ORIGINAL PAPER

ENERGY OF DIRECTED GRAPHS

SERIFE BUYUKKOSE[1], ANIL ALTINKAYA[1]

Manuscript received: 14.07.2017; Accepted paper: 06.08.2017;
Published online: 30.12.2017.

Abstract. *In this study, we defined graph energy of a directed graph and we obtained a new formula using cycle and chains for directed graphs.*
Keywords: *Directed graph, Graph energy, Eigenvalue, Chain, Cycle.*

1. INTRODUCTION

A graph G consists of a set of objects $V = \{v_1, v_2, v_3, \dots\}$ called vertices (also called points or nodes) and other set $E = \{e_1, e_2, e_3, \dots\}$ whose elements are called edges (also called lines or arcs) [3].

The set $V(G)$ is called the vertex set of G and $E(G)$ is the edge set. Usually the graph is denoted as $G = (V, E)$ [3].

Let G be a graph and $\{u, v\}$ an edge of G. Since $\{u, v\}$ is 2-element set, we may write $\{v, u\}$ instead of $\{u, v\}$. It's often more convenient to represent this edge by uv or vu [3].

If $e = uv$ is and edge of a graph G, then we say that u and v are adjacent in G and that e joins u and v [3].

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called directed graph. The directed graph with n-vertices is denoted by Γ_n [4].

A chain in Γ_n is an ordered list of distinct vertices $C = \{c_1, c_2, \dots, c_r\}$ such that $f(c_j) = c_{j+1}$for $1 \le j < r$ but $f(c_r) \ne c_1$. A cycle in Γ_n is an ordered list of distinct vertices $Z = \{z_1, z_2, \dots, z_r\}$ such that $f(z_j) = z_{j+1}$ for $1 \le j < r$ and $f(z_r) = z_1$. In either case, we call r the length of the chain or cycle and write $r = len\ C$ or $r = len\ Z$ [5].

The graph is called mixed graph if it's contain cycle, chain or loop.

If G is a graph on n-vertices and $\lambda_1, \lambda_2, \dots, \lambda_n$ are its eigenvalues, then the energy of G is $\varepsilon = \varepsilon(G) = \sum_{j=1}^{n} |\lambda_j|$ [1].

Let v be a generalized eigenvector of A for the eigenvalue λ and let p be the smallest positive integer such that $(A - \lambda I)^p v = 0$. Then the ordered set

$$\{(A - \lambda I)^{p-1} v, (A - \lambda I)^{p-2} v, \dots, (A - \lambda I) v, v\} \tag{1}$$

is a chain of generalized eigenvectors of A corresponding to λ. Observe that the first elements of the list, $(A - \lambda I)^{p-1} v$, is an ordinary eigenvector.

[1] Gazi University, Faculty of Sciences, Department of Mathematics, 06500 Ankara, Turkey.
E-mail: sbuyukkose@gazi.edu.tr ; anilaltinkaya@gazi.edu.tr ;

In the literature, many authors refer to the list of generalized eigenvectors in (1) as a cycle of generalized eigenvectors. In the context of this paper, it is better to call it a chain [5].

A partition of Γ_n is a collection of disjoint cycles and chains whose union is Γ_n. A proper partition of Γ_n is a partition

$$P = \{Z_1, \ldots, Z_r, C_1, \ldots, C_s\},$$

where $Z_1, \ldots, Z_r$ are cycles and $C_1, \ldots, C_s$ are chains satisfying the following properties:

1. Each cycle in Γ_n is equal to Z_i for some i.
2. If $\Gamma_n{}^{(i)}$ is the subgraph of Γ_n obtained by deleting the vertices in the cycles $Z_1, \ldots, Z_r$ and in the chains $C_1, \ldots, C_i$, then C_{i+1} is a chain of maximal length in $\Gamma_n{}^{(i)}$ [5].

In this paper we give one equality for mixed directed graph energy using chain and cycle.

2. ENERGY OF DIRECTED GRAPHS

In this section, firstly ,we give some lemmas. Secondly, we found the formula for the energy of directed graphs with include cycle and chain.

Lemma 2.1 [5] Let $C = \{c_1, c_2, \ldots, c_m\}$ be any chain in a proper partition of Γ_n. Then exactly one of the following occurs:

1. The terminal point c_m of the chain is a terminal point of the graph Γ_n.
2. The point $f(c_m)$ is a merge point of Γ_n.

Furthermore, if $f(c_m)$ is a merge point and $f(c_m)$ belongs to another chain $C' = \{c_1', c_2', \ldots, c_m'\}$ then $f(c_m) = f(c_k')$ where $k \geq m$. Consequently, if $f(c_m)$ is a merge point belonging to either a cycle or a chain, then there is a unique vertex z in the cycle or chain containing $f(c_r)$ such that $f^m(z) = f^m(c_1) = f(c_m)$ where f^m is the composition of f with itself m times.

Lemma 2.2 [5] Every eigenvalue of A_n is either 0 or a root of unity.

Lemma 2.3 [5] Let $Z = \{z_1, z_2, \ldots, z_m\}$ be any cycle in the partition P, and let ω = exp(2πi/l) be a primitive lth root of unity. Then the vector

$$v_k = \sum_{j=1}^{l} w^{-kj} e_{z_j} \ldots \tag{2}$$

is an eigenvector of A_n for the eigenvalue ωk. Furthermore,

$$\text{span}\{v_1, v_2, \ldots, v_l\} = \text{span}\{e_{z_1}, \ldots, e_{z_l}\} \ldots \tag{3}$$

We will say that the eigenvector v_k is attached to the vertex z_k.

Lemma 2.4 [5] In a proper partition $P = \{Z_1, \ldots, Z_r, C_1, \ldots, C_s\}$ of Γ_n, the set of all eigenvectors attached to the vertices in the cycles $Z_1, \ldots, Z_r$ is a linearly independent set.

Lemma 2.5 [5] Let $C = \{c_1, c_2, \dots, c_m\}$ be any chain in a proper partition of Γ_n.

1. If c_m is a terminal point of the graph Γ_n, then $\{e_{c_m}, e_{c_{m-1}}, \dots, e_{c_2}, e_{c_1}\}$ is a chain of generalized eigenvectors of A_n for the eigenvalue 0.

2. If $f(c_m)$is a merge point of Γ_n, let z be the vertex in the cycle or chain containing $f(c_m)$ such that $f^m(z) = f(c_m)$. (z exists by Lemma 2.1.) Then $\{e_{c_m} - e_{f^{m-1}(z)}, \dots, e_{c_3} - e_{f^2(z)}, e_{c_2} - e_{f(z),} e_{c_1} - e_z\}$ is a chain of generalized eigenvectors of A_n for the eigenvalue 0.

In the first case, we say that the vector eci is attached to the vertex c_i. In the second case, we say the vector $e_{c_i} - e_{f^{i-1}(z)}$ is attached to the vertex c_i .

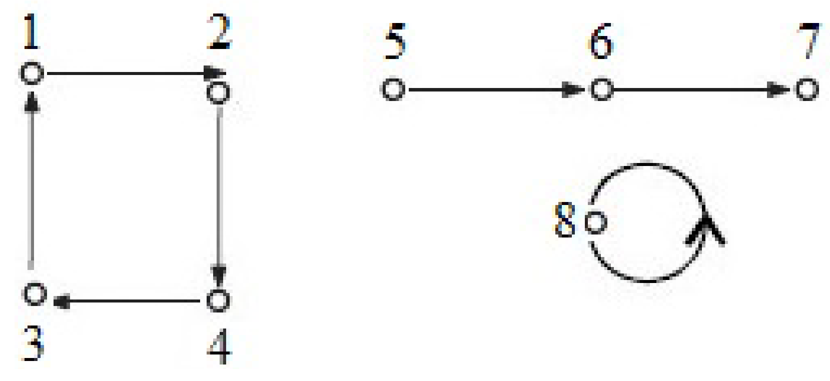
By convention, the first element of a chain of generalized eigenvectors, as in Eq. (1) is the eigenvector, but the eigenvector corresponds to the last element of the chain $\{c_1, c_2, \dots, c_m\}$ in Lemma 2.5. So, the order of indices in the subscripts is reversed.

Theorem 2.6. [2] Let Γ_n be any directed graph with cycles and chains. This graph has r-cycle and s-chains. Then the energy of this graph equal to $\sum_{i=1}^{r} lenZ_i$, where $Z_i s$ are cycles.

Note. Since chains corresponds to "0" roots, $\sum lenC_i$ is not calculated.

Proof: From Lemma 2.2, every eigenvalue of A_n is either 0 or a root of unity. Using Lemma 2.3, since the eigenvalue attached the cycles which is root of unity, i.e. $|\lambda_j| = 1, \forall j = 1,2, \dots, r$, "1" repeat the time of length-cycle in the formula. Further using Lemma 2.5, since the "0" eigenvalue attached the chain, i.e. $|\lambda_j| = 0$, $\forall j = 1,2, \dots, s$, "0" repeat the time of length-chain in the formula. Finally, from this lemmas, we found $\varepsilon(G) = \sum_{j=1}^{r} lenz_i$.

Example 2.7. Find the energy of the following directed graph.



Solution: Firstly, we construct chain and cycle set of above graph. Cycle sets are $Z_1 = \{1, 2, 3, 4\}$, $Z_2 = \{8\}$ and chain set is $C = \{5, 6, 7\}$.

We use the Theorem 2.6.

$$\sum_{i=1}^{2} lenZ_i = lenZ_1 + lenZ_2 = 4 + 1 = 5.$$

REFERENCES

[1] Gutman, I., *Graph Energy*, Springer, New York, 2012.
[2] Altinkaya, A., Buyukkose, S., Master Thesis *Energy of Directed Graphs*, Gazi University, 2017.
[3] Vasudev, C., *Graph Theory With Applications(First Edition),* New Age International Publishers, New Delhi, 1-5, 2006.
[4] Diestel, R., *Graph theory(Electronic Edition),*Springer-Verlag, New York, 2000.
[5] Cardon, D., Tuckfield, B., *Linear Algebra and Its Applications*, **435**, 2942, 2011.