ORIGINAL PAPER

ANALYTICAL TREATMENT OF SINGULAR INTEGRAL EQUATIONS VIA LAPLACE DECOMPOSITION METHOD

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Abstract. *Many mathematical models are contributed to give rise to of linear and nonlinear integral equations. This paper deals with the application of coupling of Laplace transform with decomposition method to solve analytically various nonlinear singular integral equations that described heat transfer. The elegant coupling gives rise to the modified versions of decomposition method which is very efficient in solving nonlinear problems of diversified nature of handling the Abel-type singular equation. It is observed that the decomposition method using Laplace transform is very efficient, easier to implements and more users friendly. Several examples are given to reconfirm the efficiency of the proposed method.*

Keywords: *Singular integral equation; Laplace transform; Adomian decomposition method.*

1. INTRODUCTION

Integral equation plays a vital role with in many disciplines of sciences, engineering and mathematics. The mathematical formulation of solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology often involve singular integral and integro-differential equations. Using of integral equations with exact parameter within many modeling physical problems is not quite easy or better to say impossible in real problems.

Volterra examined the nonlinear Volterra integral equation of the form [1-4]

$$u(x) = f(x) + \lambda \int_0^x K(x,t) F(u(t)) dx, \tag{1}$$

Where $F(u(x))$ is a nonlinear function of the solution $u(x), K(x,t)$ and is the kernel of the integral equation. The Volterra integral equations appear in many physical applications such as neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating. It well knows that the nonlinear Volterra integral equation (1) is usually handled by series solution methods, such as the Taylor series method and the successive approximations method. The traditional Laplace transform method by itself cannot be used in this case because of the nonlinearity of this equation. However, it is possible to overcome this difficulty by combining the Laplace transform with the powerful Adomian decomposition method. In the three decades, many powerful and simple methods have been proposed and applied successfully to approximate various type of singular integral and

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integro-differential equations with a wide range of applications. ADM was first proposed by G. Adomian [5]; unlike classical techniques, nonlinear equations are solved easily and more accurately via ADM. Laplace Adomian's Decomposition Method (LADM) were first introduced by Suheil A. Khuri [6-7], and have been successfully used to find the solution of differential equations [8-13]. This Method has been applied successfully to find the exact solution of the Bratu and Duffing equation in [9]. The Significant advantage of this method is its capability of combining the two powerful methods to obtain exact solution for non-linear phenomena. It is the aim of this work to develop a combined form of the Laplace transform method with the Adomian decomposition method to establish exact solutions or approximate solutions of high degree of accuracy for the nonlinear singular integral equations.

2. THE LAPLACE DECOMPOSITION METHOD

In this work, we will assume that the kernel $K(x,t)$ of (1) is a difference kernel expressed by a difference $x-t$. The nonlinear volterra integral Eq. (1) can be expressed as [14-15]

$$u(x) = f(x) + \lambda \int_0^x K(x-t)F(u(t))dx, \tag{2}$$

Applying Laplace transform to both sides of (2) gives

$$U(s) = \mathcal{L}\{f(x)\} + \lambda\mathcal{L}\{K(x-t)\}\mathcal{L}\{F\big(u(t)\big)\}, \tag{3}$$

The Adomian polynomials can be used to handle Eq. (3) and to address the nonlinear term $F\big(u(x)\big)$. We first represent the nonlinear term $U(s)$ at the left hand side by an infinite series of components given by

$$U(s) = \sum_{n=0}^{\infty} U_n(s) \tag{4}$$

$U_n(s)\,, n \geq 0$ will be obtained recursively, and the nonlinear term $F(u(x))$ at the right hand side will be represented by an infinite series of Adomian polynomials as

$$F\big(u(x)\big) = \sum_{n=0}^{\infty} A_n(x) \tag{5}$$

Where $A_n(x)\,, n \geq 0$ and defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right] \quad n = 0,1,2\ ... \tag{6}$$

Putting (4) and (5) into (3) leads to

$$\sum_{n=0}^{\infty} U_n(s) = \mathcal{L}\{f(x)\} + \gamma\mathcal{L}\{K(x-t)\}\mathcal{L}\{\sum_{n=0}^{\infty} A_n(x)\} \tag{7}$$

The recursive relation according to Adomian decomposition method is

$$U_0(s) = \mathcal{L}\{f(x)\},$$

$$U_{r+1}(s) = \gamma\mathcal{L}\{K(x-t)\}\mathcal{L}\{A_r(x)\}, r \geq 0. \tag{8}$$

After using inverse Laplace transform to first part of (8) obtains $u_0(x)$, that will define A_0, within $A_0(x)$ we will evaluate $u_1(x)$. The finding of $u_0(x)$ and $u_1(x)$, we will evaluate $A_1(x)$ that will allow us to determine $u_2(x)$ and so on. Then series solution using (4) may converge to exact if solution exists. Otherwise, the series solution can be used for numerical purposes.

3. NUMERICAL EXAMPLES

Example 3.1 Consider the nonlinear singular volterra integral equation [16]

$$u(x) = x - \frac{16}{15}x^{\frac{5}{2}} + \int_0^x \frac{u^2(t)}{\sqrt{x-t}}dt \tag{9}$$

By taking the Laplace transform of both sides, we obtain

$$U(S) = \frac{2}{s^2} + \frac{1}{s^{\frac{7}{2}}}\sqrt{\pi} + \mathcal{L}\left\{x^{-\frac{1}{2}}\right\}\mathcal{L}\{u^2(x)\} \tag{10}$$

The Adomian decomposition method admits the use of

$$U_0(S) = \frac{2}{s^2} + \frac{1}{s^{\frac{7}{2}}}\sqrt{\pi}$$

$$`U_{k+1}(S) = \mathcal{L}\left\{x^{-\frac{1}{2}}\right\}\mathcal{L}\{A_k(x)\} \tag{11}$$

Where $A_k(x)$ are the Adomian polynomials for the nonlinear term $u^2(x)$. The Adomian method assumes that the linear term $u(x)$ be decomposed by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{12}$$

And the nonlinear term $u^2(x)$ by the series

$$u^2(x) = \sum_{n=0}^{\infty} A_n(x) \tag{13}$$

In what follows we list some of the Adomian polynomials for $u^2(x)$

$$\begin{aligned} A_0 &= u_0^2 \\ A_1 &= 2u_0u_1 \\ A_2 &= 2u_0u_2 + u_1^2 \\ A_3 &= 2u_0u_3 + 2u_1u_2 \end{aligned}$$

And so on for other polynomials. Using the recurrence relation (11) And using the inverse Laplace transform of U_0, we find

$$u_0(x) = x - \frac{16}{15}x^{\frac{5}{2}} \tag{14}$$

Using this result in U_1 of (11) and using the inverse Laplace transform we find $u_1(x)$,

$$u_1(x) = \frac{16}{15}x^{\frac{5}{2}} - \frac{7}{12}\pi x^4 + \frac{131072}{155925}x^{\frac{11}{2}} \tag{15}$$

The noise terms $\pm\frac{16}{15}x^{\frac{5}{2}}$ appear between $u_0(x)$ and $u_1(x)$. By cancelling the noise terms, the exact solution is therefore given by

$$u(x) = x. \tag{16}$$

Example 3.2 Consider the nonlinear singular volterra integral equation [16]

$$u(x) = 1 - x - \frac{2}{15}x^{\frac{1}{2}}(8x^2 - 20x + 15) + \int_0^x \frac{u^2(t)}{\sqrt{x-t}}dx \tag{17}$$

By taking the Laplace transform of both sides we obtain

$$U(s) = \frac{1}{s} - \frac{1}{s^2} - 2\sqrt{\pi}\left(\frac{1}{s^{\frac{7}{2}}} - \frac{1}{s^{\frac{5}{2}}} + \frac{1}{2s^{\frac{3}{2}}}\right) + \mathcal{L}\{x^{\frac{-1}{2}}\}\mathcal{L}\{u^2(x)\} \tag{18}$$

The Adomian decomposition method admits the use of

$$U_0(S) = \frac{1}{s} - \frac{1}{s^2},$$
$$U_1(S) = -2\sqrt{\pi}\left(\frac{1}{s^{\frac{7}{2}}} - \frac{1}{s^{\frac{5}{2}}} + \frac{1}{2s^{\frac{3}{2}}}\right) + \mathcal{L}\{x^{\frac{-1}{2}}\}\mathcal{L}\{A_0(x)\},$$
$$U_{k+1}(S) = \mathcal{L}\{x^{\frac{-1}{2}}\}\mathcal{L}\{A_k(x)\}, n \geq 1. \tag{19}$$

Where $A_k(x)$ are the polynomials for the nonlinear term $u^2(x)$. The Adomian method assumes that the linear term $u(x)$ be decomposed by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{20}$$

And the nonlinear term $u^2(x)$ by the series

$$u^2(x) = \sum_{n=0}^{\infty} A_n(x) \tag{21}$$

In what follows we list some of the Adomian polynomials for $u^2(x)$

$$A_0 = u_0^2$$
$$A_1 = 2u_0u_1$$
$$A_2 = 2u_0u_2 + u_1^2$$

And so on for other polynomials. Using the recurrence relation (19) and using the inverse Laplace transform of U_0, consequently we have

$$u_0(x) = 1 - x, \tag{22}$$
$$u_1(x)=0,$$

The exact solution is therefore given by

$$u(x) = 1 - x. \tag{23}$$

Example 3.3 Singular Integral equation of Heat Radiation in a Semi-infinite Solid [11, 16]

Consider the Abel-type nonlinear singular integral equation given by

$$u(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t) - u^n(t)}{\sqrt{x-t}} dt. \tag{24}$$

where $u(x)$ gives the temperature at the surface for all time. The physical problem which motivated consideration of (24) is that of determining the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation. At the surface, energy is supplied according to the given function $f(t)$, while radiated energy escapes in proportion to $u^n(t)$.

Equation (28) can be rewritten as

$$u(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt - \frac{1}{\sqrt{\pi}} \int_0^x \frac{u^n(t)}{\sqrt{x-t}} dt \tag{25}$$

We select $f(x) = \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}}$ and $n = 2$. Based on this selection Equation (25) takes the form

$$u(x) = x^2 - \frac{1}{\sqrt{\pi}} \int_0^x \frac{u^2(t)}{\sqrt{x-t}} dt. \tag{26}$$

By taking the Laplace transform of both sides we obtain

$$U(s) = \frac{2}{s^3} - \frac{1}{\sqrt{\pi}} \mathcal{L}\{x^{\frac{-1}{2}}\}\mathcal{L}\{u^2(x)\}. \tag{27}$$

The recursive relation is given by

$$U_0(S) = \frac{2}{s^3},$$

$$U_{k+1}(S) = -\frac{1}{\sqrt{\pi}} \mathcal{L}\{x^{\frac{-1}{2}}\}\mathcal{L}\{A_k(x)\}, n \geq 0. \tag{28}$$

Where $A_k(x)$ are the polynomials for the nonlinear term $u^2(x)$.

According to described procedure above, consequently we have

$u_0(x) = x^2,$

$u_1(x) = -0.458515979x^{\frac{9}{2}},$

$u_2(x) = 0.3404761905x^7,$

$$u_3(x) = -0.2853621586x^{\frac{19}{2}},$$

$$u_4(x) = 0.2522457226x^{12},$$

and so on. The series solution is given by

$$u(x) = x^2 - 0.458515979x^{\frac{9}{2}} + 0.3404761905x^7 - 0.2853621586x^{\frac{19}{2}} + 0.2522457226x^{12} \ldots .$$

4. CONCLUSION

In this paper Laplace decomposition method is effectively used to handle nonlinear Abel-type integral equations that describe the temperature distribution of the surface of a projectile moving through a laminar layer. The proposed method presents a useful way to develop an analytic treatment for such kinds of nonlinear singular integral equations. The proposed scheme can be applied for other nonlinear equations of physics applications.

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