)[{r_2}^{m-n} - {r_1}^{m-n}] \\
&= (-2)^n(\propto \beta)\sqrt{k^2+8}({r_1}^{m-n} - {r_2}^{m-n} - 2{r_1}^{m-n}) \\
&= (-2)^n(\propto \beta)\sqrt{k^2+8}\left(j_{k,m-n} - 2{r_1}^{m-n}\right)
\end{aligned}$$

$$= (-2)^n(\propto \beta)\sqrt{k^2+8} \quad \left[j_{k,m-n} - 2\,\frac{k+\sqrt{k^2+8}}{2}\right]^{m-n}$$
$$= (-2)^n(\propto \beta)\sqrt{k^2+8}\,\left[\,j_{k,m-n} - 2^{m-n+1}\left(k+\sqrt{k^2+8}\right)\right]^{m-n}$$

as desired.

Theorem 3.5(Catalan's identity for k- Jacobsthal- Lucas quaternions)

$$Qj_{k,n-r}Qj_{k,n+r} - Qj_{k,n}{}^2 = -(\propto \beta)(-2)^{n-r}\left[j_{k,r}{}^2 - (-2)^{r+2}\right] \qquad (3.2)$$

where $n, r \in N, n > r$ and where $j_{k,r}$ denotes r th k- Jacobsthal-Lucas number and

$$\propto = 1 + i_1 r_1 + i_2 r_1{}^2 + i_3 r_1{}^3,$$

$$\beta = 1 + i_1 r_2 + i_2 r_2{}^2 + i_3 r_2{}^3.$$

Proof: Again considering the Binet- like formula for the k- Jacobsthal-Lucas quaternions, we have

$$\begin{aligned}Qj_{k,n-r}Qj_{k,n+r} - Qj_{k,n}{}^2 &= (\propto r_1^{n-r} + \beta r_2^{n-r})(\propto r_1^{n+r} + \beta r_2^{n+r}) - (\propto r_1^n + \beta r_2^n)^2\\
&= (\propto \beta)(r_1 r_2)^r\left(\frac{r_1^r}{r_2{}^r} + \frac{r_2^r}{r_1{}^r} - 2\right)\\
&= (\propto \beta)(-2)^n\left(\frac{r_1^{2r} + r_2^{2r} - 2r_1{}^r r_2^r}{(r_1 r_2)^r}\right)\\
&= (\propto \beta)(-2)^n\left(\frac{r_1^{2r} + r_2^{2r} - 2(r_1 r_2)^{\,r}}{(-2)^r}\right)\\
&= -(\propto \beta)(-2)^{n-r}(r_1^{2r} + r_2^{2r} - 2(r_1 r_2)^{\,r})\\
&= -(\propto \beta)(-2)^{n-r}((r_1^r + r_2^r)^2 - 4(r_1 r_2)^{\,r})\\
&= -(\propto \beta)(-2)^{n-r}(j_{k,r}{}^2 - 4(-2)^{\,r})\\
&= -(\propto \beta)(-2)^{n-r}(j_{k,r}{}^2 - (-2)^{\,r+2})\end{aligned}$$

as required.

Corollary 3.6 (Cassini's Identity)

$$Qj_{k,n-1}Qj_{k,n+1} - Qj_{k,n}^2 = (\propto \beta)(-2)^{n-1}(k^2+8)$$

Proof: Substituting r = 1 in Catalan's identity the prof is easily seen.

Now we give the matrix representation of k-Jacobsthal-Lucas quaternions.

Theorem 3.7. Let $n \geq 1$the integer. Then

$$\begin{bmatrix} k & 2\\ 1 & 0\end{bmatrix}^n \begin{bmatrix} Qj_{k,2} & Qj_{k,1}\\ Qj_{k,1} & Qj_{k,0}\end{bmatrix} = \begin{bmatrix} Qj_{k,n+2} & Qj_{k,n+1}\\ Qj_{k,n+1} & Qj_{k,n}\end{bmatrix}. \qquad (3.3)$$

Proof: We use induction on n. If $n = 1$ then the result is obviously true. Assume that it is true n-1 i.e.,

$$\begin{bmatrix} k & 2\\ 1 & 0\end{bmatrix}^{n-1} \begin{bmatrix} Qj_{k,2} & Qj_{k,1}\\ Qj_{k,1} & Qj_{k,0}\end{bmatrix} = \begin{bmatrix} Qj_{k,n+1} & Qj_{k,n}\\ Qj_{k,n} & Qj_{k,n-1}\end{bmatrix}.$$

We shall show that it is true for n.

$$\begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} Qj_{k,2} & Qj_{k,1} \\ Qj_{k,1} & Qj_{k,0} \end{bmatrix} = \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Qj_{k,n+1} & Qj_{k,n} \\ Qj_{k,n} & Qj_{k,n-1} \end{bmatrix}$$
$$= \begin{bmatrix} kQj_{k,n+1} + 2Qj_{k,n} & kQj_{k,n} + 2Qj_{k,n-1} \\ Qj_{k,n+1} & Qj_{k,n} \end{bmatrix}$$
$$= \begin{bmatrix} Qj_{k,n+2} & Qj_{k,n+1} \\ Qj_{k,n+1} & Qj_{k,n} \end{bmatrix}$$

which ends the proof.

Corollary 3.8.

$$Qj_{k,n+2}Qj_{k,n} - {Qj_{k,n+1}}^2 = (-2)^n\left(Qj_{k,2}Qj_{k,0} - {Qj_{k,1}}^2\right),$$

where

$$Qj_{k,0} = 2 + i_1k + i_2(k^2 + 4) + i_3(k^3 + 6k)$$
$$Qj_{k,1} = k + i_1(k^2 + 4) + i_2(k^3 + 6k) + i_3(k^4 + 8k^2 + 8).$$
$$Qj_{k,2} = (k^2 + 4) + i_1(k^3 + 6k) + i_2(k^4 + 8k^2 + 8) + i_3(k^5 + 10k^3 + 20k)$$

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