ORIGINAL PAPER

COMPARISON RESULTS OF LINEAR DIFFERENTIAL EQUATIONS WITH FUZZY BOUNDARY VALUES

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Abstract. *In this paper, comparison results of linear differential equations with fuzzy boundary conditions are examined. It is solved the problem of Liu [17] by using the solution method of Gasilov et al's [12]. Here, the solution is same as the solutions (1,1) and (2,2) of Liu for the case of positive constant coefficient and the solution is same as the solutions (1,2) and (2,1) of Liu for the case of negative constant coefficient. If the boundary values are symmetric triangular fuzzy numbers, the value of the solution at any time is a symmetric triangular fuzzy number. Several examples are solved about the worked problems.*

Keywords: *Fuzzy boundary value problems; Boundary value problems; Second order fuzzy differential equations*

1. INTRODUCTION

There are several approaches to studying fuzzy differential equations [1, 6, 9, 13, 16, 18]. The first approach was the use of Hukuhara derivative. This approach has a drawback: the solution becomes fuzzier as time goes by [3, 10]. Also, Bede [4] proved that a large class of fuzzy boundary value problems have not a solution under the Hukuhara derivative concept. The strongly generalized derivative was introduced [2] and studied in [3, 5, 8, 14, 19]. Recently, Khastan and Nieto [15] have found solutions for a large enough class of fuzzy boundary value problems with the strongly generalized derivative.

The second approach generate the fuzzy solution from the crips solution. This approach can be three ways. The first way is the extension principle, where firstly, the initial value is taken a real constant and crips problem is solved. Then, the real constant in the solution is replaced with the initial fuzzy value [6, 7]. The second way is the concept of differential inclusion, where by taking an α-cut of initial value, the differential equation is converted to a differential inclusion and the solution is accepted as the α-cut of fuzzy solution [13]. In the third way, fuzzy solution is considered to be a set of crips problems [11].

In this paper, a investigation is made on comparison results of linear differential equations with fuzzy boundary conditions. It is solved the problem of Liu [17] by using the solution method of Gasilov et al's [12].

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2. PRELIMINARIES

2.1. DEFINITION

A fuzzy number is a function $u: \mathbb{R} \to [0,1]$ satisfying the following properties:

i) u is normal,
ii) u is convex fuzzy set,
iii) u is upper semi-continuous on $\mathbb{R}$,
iv) $cl\{x\epsilon\mathbb{R}|u(x) > 0\}$ is compact where cl denotes the closure of a subset [14].

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

2.2. DEFINITION

Let $u \in \mathbb{R}_F$. The α-level set of u, denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is

$$[u]^\alpha = \{x\epsilon\mathbb{R}|u(x) \geq \alpha\}.$$

If α=0, the support of u is defined

$$[u]^0 = cl\{x\epsilon\mathbb{R}|u(x) > 0\}.$$

The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively [15].
The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

2.3. REMARK

The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

i) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

ii) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

iii) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$ [14].

2.4. DEFINITION

If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the level sets of $[A]^\alpha$ is

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right] \text{ [17].}$$

3. SECOND ORDER FUZZY LINEAR DIFFERENTIAL EQUATIONS

Consider the fuzzy boundary value problem

$$x''(t) + a_1(t)x' + a_2(t)x = f(t) \tag{3.1}$$

$$x(0) = A,\ x(T) = B, \tag{3.2}$$

where A and B are fuzzy numbers, $A = a_{cr} + a$, $B = b_{cr} + b$, a_{cr} and b_{cr} denote the crips parts of A and B; a and b denote the uncertain parts. The problem is splited to following two problems:

1) Associated crips problem:

$$x''(t) + a_1(t)x' + a_2(t)x = f(t), \tag{3.3}$$

$$x(0) = a_{cr}, \quad x(T) = b_{cr}. \tag{3.4}$$

2) Homogeneous problem with fuzzy boundary values:

$$x''(t) + a_1(t)x' + a_2(t)x = 0, \tag{3.5}$$

$$x(0) = a, \quad x(T) = b. \tag{3.6}$$

The solution of the problem (3.1)-(3.2) is of the form

$$x(t) = x_{cr}(t) + x_{un}(t),$$

where $x_{cr}(t)$ is the solution of the non-homogeneous crips problem (3.3)-(3.4) and $x_{un}(t)$ is the solution of the homogeneous problem (3.5)-(3.6). Let be $x_1(t)$ and $x_2(t)$ linear independent solutions of the differential equations (3.5),

$$s(t) = [x_1(t) \quad x_1(t)\ ],\ M = \begin{bmatrix} x_1(0) & x_2(0) \\ x_1(\ell) & x_2(\ell) \end{bmatrix},$$

$$W(t) = s(t)M^{-1} = [w_1(t), w_2(t)].$$

Then, the solution of the fuzzy boundary problem (3.1)-(3.2) is

$$x(t) = x_{cr}(t) + w_1(t)a + w_2(t)b \ \ [\,12].$$

A. *THE CASE OF POSITIVE CONSTANT COEFFICIENTS*

Consider the fuzzy boundary value problem

$$x''(t) = \lambda x(t) \tag{3.7}$$

$$x(0) = A, \quad x(\ell) = B \tag{3.8}$$

where, $\lambda > 0$. Boundary conditions $A = \left(\underline{a}, \frac{\overline{a}+\underline{a}}{2}, \overline{a}\right)$ and $B = \left(\underline{b}, \frac{\overline{b}+\underline{b}}{2}, \overline{b}\right)$ are symmetric triangular fuzzy numbers, the α -level sets of $[A]^\alpha$ and $[B]^\alpha$ are

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right],$$

$$[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right],$$

respectively.

1) Crips problem:

$$x''(t) = \lambda x(t), \tag{3.9}$$

$$x(0) = \frac{\overline{a}+\underline{a}}{2}, \quad x(\ell) = \frac{\overline{b}+\underline{b}}{2}. \tag{3.10}$$

The solution of the differential equation (3.9) is

$$x_{cr}(t) = c_1 e^{\sqrt{\lambda}t} + c_2 e^{-\sqrt{\lambda}t} .$$

Using the boundary conditions (3.10)

$$x_{cr}(0) = c_1 + c_2 = \frac{\overline{a}+\underline{a}}{2} ,$$

$$x_{cr}(\ell) = c_1 e^{\sqrt{\lambda}\ell} + c_2 e^{-\sqrt{\lambda}\ell} = \frac{\overline{b}+\underline{b}}{2}$$

and from this

$$c_1 = \frac{\left(\frac{\overline{b}+\underline{b}}{2}\right) - \left(\frac{\overline{a}+\underline{a}}{2}\right)e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} , \quad c_2 = \frac{\left(\frac{\overline{a}+\underline{a}}{2}\right)e^{\sqrt{\lambda}\ell} - \left(\frac{\overline{b}+\underline{b}}{2}\right)}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}}$$

are obtained. Then the solution of the boundary value problem (3.9)-(3.10) is obtained as

$$x_{cr}(t) = \frac{\left(\frac{\overline{b}+\underline{b}}{2}\right) - \left(\frac{\overline{a}+\underline{a}}{2}\right)e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} e^{\sqrt{\lambda}t} + \frac{\left(\frac{\overline{a}+\underline{a}}{2}\right)e^{\sqrt{\lambda}\ell} - \left(\frac{\overline{b}+\underline{b}}{2}\right)}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} e^{-\sqrt{\lambda}t},$$

$$x_{cr}(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}}\left\{\left(-e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\frac{\overline{a}+\underline{a}}{2}\right) + \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)\left(\frac{\overline{b}+\underline{b}}{2}\right)\right\}.$$

2) The problem with fuzzy boundary values:

$$x''(t) = \lambda x(t), \tag{3.11}$$

$$x(0) = \left(\frac{\underline{a}-\overline{a}}{2}, 0, \frac{\overline{a}-\underline{a}}{2}\right), x(\ell) = \left(\frac{\underline{b}-\overline{b}}{2}, 0, \frac{\overline{b}-\underline{b}}{2}\right) \tag{3.12}$$

$x_1(t) = e^{\sqrt{\lambda}t}$, $x_2(t) = e^{-\sqrt{\lambda}t}$ are linear independent solutions for the differential equation (3.11). Then,

$$s(t) = \begin{bmatrix} e^{\sqrt{\lambda}t} & e^{-\sqrt{\lambda}t} \end{bmatrix}, M = \begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\ell} & e^{-\sqrt{\lambda}\ell} \end{bmatrix}.$$

$$W(t) = s(t)M^{-1}$$

Since

$$M^{-1} = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \begin{bmatrix} -e^{-\sqrt{\lambda}\ell} & 1 \\ e^{\sqrt{\lambda}\ell} & -1 \end{bmatrix},$$

$$W(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \begin{bmatrix} e^{\sqrt{\lambda}t} & e^{-\sqrt{\lambda}t} \end{bmatrix} \begin{bmatrix} -e^{-\sqrt{\lambda}\ell} & 1 \\ e^{\sqrt{\lambda}\ell} & -1 \end{bmatrix}$$

$$W(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \begin{bmatrix} -e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)} & e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t} \end{bmatrix}$$

is obtained, where

$$w_1(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \left(-e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)}\right),$$

$$w_2(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right).$$

Then, the solution of the fuzzy boundary value problem (3.11)-(3.12) is

$$x_{un}(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \left\{ \left(-e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)}\right) \left(\frac{\underline{a}-\overline{a}}{2}, 0, \frac{\overline{a}-\underline{a}}{2}\right) + \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) \left(\frac{\underline{b}-\overline{b}}{2}, 0, \frac{\overline{b}-\underline{b}}{2}\right) \right\}.$$

Since the solution of the fuzzy boundary value problem (3.7)-(3.8)

$$x(t) = x_{cr}(t) + x_{un}(t),$$

we have

$$x(t) = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \left\{ \left(-e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)}\right) \left(\underline{a}, \frac{\overline{a}+\underline{a}}{2}, \overline{a}\right) + \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) \left(\underline{b}, \frac{\overline{b}+\underline{b}}{2}, \overline{b}\right) \right\}.$$

Then, the α-cut of the solution $x(t)$ is obtained as

$$[x(t)]^\alpha = \left[\underline{x}_\alpha(t), \overline{x}_\alpha(t)\right],$$

$$[x(t)]^\alpha = \frac{1}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}} \left\{ \left(-e^{\sqrt{\lambda}(t-\ell)} + e^{-\sqrt{\lambda}(t-\ell)}\right) \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right] \right.$$
$$\left. + \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right] \right\}. \quad (3.13)$$

3.1 PROPOSITION

If $(\overline{b}-\underline{b})-(\overline{a}-\underline{a})e^{-\sqrt{\lambda}\ell}>0$, the solution $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ of (3.7)-(3.8) is a valid α-level set.

Proof. Using $w_1(t)>0$, $w_2(t)>0$ and fuzzy arithmetic, we have

$$[x(t)]^{\alpha}=\frac{1}{e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}}\left\{\left[\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\right.\right.$$
$$+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right),\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)$$
$$\left.\left.+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right]\right\}.$$

Then,

$$\underline{x}_{\alpha}(t)=\frac{1}{e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}}\left\{\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\right.$$
$$\left.+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right\},$$

$$\overline{x}_{\alpha}(t)=\frac{1}{e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}}\left\{\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\right.$$
$$\left.+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right\}.$$

If $\frac{\partial\underline{x}_{\alpha}(t)}{\partial\alpha}\geq 0$, $\frac{\partial\overline{x}_{\alpha}(t)}{\partial\alpha}\leq 0$ and $\overline{x}_{\alpha}(t)-\underline{x}_{\alpha}(t)\geq 0$, $[x(t)]^{\alpha}$ is a valid α-level set. Then, it must be

$$\left\{e^{\sqrt{\lambda}t}\left(\left(\frac{\overline{b}-\underline{b}}{2}\right)-\left(\frac{\overline{a}-\underline{a}}{2}\right)e^{-\sqrt{\lambda}\ell}\right)+e^{-\sqrt{\lambda}t}\left(\left(\frac{\overline{a}-\underline{a}}{2}\right)e^{\sqrt{\lambda}\ell}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\right)\right\}\geq 0\ .$$

From this,

$$e^{-\sqrt{\lambda}t}\left\{e^{2\sqrt{\lambda}t}\left((\overline{b}-\underline{b})-(\overline{a}-\underline{a})e^{-\sqrt{\lambda}\ell}\right)+\left((\overline{a}-\underline{a})e^{\sqrt{\lambda}\ell}-(\overline{b}-\underline{b})\right)\right\}\geq 0,$$

Let be

$$f(t)=e^{2\sqrt{\lambda}t}\left((\overline{b}-\underline{b})-(\overline{a}-\underline{a})e^{-\sqrt{\lambda}\ell}\right)+\left((\overline{a}-\underline{a})e^{\sqrt{\lambda}\ell}-(\overline{b}-\underline{b})\right).$$

$$f(0)=(\overline{a}-\underline{a})\left(e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}\right)>0$$

$$f'(t)=2\sqrt{\lambda}\left((\overline{b}-\underline{b})-(\overline{a}-\underline{a})e^{-\sqrt{\lambda}\ell}\right)e^{2\sqrt{\lambda}t}>0\ ,$$

if

$$\left(\overline{b}-\underline{b}\right)-\left(\overline{a}-\underline{a}\right)e^{-\sqrt{\lambda}\ell}>0.$$

Then, if $\left(\overline{b}-\underline{b}\right)-\left(\overline{a}-\underline{a}\right)e^{-\sqrt{\lambda}\ell}>0$, $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ is a valid α-level set.

3.2 PROPOSITION

For any $t\in[0,\ell]$, the solution $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ of (3.7)-(3.8) is a symmetric triangle fuzzy number.

Proof. Since

$$\underline{x}_1(t)=\frac{1}{e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}}\left\{\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\frac{\overline{a}+\underline{a}}{2}\right)+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\frac{\overline{b}+\underline{b}}{2}\right)\right\}=\overline{x}_1(t)$$

and

$$\underline{x}_1(t)-\underline{x}_{\alpha}(t)=\frac{1}{e^{\sqrt{\lambda}\ell}-e^{-\sqrt{\lambda}\ell}}\left\{\left(-e^{\sqrt{\lambda}(t-\ell)}+e^{-\sqrt{\lambda}(t-\ell)}\right)\left(\left(\frac{\overline{a}-\underline{a}}{2}\right)(1-\alpha)\right)\right.$$
$$\left.+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(\left(\frac{\overline{b}-\underline{b}}{2}\right)(1-\alpha)\right)\right\}=\overline{x}_{\alpha}(t)-\overline{x}_1(t),$$

the solution $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ of (3.7)-(3.8) is a symmetric triangle fuzzy number.

3.3 EXAMPLE

Consider the fuzzy boundary value problem

$$x''(t)=x(t)\quad,t\in\left(0,\frac{3\pi}{2}\right) \tag{3.14}$$

$$x(0)=\left(1,\frac{3}{2},2\right),\quad x\left(\frac{3\pi}{2}\right)=\left(3,\frac{7}{2},4\right) \tag{3.15}$$

where $\left[\left(1,\frac{3}{2},2\right)\right]^{\alpha}=\left[1+\frac{1}{2}\alpha,2-\frac{1}{2}\alpha\right]$, $\left[\left(3,\frac{7}{2},4\right)\right]^{\alpha}=\left[3+\frac{1}{2}\alpha,4-\frac{1}{2}\alpha\right]$.

1)Crips problem:

$$x''(t)=x(t) \tag{3.16}$$

$$x(0)=\frac{3}{2},\quad x\left(\frac{3\pi}{2}\right)=\frac{7}{2} \tag{3.17}$$

The solution of the differential equation (3.16) is

$$x_{cr}(t)=c_1e^{t}+c_2e^{-t}\ .$$

Using the boundary conditions (3.17), we have

$$c_1=\frac{\frac{7}{2}-\frac{3}{2}e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\ ,\ c_2=\frac{\frac{3}{2}e^{\frac{3\pi}{2}}-\frac{7}{2}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\ .$$

Then, the solution of the boundary value problem (3.16)-(3.17) is

$$x_{cr}(t) = \frac{\frac{7}{2}-\frac{3}{2}e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}e^{t} + \frac{\frac{3}{2}e^{\frac{3\pi}{2}}-\frac{7}{2}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\ e^{-t},$$

$$x_{cr}(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\frac{7}{2}(e^{t}-e^{-t}) + \frac{3}{2}\left(e^{\frac{3\pi}{2}-t} - e^{t-\frac{3\pi}{2}}\right)\right\}.$$

3) The problem with fuzzy boundary values:

$$x''(t) = x(t), \tag{3.18}$$

$$x(0) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right), x\left(\frac{3\pi}{2}\right) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right). \tag{3.19}$$

$x_1(t) = e^{t}$, $x_2(t) = e^{-t}$ are linear independent solutions for the differential equation (3.18). Then

$$s(t) = [e^{t} \quad e^{-t}],\ M = \begin{bmatrix} 1 & 1 \\ e^{\frac{3\pi}{2}} & e^{-\frac{3\pi}{2}} \end{bmatrix}$$

and

$$W(t) = s(t)M^{-1}.$$

Since

$$M^{-1} = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\begin{bmatrix} -e^{-\frac{3\pi}{2}} & 1 \\ e^{\frac{3\pi}{2}} & -1 \end{bmatrix},$$

we have

$$W(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}[e^{t} \quad e^{-t}]\begin{bmatrix} -e^{-\frac{3\pi}{2}} & 1 \\ e^{\frac{3\pi}{2}} & -1 \end{bmatrix},$$

$$W(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left[-e^{t-\frac{3\pi}{2}} + e^{\frac{3\pi}{2}-t} \quad e^{t}-e^{-t}\right],$$

where

$$w_1(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left(-e^{t-\frac{3\pi}{2}} + e^{\frac{3\pi}{2}-t}\right),\ w_2(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}(e^{t}-e^{-t}).$$

Then, the solution of the fuzzy boundary value problem (3.18)-(3.19) is

$$x_{un}(t) = \frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left(e^{\frac{3\pi}{2}-t} - e^{t-\frac{3\pi}{2}}\right)\left(-\frac{1}{2}, 0, \frac{1}{2}\right) + (e^{t}-e^{-t})\left(-\frac{1}{2}, 0, \frac{1}{2}\right)\right\}.$$

Since

$$x(t) = x_{cr}(t) + x_{un}(t),$$

the solution of the fuzzy boundary value problem (3.14)-(3.15) is

$$x(t)=\frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left(1,\frac{3}{2},2\right)+(e^{t}-e^{-t})\left(3,\frac{7}{2},4\right)\right\}.$$

Then, the α-level set $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ of the soluton $x(t)$ is

$$[x(t)]^{\alpha}=\frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left[1+\frac{1}{2}\alpha,2-\frac{1}{2}\alpha\right]+(e^{t}-e^{-t})\left[3+\frac{1}{2}\alpha,4-\frac{1}{2}\alpha\right]\right\}.$$

Using $w_1(t)>0$, $w_2(t)>0$ and fuzzy arithmetic, we have

$$[x(t)]^{\alpha}=\frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left[\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left(1+\frac{1}{2}\alpha\right)+(e^{t}-e^{-t})\left(3+\frac{1}{2}\alpha\right),\right.\right.$$

$$\left.\left.\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left(2-\frac{1}{2}\alpha\right)+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(4-\frac{1}{2}\alpha\right)\right]\right\}.$$

Then,

$$\underline{x}_{\alpha}(t)=\frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left(1+\frac{1}{2}\alpha\right)+(e^{t}-e^{-t})\left(3+\frac{1}{2}\alpha\right)\right\}$$

$$\overline{x}_{\alpha}(t)=\frac{1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\left\{\left(e^{\frac{3\pi}{2}-t}-e^{t-\frac{3\pi}{2}}\right)\left(2-\frac{1}{2}\alpha\right)+\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\left(4-\frac{1}{2}\alpha\right)\right\}$$

According to the proposition 3.1, since $1>e^{-\frac{3\pi}{2}}$, the solution $[x(t)]^{\alpha}=\left[\underline{x}_{\alpha}(t),\overline{x}_{\alpha}(t)\right]$ of the fuzzy boundary value problem (3.14)-(3.15) is a valid α-level set and the solution is a symmetric triangle fuzzy number.

B. THE CASE OF NEGATIVE CONSTANT COEFFICIENTS

Consider the fuzzy boundary value problem

$$x''(t)=-\lambda x(t) \tag{3.20}$$

$$x(0)=A,\quad x(\ell)=B \tag{3.21}$$

where, $\lambda>0$. Boundary conditions $A=\left(\underline{a},\ \frac{\overline{a}+\underline{a}}{2},\overline{a}\right)$ and $B=\left(\underline{b},\ \frac{\overline{b}+\underline{b}}{2},\overline{b}\right)$ are symmetric triangular fuzzy numbers, the α -level sets of $[A]^{\alpha}$ and $[B]^{\alpha}$ are

$$[A]^{\alpha}=\left[\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right],$$

$$[B]^{\alpha}=\left[\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right],$$

respectively.

1) Crips problem:

$$x''(t) = -\lambda x(t), \tag{3.22}$$

$$x(0) = \frac{\overline{a}+\underline{a}}{2}, \quad x(\ell) = \frac{\overline{b}+\underline{b}}{2}. \tag{3.23}$$

The solution of the differential equation is

$$x_{cr}(t) = c_1 cos\left(\sqrt{\lambda}t\right) + c_2 sin\left(\sqrt{\lambda}t\right).$$

Using the boundary conditions (3.23)

$$x_{cr}(0) = c_1 = \frac{\overline{a}+\underline{a}}{2},$$

$$x_{cr}(\ell) = \frac{\overline{a}+\underline{a}}{2} cos\left(\sqrt{\lambda}\ell\right) + c_2 sin\left(\sqrt{\lambda}\ell\right) = \frac{\overline{b}+\underline{b}}{2},$$

$$c_2 = \frac{\left(\frac{\overline{b}+\underline{b}}{2}\right) - \left(\frac{\overline{a}+\underline{a}}{2}\right) cos\left(\sqrt{\lambda}\ell\right)}{sin\left(\sqrt{\lambda}\ell\right)}$$

are obtained. Then, the solution of the boundary value problem (3.22)-(3.23) is

$$x_{cr}(t) = \left(\frac{\overline{a}+\underline{a}}{2}\right) cos\left(\sqrt{\lambda}t\right) + \frac{\left(\frac{\overline{b}+\underline{b}}{2}\right) - \left(\frac{\overline{a}+\underline{a}}{2}\right) cos\left(\sqrt{\lambda}\ell\right)}{sin\left(\sqrt{\lambda}\ell\right)} sin\left(\sqrt{\lambda}t\right), \ \ell \neq \frac{n\pi}{\sqrt{\lambda}}, n \in \mathbb{N}$$

$$= \left(\frac{\overline{a}+\underline{a}}{2}\right)\left(cos\left(\sqrt{\lambda}t\right) - \frac{sin\left(\sqrt{\lambda}t\right) cos\left(\sqrt{\lambda}\ell\right)}{sin\left(\sqrt{\lambda}\ell\right)}\right) + \left(\frac{\overline{b}+\underline{b}}{2}\right)\left(\frac{sin\left(\sqrt{\lambda}t\right)}{sin\left(\sqrt{\lambda}\ell\right)}\right).$$

2) The problem with fuzzy boundary values:

$$x''(t) = -\lambda x(t) \tag{3.24}$$

$$x(0) = \left(\frac{\underline{a}-\overline{a}}{2}, 0, \frac{\overline{a}-\underline{a}}{2}\right), x(\ell) = \left(\frac{\underline{b}-\overline{b}}{2}, 0, \frac{\overline{b}-\underline{b}}{2}\right) \tag{3.25}$$

$x_1(t) = cos\left(\sqrt{\lambda}t\right)$, $x_2(t) = sin\left(\sqrt{\lambda}t\right)$ are linear independent solutions for the differential equation (3.24). Then,

$$s(t) = \left[cos\left(\sqrt{\lambda}t\right) \quad sin\left(\sqrt{\lambda}t\right)\right], \qquad M = \begin{bmatrix} 1 & 1 \\ cos\left(\sqrt{\lambda}\ell\right) & sin\left(\sqrt{\lambda}\ell\right) \end{bmatrix}$$

and

$$W(t) = s(t)M^{-1}.$$

Since

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{cos(\sqrt{\lambda}\ell)}{sin(\sqrt{\lambda}\ell)} & \frac{1}{sin(\sqrt{\lambda}\ell)} \end{bmatrix},$$

$$W(t) = \begin{bmatrix} cos(\sqrt{\lambda}t) & sin(\sqrt{\lambda}t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{cos(\sqrt{\lambda}\ell)}{sin(\sqrt{\lambda}\ell)} & \frac{1}{sin(\sqrt{\lambda}\ell)} \end{bmatrix}$$

$$= \begin{bmatrix} cos(\sqrt{\lambda}t) - \frac{sin(\sqrt{\lambda}t)cos(\sqrt{\lambda}\ell)}{sin(\sqrt{\lambda}\ell)} & \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{sin\left(\sqrt{\lambda}(\ell - t)\right)}{sin(\sqrt{\lambda}\ell)} & \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)} \end{bmatrix}$$

is obtained, where

$$w_1(t) = \frac{sin\left(\sqrt{\lambda}(\ell-t)\right)}{sin(\sqrt{\lambda}\ell)}, \quad w_2(t) = \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)}.$$

Thus,

$$x_{un}(t) = \frac{sin\left(\sqrt{\lambda}(\ell-t)\right)}{sin(\sqrt{\lambda}\ell)}\left(\frac{\underline{a}-\overline{a}}{2}, 0, \frac{\overline{a}-\underline{a}}{2}\right) + \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)}\left(\frac{\underline{b}-\overline{b}}{2}, 0, \frac{\overline{b}-\underline{b}}{2}\right).$$

Since

$$x(t) = x_{cr}(t) + x_{un}(t),$$

we have

$$x(t) = \frac{sin\left(\sqrt{\lambda}(\ell-t)\right)}{sin(\sqrt{\lambda}\ell)}\left(\underline{a}, \frac{\overline{a}+\underline{a}}{2}, \overline{a}\right) + \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)}\left(\underline{b}, \frac{\overline{b}+\underline{b}}{2}, \overline{b}\right).$$

The α-cut of the solution $x(t)$ is

$$[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$$

$$[x(t)]^{\alpha} = \frac{sin\left(\sqrt{\lambda}(\ell-t)\right)}{sin(\sqrt{\lambda}\ell)}\left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right] + \frac{sin(\sqrt{\lambda}t)}{sin(\sqrt{\lambda}\ell)}\left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right]. \tag{3.26}$$

3.4 PROPOSITION

$t \in [0, \pi]$, the solution $[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$ of (3.20)-(3.21) is no longer a valid α-level set as

$$t > \frac{1}{\sqrt{\lambda}} cot^{-1}\left(\frac{(\overline{a}-\underline{a})cos(\sqrt{\lambda}\ell)-(\overline{b}-\underline{b})}{(\overline{a}-\underline{a})sin(\sqrt{\lambda}\ell)}\right).$$

Proof. For $t \in [0,\pi]$, using $w_1(t) > 0$, $w_2(t) > 0$ and fuzzy arithmetic, we have

$$[x(t)]^\alpha = \left[\frac{sin\left(\sqrt{\lambda}(\ell - t)\right)}{sin\left(\sqrt{\lambda}\ell\right)}\left(\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right) + \frac{sin\left(\sqrt{\lambda}t\right)}{sin\left(\sqrt{\lambda}\ell\right)}\left(\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right),\right.$$
$$\left.\frac{sin\left(\sqrt{\lambda}(\ell-t)\right)}{sin\left(\sqrt{\lambda}\ell\right)}\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \frac{sin\left(\sqrt{\lambda}t\right)}{sin\left(\sqrt{\lambda}\ell\right)}\left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right].$$

Using

$$sin\left(\sqrt{\lambda}(\ell - t)\right) = cos\left(\sqrt{\lambda}t\right)sin\left(\sqrt{\lambda}\ell\right) - sin\left(\sqrt{\lambda}t\right)cos\left(\sqrt{\lambda}\ell\right),$$

$$\underline{x}_\alpha(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos\left(\sqrt{\lambda}t\right) + \underline{c}sin\left(\sqrt{\lambda}t\right),$$

$$\overline{x}_\alpha(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos\left(\sqrt{\lambda}t\right) + \overline{c}sin\left(\sqrt{\lambda}t\right)$$

is obtained, where

$$\underline{c} = \frac{\left[\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)-\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos\left(\sqrt{\lambda}\ell\right)\right]}{sin\left(\sqrt{\lambda}\ell\right)}, \quad \overline{c} = \frac{\left[\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)-\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos\left(\sqrt{\lambda}\ell\right)\right]}{sin\left(\sqrt{\lambda}\ell\right)}.$$

If $\frac{\partial \underline{x}_\alpha(t)}{\partial \alpha} \geq 0$, $\frac{\partial \overline{x}_\alpha(t)}{\partial \alpha} \leq 0$ and $\overline{x}_\alpha(t) - \underline{x}_\alpha(t) \geq 0$, $[x(t)]^\alpha$ is a valid α-level set. Then, it must be

$$\left(\overline{a} - \underline{a}\right)cos\left(\sqrt{\lambda}t\right) + \frac{\left(\overline{b}-\underline{b}\right)-\left(\overline{a}-\underline{a}\right)cos\left(\sqrt{\lambda}\ell\right)}{sin\left(\sqrt{\lambda}\ell\right)}sin\left(\sqrt{\lambda}t\right) \geq 0 \ .$$

From this,

$$cot\left(\sqrt{\lambda}t\right) \geq \frac{\left(\overline{a}-\underline{a}\right)cos\left(\sqrt{\lambda}\ell\right)-\left(\overline{b}-\underline{b}\right)}{sin\left(\sqrt{\lambda}\ell\right)\left(\overline{a}-\underline{a}\right)},$$

and since $t \in [0,\pi]$

$$t \leq cot^{-1}\left(\frac{\left(\overline{a}-\underline{a}\right)cos\left(\sqrt{\lambda}\ell\right)-\left(\overline{b}-\underline{b}\right)}{sin\left(\sqrt{\lambda}\ell\right)\left(\overline{a}-\underline{a}\right)}\right)$$

is obtained. Then, the solution $[x(t)]^\alpha = \left[\underline{x}_\alpha(t), \overline{x}_\alpha(t)\right]$ of (3.20)-(3.21) is no longer a valid α-level set as

$$t > \frac{1}{\sqrt{\lambda}}cot^{-1}\left(\frac{\left(\overline{a}-\underline{a}\right)cos\left(\sqrt{\lambda}\ell\right)-\left(\overline{b}-\underline{b}\right)}{\left(\overline{a}-\underline{a}\right)sin\left(\sqrt{\lambda}\ell\right)}\right).$$

3.5 PROPOSITION

For any $t \in [0, \ell]$, the solution $[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$ of (3.20)-(3.21) is a symmetric triangle fuzzy number.

Proof. Since

$$sin\left(\sqrt{\lambda}(\ell - t)\right) = cos(\sqrt{\lambda}t)sin(\sqrt{\lambda}\ell) - sin(\sqrt{\lambda}t)cos(\sqrt{\lambda}\ell),$$

$$\underline{x}_1(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos(\sqrt{\lambda}t) + \underline{c}sin(\sqrt{\lambda}t),$$

$$\overline{x}_{\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos(\sqrt{\lambda}t) + \overline{c}sin(\sqrt{\lambda}t)$$

is obtained, where

$$\underline{c} = \frac{\left[\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)-\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos(\sqrt{\lambda}\ell)\right]}{sin(\sqrt{\lambda}\ell)}, \quad \overline{c} = \frac{\left[\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)-\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)cos(\sqrt{\lambda}\ell)\right]}{sin(\sqrt{\lambda}\ell)}.$$

Then,

$$\underline{x}_1(t) = \left(\frac{\overline{a}+\underline{a}}{2}\right)cos(\sqrt{\lambda}t) + \frac{\left[\left(\frac{\overline{b}+\underline{b}}{2}\right)-\left(\frac{\overline{a}+\underline{a}}{2}\right)cos(\sqrt{\lambda}\ell)\right]}{sin(\sqrt{\lambda}\ell)}sin(\sqrt{\lambda}t) = \overline{x}_1(t)$$

and

$$\underline{x}_1(t) - \underline{x}_{\alpha}(t) = (1-\alpha)\left\{\left(\frac{\overline{a}-\underline{a}}{2}\right)cos(\sqrt{\lambda}t) + \frac{\left[\left(\frac{\overline{b}-\underline{b}}{2}\right)-\left(\frac{\overline{a}-\underline{a}}{2}\right)cos(\sqrt{\lambda}\ell)\right]}{sin(\sqrt{\lambda}\ell)}sin(\sqrt{\lambda}t)\right\}$$

$$= \overline{x}_{\alpha}(t) - \overline{x}_1(t).$$

The solution $[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$ of (3.20)-(3.21) is a symmetric triangle fuzzy number.

3.6 EXAMPLE

Consider the fuzzy boundary value problem

$$x''(t) = -x(t) \quad , t \in \left(0, \frac{\pi}{2}\right) \tag{3.27}$$

$$x(0) = \left(1, \frac{3}{2}, 2\right), \quad x\left(\frac{\pi}{2}\right) = \left(3, \frac{7}{2}, 4\right) \tag{3.28}$$

1) Crips problem:

$$x''(t) = -x(t) \tag{3.29}$$

$$x(0) = \frac{3}{2}, \quad x\left(\frac{\pi}{2}\right) = \frac{7}{2} \tag{3.30}$$

The solution of the differential equation (3.29) is

$$x_{cr}(t) = c_1 cos(t) + c_2 sin(t)\ .$$

Using the boundary values (3.30), we have

$$x_{cr}(0) = c_1 = \frac{3}{2}\ ,\ x_{cr}\left(\frac{\pi}{2}\right) = c_2 = \frac{7}{2}\ .$$

Then, the solution of the boundary value problem (3.29)-(3.30) is

$$x_{cr}(t) = \frac{3}{2} cos(t) + \frac{7}{2} sin(t).$$

2) The problem with fuzzy boundary values:

$$x''(t) = -x(t) \tag{3.31}$$

$$x(0) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right), x\left(\frac{3\pi}{2}\right) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right) \tag{3.32}$$

$x_1(t) = cos(t)$, $x_2(t) = sin(t)$ are linear independent solutions for the differential equation (3.31). Then,

$$s(t) = [cos(t) \quad sin(t)],\ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$W(t) = s(t)M^{-1}$$

$$W(t) = [cos(t) \quad sin(t)] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W(t) = [cos(t) \quad sin(t)]$$

are obtained, where

$$w_1(t) = cos(t),\ \ w_2(t) = sin(t).$$

Then, the solution of the fuzzy boundary value problem (3.31)-(3.32) is

$$x_{un}(t) = cos(t)\left(-\frac{1}{2}, 0, \frac{1}{2}\right) + sin(t)\left(-\frac{1}{2}, 0, \frac{1}{2}\right).$$

Since

$$x(t) = x_{cr}(t) + x_{un}(t),$$

the solution of the fuzzy boundary value problem (3.27)-(3.28) is

$$x(t) = cos(t)\left(1, \frac{3}{2}, 2\right) + sin(t)\left(3, \frac{7}{2}, 4\right).$$

Then, the α-level set $[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$ of the soluton $x(t)$ is

$$[x(t)]^{\alpha} = cos(t)\left[1+\tfrac{1}{2}\alpha, 2-\tfrac{1}{2}\alpha\right] + sin(t)\left[3+\tfrac{1}{2}\alpha, 4-\tfrac{1}{2}\alpha\right].$$

Using $w_1(t) > 0$, $w_2(t) > 0$ for $t \in \left(0, \frac{\pi}{2}\right)$ and fuzzy arithmetic, we have

$$[x(t)]^{\alpha} = \left[cos(t)\left(1+\tfrac{1}{2}\alpha\right) + sin(t)\left(3+\tfrac{1}{2}\alpha\right), cos(t)\left(2-\tfrac{1}{2}\alpha\right) + sin(t)\left(4+\tfrac{1}{2}\alpha\right)\right].$$

Then,

$$\underline{x}_{\alpha}(t) = cos(t)\left(1+\tfrac{1}{2}\alpha\right) + sin(t)\left(3+\tfrac{1}{2}\alpha\right),$$

$$\overline{x}_{\alpha}(t) = cos(t)\left(2-\tfrac{1}{2}\alpha\right) + sin(t)\left(4+\tfrac{1}{2}\alpha\right).$$

According to the proposition 3.4, when $t > cot^{-1}(-1)$, the solution $[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right]$ of the fuzzy boundary value problem (3.27)-(3.28) is not a valid α-level set. Also, the solution is a symmetric triangle fuzzy number.

4. CONCLUSIONS

In this paper, it is solved the problem of Liu [17] by using the solution method of Gasilov et al's [12]. Here, the solution is same as the solutions (1,1) and (2,2) of Liu for the case of positive constant coefficient and the solution is same as the solutions (1,2) and (2,1) of Liu for the case of negative constant coefficient. In Gasilov et al's paper [12], the α-cuts of the solution $x(t)$ defined in the form

$$[x(t)]^{\alpha} = \left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right] = x_{cr}(t) + x_{un,\alpha}(t)$$

is not a valid α-level set, where $x_{un,\alpha}(t) = (1-\alpha)\left[\underline{x}_{un}(t), \overline{x}_{un}(t)\right]$**.** This, it can be see on the examples in the paper [12].

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