ORIGINAL PAPER

NON-NEWTONIAN IMPROPER INTEGRALS

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Abstract: *In this study, non-Newtonian improper integrals were introduced and their convergence conditions were investigated. Furthermore, some main theorems such as the second mean value theorem and intermediate value theorem were proved in the non-Newtonian sense to be given convergence tests.*

Keywords: *Non-Newtonian improper integrals, Non-Newtonian calculus, Convergence tests.*

1. INTRODUCTION

The non-Newtonian calculus provides a wide diversity of mathematical tools for use in engineering, mathematics and science. The notion of non-Newtonian calculus was firstly introduced and worked by Grossman and Katz. They published the book about fundamentals of non-Newtonian calculus and which includes some special calculuses such as geometric, harmonic, bigeometric etc. [1]. Non-Newtonian calculus was used by Meginniss to create a theory of probability that is adapted to human behavior and decision making [2]. Rybaczuk and Stopel used the bigeometric calculus on fractals and material science [3]. In a study which is made by Aniszewska and Rybaczuk, the bigeometric calculus was used on a multiplicative Lorenz system [4]. Uzer investigated multiplicative type complex calculus as alternative to classical calculus [5]. In a study which is made by Bashirov and Rıza, differentiation analysed as complex multiplicative [6]. The non-Newtonian calculus was used in the study of biomedical image analysis by Florack and van Assen [7]. Çakmak and Başar obtained some results and sequence spaces with respect to non-Newtonian calculus [8]. In a study which is made by Tekin and Başar, sequence spaces were examined on non-Newtonian complex field [9]. Duyar, Sağır and Oğur got some basic topologic properties on non-Newtonian real line [10]. Duyar and Erdoğan investigated non-Newtonian real number series and obtained convergence tests for them [11].

As a result of these studies, it has arisen the need of examination the improper integrals on non-Newtonian calculus. Hence, in this study, we introduce the non-Newtonian improper integrals and show some convergence tests for them.

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2. GENERAL INFORMATIONS

2.1. α -ARITHMETIC

Definition 1: A generator is one-to-one function whose domain is $\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$. The range of generator α is called non-Newtonian real line and we denote it by $\mathbb{R}(N)_\alpha$. By $\alpha-$arithmetic we mean the arithmetic whose realm $\mathbb{R}(N)_\alpha$ and whose operations and ordering relation are defined as follows:

α-addition $\quad y \dot{+} z = \alpha\left\{\alpha^{-1}(y) + \alpha^{-1}(z)\right\}$

α-subtraction $\quad y \dot{-} z = \alpha\left\{\alpha^{-1}(y) - \alpha^{-1}(z)\right\}$

α-multiplication $\quad y \dot{\times} z = \alpha\left\{\alpha^{-1}(y) \times \alpha^{-1}(z)\right\}$

α-division $\quad y \dot{/} z = \alpha\left\{\alpha^{-1}(y) / \alpha^{-1}(z)\right\}$

α-order $\quad y \dot{<} z \left(y \dot{\leq} z\right) \Leftrightarrow \alpha^{-1}(y) < \alpha^{-1}(z) \ \left(\alpha^{-1}(y) \leq \alpha^{-1}(z)\right)$.

In this case, it is said that α generates $\alpha-$arithmetic. For example, the identity function I generates the classical arithmetic and the exponential function $\exp$ generates geometric arithmetic. Each generator generates exactly one arithmetic and, conversely, each arithmetic is generated by exactly one generator [1].

Definition 2: The $\alpha-$positive numbers are the numbers in $\mathbb{R}(N)_\alpha$ such that $x \dot{>} \dot{0}$, similarly the $\alpha-$negative numbers are the numbers in $\mathbb{R}(N)_\alpha$ such that $x \dot{<} \dot{0}$. $\alpha-$zero and $\alpha-$one numbers are denoted by $\dot{0} = \alpha(0)$ and $\dot{1} = \alpha(1)$ respectively. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. Hence $\alpha-$integers are as follows:

$$\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots .$$

For each integer n, we set $\dot{n} = \alpha(n)$. If $\dot{n}$ is an $\alpha-$positive integer, then it is n times sum of $\dot{1}$[1, 11].

Definition 3: $\alpha-$absolute value of a number $x \in \mathbb{R}(N)_\alpha$ is defined by

$$|x|_\alpha = \begin{cases} x & \text{if} \quad x \dot{>} \dot{0} \\ \dot{0} & \text{if} \quad x = \dot{0} \\ \dot{0} \dot{-} x & \text{if} \quad x \dot{<} \dot{0} \end{cases}.$$

This value is equivalent to the expression $\alpha\left(\left|\alpha^{-1}(x)\right|\right)$ [1, 8, 10].

Definition 4: A closed $\alpha-$interval on $\mathbb{R}(N)_\alpha$ is represented by

$$\dot{[}a,b\dot{]}=\left\{x\in\mathbb{R}(N)_\alpha : a\dot{\leq} x\dot{\leq} b\right\}$$
$$=\left\{x\in\mathbb{R}(N)_\alpha : \alpha^{-1}(a)\leq\alpha^{-1}(x)\leq\alpha^{-1}(b)\right\}=\alpha\left(\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]\right),$$

similarly an open $\alpha-$interval $\dot{(}a,b\dot{)}$ can be represented. It is said that an $\alpha-$interval has $\alpha-$extent $b\dot{-}a$ [1, 10, 11].

Definition 5: Let $\{u_n\}$ be an infinite sequence of the numbers in $\mathbb{R}(N)_\alpha$. If each open $\alpha-$interval containing an element u includes all elements except for a finite numbers of elements of the sequence $\{u_n\}$, then it is said that the sequence $\{u_n\}$ $\alpha-$converges to u and the element u is called as $\alpha-$limit of the sequence $\{u_n\}$. This is denoted by ${}^{\alpha}\lim_{n\to\infty} u_n=u$. This convergence becomes the classic convergence if $\alpha=I$. Classic and geometric convergence are equivalent in the sense that a positive number sequence $\{p_n\}$ converges as geometric to a positive number p iff $\{p_n\}$ converges as classic to p [1, 9, 5, 11].

Proposition 1: $|x\dot{+}y|_\alpha \dot{\leq} |x|_\alpha \dot{+} |y|_\alpha$ (non-Newtonian triangle inequality) and $|x\dot{\times}y|_\alpha = |x|_\alpha \dot{\times} |y|_\alpha$ are hold for any $x,y\in\mathbb{R}(N)_\alpha$ [8].

*2.2. *-CALCULUS*

Let α and β be arbitrary chosen generators which image the set $\mathbb{R}$ to A and B respectively. *-Calculus is defined as an ordered pair of the arithmetics $(\alpha-\text{arithmetic}, \beta-\text{arithmetic})$ and the following notations are used:

	$\alpha-$arithmetic	$\beta-$arithmetic
Universe(Realm)	$A\left(=\mathbb{R}(N)_\alpha\right)$	$B\left(=\mathbb{R}(N)_\beta\right)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}\left(or\ -\alpha\right)$	$\ddot{/}\left(or\ -\beta\right)$
Ordering	$\dot{<}$	$\ddot{<}$

$\alpha-$arithmetic is used on inputs and $\beta-$arithmetic is used on outputs. In particular, the changes of inputs and outputs are measured by $\alpha-$arithmetic and $\beta-$arithmetic, respectively. The operators in *-calculus are applied to functions whose inputs and outputs belong to A and B, respectively.

If the generators α and β are chosen as one of I and $\exp$, the following special calculuses are obtained.

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function ι (iota) that possesses the following three properties:

1. ι is one to one,
2. ι is on A and onto B,
3. For any number u and v in A,

$$\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v),$$
$$\iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v),$$
$$\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v),$$
$$\iota(u \dot{/} v) = \iota(u) \ddot{/} \iota(v), \ v \neq \dot{0}$$
$$u \dot{<} v \Leftrightarrow \iota(u) \ddot{<} \iota(v).$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every number x in A, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n. Any statement in $\alpha-$arithmetic can easily transformed into a statement in $\beta-$arithmetic thanks to the isomorphism ι[1].

Definition 6: The *-limit at a point $a \in A$ of a function f is, if it exists, the unique number b in the set B which is $\beta-$converged by outputs sequence $\{f(a_n)\}$ for every infinite sequence $\{a_n\}$ whose terms are distinct from a and which $\alpha-$converges to a. In this case,

$$* - \lim_{x \to a} f(x) = b$$

is written[1].

Definition 7: A function f is *-continuous at a point $a \in A$ iff this point a is an input for f and $* - \lim_{x \to a} f(x) = f(a)$ [1].

Definition 8: If the following *-limit exists, we denote it by $\left[\overset{*}{D} f\right](a)$ and call it the *-derivative of f at a, and say that f is *-differentiable at a:

$$* - \lim_{x \to a} \left\{ \left[f(x) \ddot{-} f(a) \right] \ddot{/} \left[\iota(x) \ddot{-} \iota(a) \right] \right\}.$$

If it exists, $\left[\overset{*}{D} f\right](a)$ is necessarily in B.

The derivative of f, denoted by $\overset{*}{D} f$, is the function that assigns each number t in A to the number $\left[\overset{*}{D} f\right](t)$, if it exists[1].

Definition 9: The *-average of a *-continuous function f on $[r,s]$ is denoted by $\overset{*}{M}{}_r^s f$ and defined to be $\beta-$limit of the $\beta-$convergent sequence whose nth term is $\beta-$average of $f(a_1),...,f(a_n)$, where $a_1,...,a_n$ is the $n-$fold $\alpha-$partition of $[r,s]$ [1].

Definition 10: The *-integral of a *-continuous function f on $[r,s]$, denoted by ${}^*\int_r^s f(x)\,d^*x$, is the number $\left[\iota(s) \dot{-} \iota(r)\right] \ddot{\times} \overset{*}{M}{}_r^s f$ in B [1].

The *-derivative and *-integral are "inversely" related in the sense indicated by the following two theorems.

Theorem 1: (First fundamental theorem of *-calculus) If f is *-continuous on $[r,s]$ and $g(x) = {}^*\int_r^x f(t)\,d^*t$ for every $x \in [r,s]$, then $\overset{*}{D} g = f$ on $[r,s]$ [1].

Theorem 2: (Second fundamental theorem of *-calculus) If $\overset{*}{D} h$ is *-continuous on $[r,s]$, then ${}^*\int_r^s \left[\overset{*}{D} h\right](x)\,d^*x = h(s) \ddot{-} h(r)$ [1].

Remark 1: Let $\bar{a} = \alpha^{-1}(a)$ for a given number $a \in A$. Let $\bar{f}(t) = \beta^{-1}\left(f\left(\alpha(t)\right)\right)$ for a function f whose inputs and outputs are in A and B, respectively. Then the following relations are true[1]:

1. $*-\lim_{x \to a} f(x)$ and $\lim_{t \to \bar{a}} \bar{f}(t)$ coexist and if they do exist

$$*-\lim_{x \to a} f(x) = \beta\left\{\lim_{t \to \bar{a}} \bar{f}(t)\right\}.$$

Furthermore, f is *-continuous at a iff $\bar{f}$ is classically continuous at $\bar{a}$.

2. The derivatives $\left[\overset{*}{D} f\right](a)$ and $\left[D\bar{f}\right](\bar{a})$ coexist and if they do exist

$$\left[\overset{*}{D} f\right](a) = \beta\left\{\left[D\bar{f}\right](\bar{a})\right\}.$$

3. If f is *-continuous on $[r,s]$, then

$$\overset{*}{M}{}_r^s f = \beta\left\{M_{\bar{r}}^{\bar{s}} \bar{f}\right\} \quad \text{and} \quad {}^*\int_r^s f(x)\,d^*x = \beta\left\{\int_{\bar{r}}^{\bar{s}} \bar{f}(x)\,dx\right\}.$$

4. Let $a \in A$ and $b \in B$. If the generators α and β are classically continuous at $\alpha^{-1}(a)$ and $\beta^{-1}(b)$, respectively, then

$$*-\lim_{x\to a} f(x) = b \quad \text{iff} \quad \lim_{x\to a} f(x) = b.$$

If α and β are classically continuous at $\alpha^{-1}(a)$ and $\beta^{-1}(f(a))$, respectively, then f is *-continuous at a iff f is classically continuous at a.

3. RESULTS AND DISCUSSION

Definition 11: Let a function $f : X \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be given. It is said that the left(right)-handed *-limit of the function f at the point $a \in \mathbb{R}(N)_\alpha$ is the number $L \in \mathbb{R}(N)_\beta$, when any number $\varepsilon \ddot{>} \ddot{0}$ is given if there exists at least a number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $\left|f(x) \ddot{-} L\right|_\beta \ddot{<} \varepsilon$ for all $x \in X$ and $a \dot{-} \delta \dot{<} x \dot{<} a \left(a \dot{<} x \dot{<} a \dot{+} \delta\right)$. This *-limit is denoted by

$$*-\lim_{x\to a^-} f(x) = L \left(*-\lim_{x\to a^+} f(x) = L \right).$$

Definition 12: Let a function $f : (a, \dot{+}\infty) \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ $\left(f : (\dot{-}\infty, a) \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta\right)$ be given. It is said that *-limit of the function f at the $\dot{+}\infty \left(\dot{-}\infty\right)$ is the number $L \in \mathbb{R}(N)_\beta$, when any number $\varepsilon \ddot{>} \ddot{0}$ is given if there exists at least a number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $\left|f(x) \ddot{-} L\right|_\beta \ddot{<} \varepsilon$ for all $x \dot{>} \delta \left(x \dot{<} \delta\right)$. This *-limit is denoted by

$$*-\lim_{x\to \dot{-}\infty} f(x) = L \left(*-\lim_{x\to \dot{+}\infty} f(x) = L \right).$$

Similar definitions can be given for $L = \ddot{+}\infty$ and $L = \ddot{-}\infty$.

Proposition 2: For $a, b \in \mathbb{R}(N)_\alpha$ the followings are hold.

1. If $b \dot{\geq} \dot{0}$, then $|a|_\alpha \dot{\leq} b \Leftrightarrow \dot{0} \dot{-} b \dot{\leq} a \dot{\leq} b$,
2. $\left| |a|_\alpha \dot{-} |b|_\alpha \right|_\alpha \dot{\leq} |a \dot{-} b|_\alpha$.

Proof: One can show this proposition easily by using definition and properties of $\alpha-$ absolute value. We omit the details.

Definition 13: Let the function $f : [a, \dot{+}\infty) \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be *-continuous on $\alpha-$interval $[a,b]$ for each number $b \dot{\geq} a$. The *-limit $*-\lim_{b\to\dot{+}\infty} {}^*\!\int_a^b f(x)\, d^*x$ is called improper

*-integral of type 1 of the function f on $\dot{[}a,\dot{+}\infty\dot{]}$ and is denoted by ${}^{*}\int\limits_{a}^{\dot{+}\infty} f\left(x\right)d^{*}x$. If $*-\lim\limits_{b\to\dot{+}\infty} {}^{*}\int\limits_{a}^{b} f\left(x\right)d^{*}x$ exists and equals to a number $L\in\mathbb{R}(N)_{\beta}$, then it is said that the improper *-integral ${}^{*}\int\limits_{a}^{\dot{+}\infty} f\left(x\right)d^{*}x$ is convergent(*-convergent). If this *-limit does not exist or equals to one of $\ddot{+}\infty$ and $\ddot{-}\infty$, then it is said that the improper *-integral ${}^{*}\int\limits_{a}^{\dot{+}\infty} f\left(x\right)d^{*}x$ is divergent.

Similarly, if the function $f:\dot{(}\dot{-}\infty,b\dot{]}\subset\mathbb{R}(N)_{\alpha}\to\mathbb{R}(N)_{\beta}$ is continuous on $\dot{[}a,b\dot{]}$ for each number $a\dot{\leq}b$, then the improper *-integral of type 1 of the function f on $\dot{[}\dot{-}\infty,b\dot{]}$ is defined by

$$ {}^{*}\int\limits_{\dot{-}\infty}^{b} f\left(x\right)d^{*}x = *-\lim_{a\to\dot{-}\infty} {}^{*}\int\limits_{a}^{b} f\left(x\right)d^{*}x. $$

If the function $f:\dot{(}\dot{-}\infty,\dot{+}\infty\dot{)}\to\mathbb{R}(N)_{\beta}$ is *-continuous on each $\alpha-$interval $\dot{[}a,b\dot{]}$ with $a\dot{<}b\left(a,b\in\mathbb{R}(N)_{\alpha}\right)$, then the improper *-integral of type 1 of the function f on $\dot{[}\dot{-}\infty,\dot{+}\infty\dot{]}$ is defined by

$$ {}^{*}\int\limits_{\dot{-}\infty}^{\dot{+}\infty} f\left(x\right)d^{*}x = *-\lim_{\substack{a\to\dot{-}\infty\\ b\to\dot{+}\infty}} {}^{*}\int\limits_{a}^{b} f\left(x\right)d^{*}x $$

for $a\to\dot{-}\infty$ and $b\to\dot{+}\infty$ are independently from each other.

Example 1: If the improper *-integral of type 1 is looked for geometric calculus, since $\alpha\left(x\right)=I\left(x\right)=x$ and $\beta\left(x\right)=e^{x}$, then $\dot{+}\infty=\lim\limits_{x\to+\infty}\alpha\left(x\right)=+\infty$, $\dot{-}\infty=\lim\limits_{x\to-\infty}\alpha\left(x\right)=-\infty$, $\ddot{+}\infty=\lim\limits_{x\to+\infty}\beta\left(x\right)=+\infty$, $\ddot{-}\infty=\lim\limits_{x\to-\infty}\beta\left(x\right)=0$ are obtained. According to this, $\mathbb{R}(N)_{\alpha}=\mathbb{R}$ and $\mathbb{R}(N)_{\beta}=(0,+\infty)$. Let the function $f:[a,+\infty)\to(0,+\infty)$ be continuous on $[a,b]$ for each $b\geq a$. The improper geometric integral of type 1 of f on $\left[a,+\infty\right]$ is

$$ \tilde{\int\limits_{a}^{+\infty}} f(x)d\tilde{x} = \lim_{b\to+\infty}\left(\tilde{\int\limits_{a}^{b}} f(x)d\tilde{x}\right) = \lim_{b\to+\infty} e^{\left(\int\limits_{a}^{b}\ln\left(f(x)\right)dx\right)} = e^{\lim\limits_{b\to+\infty}\left(\int\limits_{a}^{b}\ln\left(f(x)\right)dx\right)} = e^{\left(\int\limits_{a}^{+\infty}\ln\left(f(x)\right)dx\right)}. $$

For anageometric calculus, since $\alpha\left(x\right)=e^{x}$ and $\beta\left(x\right)=I\left(x\right)=x$, $\mathbb{R}(N)_{\alpha}=(0,+\infty)$ and $\mathbb{R}(N)_{\beta}=\mathbb{R}$. Let the function $f:[a,+\infty)\subset(0,+\infty)\to\mathbb{R}$ be continuous on $[a,b]$ for each $b\geq a$. The improper anageometric integral of type 1 of f on $\left[a,+\infty\right]$ is

$$ \int\limits_{\underset{\sim}{a}}^{+\infty} f(x)d\underset{\sim}{x} = \lim_{b\to+\infty}\int\limits_{\underset{\sim}{a}}^{b} f(x)d\underset{\sim}{x} = \lim_{b\to+\infty}\left(\int\limits_{\ln a}^{\ln b} f(e^{x})dx\right). $$

For bigeometric calculus, since $\alpha(x)=\beta(x)=e^x$, $\mathbb{R}(N)_\alpha=\mathbb{R}(N)_\beta=(0,+\infty)$. Let the function $f:[a,+\infty)\subset(0,+\infty)\to(0,+\infty)$ be continuous on $[a,b]$ for each $b\geq a$. The improper bigeometric integral of type 1 of f on $[a,+\infty]$ is

$$\tilde{\int}_{\tilde{a}}^{\widetilde{+\infty}} f(x)\tilde{d}x=\lim_{b\to+\infty}\tilde{\int}_{\tilde{a}}^{\tilde{b}} f(x)\tilde{d}x=\lim_{b\to+\infty} e^{\left(\int_{\ln a}^{\ln b}\ln f(e^x)dx\right)}.$$

Remark 2: Since existence of ${}^*\int_{\dot{-}\infty}^{\dot{+}\infty} f(x)\,d^*x = *-\lim\limits_{\substack{a\to\dot{-}\infty\\ b\to\dot{+}\infty}} {}^*\int_a^b f(x)\,d^*x$ is equivalent to existence of $*-\lim\limits_{a\to\dot{-}\infty} {}^*\int_a^c f(x)\,d^*x$ and $*-\lim\limits_{b\to\dot{+}\infty} {}^*\int_c^b f(x)\,d^*x$ for all $c\in\mathbb{R}(N)_\alpha$, the integral on the left is meaningful except that the two terms in the rightmost sum of the following equalities are infinite from the different signs,

$$\begin{aligned}{}^*\int_{\dot{-}\infty}^{\dot{+}\infty} f(x)\,d^*x &= *-\lim_{a\to\dot{-}\infty} {}^*\int_a^c f(x)\,d^*x \,\dot{+}\, *-\lim_{b\to\dot{+}\infty} {}^*\int_c^b f(x)\,d^*x\\ &= {}^*\int_{\dot{-}\infty}^{c} f(x)\,d^*x \,\dot{+}\, {}^*\int_{c}^{\dot{+}\infty} f(x)\,d^*x.\end{aligned}$$

Example 2: We investigate convergence condition of improper integrals $\int_1^{\widetilde{+\infty}} e^{-x}\tilde{d}x$, $\tilde{\int}_{\tilde{1}}^{\widetilde{+\infty}} e^{-x}\tilde{d}x$ and $\int_1^{+\infty} e^{-x}dx$.

Solution 1: Since

$$\int_1^{\widetilde{+\infty}} e^{-x}\tilde{d}x=\lim_{b\to+\infty}\int_1^{\tilde{b}} e^{-x}\tilde{d}x=\lim_{b\to+\infty} e^{\left(\int_1^b \ln e^{-x}dx\right)}=\lim_{b\to+\infty} e^{\left(-\frac{b^2}{2}+\frac{1}{2}\right)}=0$$

$$\tilde{\int}_{\tilde{1}}^{\widetilde{+\infty}} e^{-x}\tilde{d}x=\lim_{b\to+\infty}\tilde{\int}_{\tilde{1}}^{\tilde{b}} e^{-x}\tilde{d}x=\lim_{b\to+\infty} e^{\int_0^{\ln b}\ln e^{-\left(e^x\right)}dx}=\lim_{b\to+\infty} e^{-b+1}=0$$

and

$$\int_1^{+\infty} e^{-x}dx=\lim_{b\to+\infty}\int_1^{b} e^{-x}dx=\lim_{b\to+\infty}\left(e^{-1}-e^{-b}\right)=e^{-1},$$

the improper geometric integral $\int_1^{\widetilde{+\infty}} e^{-x}\tilde{d}x$ and the improper bigeometric integral $\tilde{\int}_{\tilde{1}}^{\widetilde{+\infty}} e^{-x}\tilde{d}x$ are divergent, but the improper classic integral $\int_1^{+\infty} e^{-x}dx$ is convergent. This shows us that convergence condition of the improper integral of type 1 of a function in classical calculus and other calculuses can be different from each other.

Example 3: The improper *-integral ${}^{*}\int_{a}^{\dot{+}\infty} \frac{\ddot{1}}{\iota(x)^{p_\beta}} \beta\, d^{*}x$ is convergent for $a \dot{>} \dot{0}$ and $p>1$:

Since

$$
\begin{aligned}
{}^{*}\int_{a}^{\dot{+}\infty} \frac{\ddot{1}}{\iota(x)^{p_\beta}} \beta\, d^{*}x &= *-\lim_{b\to\dot{+}\infty} {}^{*}\int_{a}^{b} \frac{\ddot{1}}{\iota(x)^{p_\beta}} \beta\, d^{*}x \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(\frac{\ddot{1}}{\iota(\alpha(x))^{p_\beta}}\beta\right) dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(\frac{\ddot{1}}{\beta\left(\alpha^{-1}\left(\alpha(x)\right)\right)^{p_\beta}}\beta\right) dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(\frac{\ddot{1}}{\beta\left(\left[\beta^{-1}\left(\beta\left(\alpha^{-1}\left(\alpha(x)\right)\right)\right)\right]^{p}\right)}\beta\right) dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(\frac{\ddot{1}}{\beta\left(x^{p}\right)}\beta\right) dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(\beta\left(\frac{1}{x^{p}}\right)\right) dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{1}{x^{p}}\, dx\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\frac{\alpha^{-1}(b)^{1-p} - \alpha^{-1}(a)^{1-p}}{1-p}\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\frac{\left[\beta^{-1}\left(\beta\left(\alpha^{-1}(b)\right)\right)\right]^{1-p} - \left[\beta^{-1}\left(\beta\left(\alpha^{-1}(a)\right)\right)\right]^{1-p}}{\beta^{-1}\left(\beta(1)\right) - \beta^{-1}\left(\beta(p)\right)}\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\frac{\beta^{-1}\left(\beta\left(\left[\beta^{-1}\left(\iota(b)\right)\right]^{1-p}\right)\right) - \beta^{-1}\left(\beta\left(\left[\beta^{-1}\left(\iota(a)\right)\right]^{1-p}\right)\right)}{\beta^{-1}\left(\beta(1)\right) - \beta^{-1}\left(\beta(p)\right)}\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\frac{\beta^{-1}\left(\beta\left[\beta^{-1}\left(\iota(b)^{(1-p)_\beta}\right) - \beta^{-1}\left(\iota(a)^{(1-p)_\beta}\right)\right]\right)}{\beta^{-1}\left[\beta\left(\beta^{-1}(\ddot{1}) - \beta^{-1}(\ddot{p})\right)\right]}\right] \\
&= *-\lim_{b\to\dot{+}\infty} \beta\left[\frac{\beta^{-1}\left(\iota(b)^{(1-p)_\beta} \ddot{-} \iota(a)^{(1-p)_\beta}\right)}{\beta^{-1}\left(\ddot{1} \ddot{-} \ddot{p}\right)}\right]
\end{aligned}
$$

$$= *-\lim_{b\to\dot{+}\infty}\left[\frac{\iota(b)^{(1-p)_\beta} \ddot{-} \iota(a)^{(1-p)_\beta}}{\ddot{1} \ddot{-} \ddot{p}}\beta\right]$$

$$= \frac{\iota(a)^{(1-p)_\beta}}{\ddot{p} \ddot{-} \ddot{1}}\beta$$

for each $b \dot{\geq} a$, then the improper *-integral ${}^*\int_a^{\dot{+}\infty} \frac{\ddot{1}}{\iota(x)^{p_\beta}}\beta\, d^*x$ is convergent for $a>0$ and $p>1$.

Definition 14: Let the function $f:\dot{[}a,B\dot{)} \to \mathbb{R}(N)_\beta$ with $a,B \in \mathbb{R}(N)_\alpha$ be *-continuous on $\alpha-$interval $\dot{[}a,b\dot{]}$ for each $b \in \dot{[}a,B\dot{)}$. Let f be also $\beta-$unbounded on left of the point B $\left(*-\lim_{b\to B^-} f(x) = \ddot{+}\infty \text{ or } *-\lim_{b\to B^-} f(x) = \ddot{-}\infty\right)$. The *-limit $*-\lim_{b\to B^-} {}^*\int_a^b f(x)\, d^*x$ is called improper *-integral of type 2 of the function f on $\dot{[}a,B\dot{]}$ and is denoted by ${}^*\int_a^B f(x)\, d^*x$. If $*-\lim_{b\to B^-} {}^*\int_a^b f(x)\, d^*x$ exists and equals to a number $L \in \mathbb{R}(N)_\beta$, the improper *-integral ${}^*\int_a^B f(x)\, d^*x$ is convergent (*-convergent). If this *-limit does not exist or equals to one of $\ddot{+}\infty$ and $\ddot{-}\infty$, then the improper *-integral ${}^*\int_a^B f(x)\, d^*x$ is divergent.

Similarly, if the function $f:\dot{(}A,b\dot{]} \to \mathbb{R}(N)_\beta$ with $A,b \in \mathbb{R}(N)_\alpha$ is *-continuous on $\alpha-$interval $\dot{[}a,b\dot{]}$ for each number $a \in \dot{(}A,b\dot{]}$ and $\beta-$unbounded on right of the point A, then the improper *-integral of type 2 of the function f on $\dot{[}A,b\dot{]}$ is defined as

$${}^*\int_A^b f(x)\, d^*x = *-\lim_{a\to A^+} {}^*\int_a^b f(x)\, d^*x.$$

Example 4: $\mathbb{R}(N)_\alpha = \mathbb{R}$ and $\mathbb{R}(N)_\beta = \mathbb{R}^+$ for geometric calculus. Let the function $f:[a,B) \to \mathbb{R}^+$ be continuous on $[a,b]$ for each $b \in [a,B)$ and unbounded on left of the point B. Then the improper geometric integral of type 2 of the function f on $[a,B]$ is found that

$$\tilde{\int_a^B} f(x)\tilde{d}x = \lim_{b\to B^-} \tilde{\int_a^b} f(x)\tilde{d}x = \lim_{b\to B^-} e^{\left(\int_a^b \ln(f(x))dx\right)} = e^{\lim_{b\to B^-}\left(\int_a^b \ln(f(x))dx\right)} = e^{\int_a^B \ln(f(x))dx}.$$

$\mathbb{R}(N)_\alpha = \mathbb{R}^+$ and $\mathbb{R}(N)_\beta = \mathbb{R}$ for anageometric calculus. Let the function $f:[a,B) \subset \mathbb{R}^+ \to \mathbb{R}$ be continuous on $[a,b]$ for each $b \in [a,B)$ and unbounded on left of the point B. Then the improper anageometric integral of type 2 of the function f on $[a,B]$ is found that

$$\underset{\sim}{\int_{a}^{B}} f(x)\underset{\sim}{d}x = \lim_{b\to B^-} \underset{\sim}{\int_{a}^{b}} f(x)\underset{\sim}{d}x = \lim_{b\to B^-} \int_{\ln a}^{\ln b} f(e^x)dx .$$

$\mathbb{R}(N)_\alpha = \mathbb{R}(N)_\beta = \mathbb{R}^+$ for bigeometric calculus. Let the function $f:[a,B)\subset\mathbb{R}^+\to\mathbb{R}^+$ be continuous on $[a,b]$ for each $b\in[a,B)$ and unbounded on left of the point B. Then the improper bigeometric integral of type 2 of the function f on $[a,B]$ is found that

$$\tilde{\int_{\tilde{a}}^{B}} f(x)\tilde{d}x = \lim_{b\to B^-} \tilde{\int_{\tilde{a}}^{b}} f(x)\tilde{d}x = \lim_{b\to B^-} e^{\int_{\ln a}^{\ln b}\ln f(e^x)dx} = e^{\lim_{b\to B^-}\left(\int_{\ln a}^{\ln b}\ln f(e^x)dx\right)}.$$

Example 5: We investigate convergence condition of the improper integrals $\tilde{\int_{0}^{1}}\frac{1}{x^2}\tilde{d}x$ and $\int_0^1 \frac{1}{x^2}dx$.

Solution 2: Since

$$\tilde{\int_{0}^{1}}\frac{1}{x^2}\tilde{d}x = \lim_{a\to 0^+}\tilde{\int_{a}^{1}}\frac{1}{x^2}\tilde{d}x = \lim_{a\to 0^+} e^{\left(\int_a^1 \ln\left(\frac{1}{x^2}\right)dx\right)} = \lim_{a\to 0^+} e^{-2(-1-a.\ln a+a)} = e^2.1.\lim_{a\to 0^+} a^{2a} = e^2$$

and

$$\int_0^1 \frac{1}{x^2}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}dx = \lim_{a\to 0^+}\left(-1+\frac{1}{a}\right) = +\infty,$$

the improper geometric integral $\tilde{\int_{0}^{1}}\frac{1}{x^2}\tilde{d}x$ is convergent, but the improper integral $\int_0^1\frac{1}{x^2}dx$ is divergent. This shows that convergence conditions of the improper integral of type 2 of a function in classical calculus and other calculuses can be different from each other.

Definition 15: Let the function $f:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_\beta$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$ where $a\in\mathbb{R}(N)_\alpha$ and $b\in\overline{\mathbb{R}}(N)_\alpha = \dot{[}\dot{-}\infty,\dot{+}\infty\dot{]}$. Let f be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_\alpha$. If

$$*-\lim_{b\to B^-}{}^{*}\!\!\int_a^b f(x)\,d^*x$$

exists and $\beta-$finite, then

$${}^{*}\!\!\int_a^B f(x)\,d^*x = *-\lim_{b\to B^-}{}^{*}\!\!\int_a^b f(x)\,d^*x$$

is written and it is said that the improper *-integral ${}^{*}\int_{a}^{B} f(x)\,d^{*}x$ is convergent(*-convergent).

Similar definition is also given for $\dot{(}A,b\dot{]}$.

Theorem 3: Let the functions $f,g:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_{\beta}$ with $a\in\mathbb{R}(N)_{\alpha}$ and $B\in\overline{\mathbb{R}}(N)_{\alpha}$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$, and be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_{\alpha}$. Let the improper *-integrals ${}^{*}\int_{a}^{B} f(x)\,d^{*}x$ and ${}^{*}\int_{a}^{B} g(x)\,d^{*}x$ are convergent. Then the improper integral of the function $\left((\lambda\ddot{\times}f)\ddot{+}(\mu\ddot{\times}g)\right)$ on $\dot{[}a,B\dot{]}$ is

$$ {}^{*}\int_{a}^{B}\left((\lambda\ddot{\times}f)\ddot{+}(\mu\ddot{\times}g)\right)(x)\,d^{*}x=\lambda\ddot{\times}\,{}^{*}\int_{a}^{B} f(x)\,d^{*}x\ddot{+}\mu\ddot{\times}\,{}^{*}\int_{a}^{B} g(x)\,d^{*}x $$

for all $\lambda,\mu\in\mathbb{R}(N)_{\beta}$.

Proof: Since the *-integral is $\beta-$additive and $\beta-$homogeneous, it is written

$$ {}^{*}\int_{a}^{b}\left((\lambda\ddot{\times}f)\ddot{+}(\mu\ddot{\times}g)\right)(x)\,d^{*}x=\lambda\ddot{\times}\,{}^{*}\int_{a}^{b} f(x)\,d^{*}x\ddot{+}\mu\ddot{\times}\,{}^{*}\int_{a}^{b} g(x)\,d^{*}x. $$

If we take *-limit as $b\to B^{-}$ in this equality, then

$$ *-\lim_{b\to B^{-}}\left[{}^{*}\int_{a}^{b}\left((\lambda\ddot{\times}f)\ddot{+}(\mu\ddot{\times}g)\right)(x)\,d^{*}x\right]=*-\lim_{b\to B^{-}}\left[\lambda\ddot{\times}\,{}^{*}\int_{a}^{b} f(x)\,d^{*}x\ddot{+}\mu\ddot{\times}^{*}\int_{a}^{b} g(x)\,d^{*}x\right] $$

and thus

$$ \begin{aligned} {}^{*}\int_{a}^{B}\left((\lambda\ddot{\times}f)\ddot{+}(\mu\ddot{\times}g)\right)(x)\,d^{*}x&=\lambda\ddot{\times}*-\lim_{b\to B^{-}}\,{}^{*}\int_{a}^{b} f(x)\,d^{*}x\ddot{+}\mu\ddot{\times}*-\lim_{b\to B^{-}}\,{}^{*}\int_{a}^{b} g(x)\,d^{*}x\\ &=\lambda\ddot{\times}^{*}\int_{a}^{B} f(x)\,d^{*}x\ddot{+}\mu\ddot{\times}\,{}^{*}\int_{a}^{B} g(x)\,d^{*}x. \end{aligned} $$

Hence theorem is proved.

Theorem 4: Let the functions $f,g:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_{\beta}$ with $a\in\mathbb{R}(N)_{\alpha}$ and $B\in\overline{\mathbb{R}}(N)_{\alpha}$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$, and be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_{\alpha}$. If $c\in\dot{[}a,b\dot{)}$, then the equality

$$ {}^{*}\int_{a}^{B} f(x)\,d^{*}x={}^{*}\int_{a}^{c} f(x)\,d^{*}x\ddot{+}\,{}^{*}\int_{c}^{B} f(x)\,d^{*}x $$

holds.

Proof: By the property of *-integral, it can be written

$$^{*}\int_{a}^{b} f(x)\,d^{*}x = {}^{*}\int_{a}^{c} f(x)\,d^{*}x \ddot{+} {}^{*}\int_{c}^{b} f(x)\,d^{*}x$$

for each $b \in \dot{[}a,B\dot{)}$ and $c \in \dot{[}a,b\dot{)}$. If we take *-limit as $b \to B^{-}$ in this equality, then proof is completed.

Theorem 5: If the function $f : \dot{[}a,B\dot{)} \to \mathbb{R}(N)_{\beta}$ with $B \in \overline{\mathbb{R}}(N)_{\alpha}$ is *-nondecreasing, then the *-limit

$$* - \lim_{x \to B^{-}} f(x) = L = {}^{\beta}\sup\left\{ f(x) : x \in \dot{[}a,B\dot{)} \right\}$$

exists. Further, if the function f is $\beta -$bounded from above, then $L \in \mathbb{R}(N)_{\beta}$, otherwise $L = \ddot{+}\infty$.

Proof: a) First, consider the function f is $\beta -$bounded from above. It is obvious that $L = {}^{\beta}\sup\left\{ f(x) : x \in \dot{[}a,B\dot{)} \right\} \in \mathbb{R}(N)_{\beta}$. We show that $* - \lim_{x \to B^{-}} f(x) = L$. Any number $\varepsilon \ddot{>} \ddot{0}$ is given. By the property of $\beta -$supremum, following statements hold:

1) $f(x) \ddot{\leq} L$ for all $x \in \dot{[}a,B\dot{)}$,

2)at least a $x_{\varepsilon} \in \dot{[}a,B\dot{)}$ exist such that $f(x_{\varepsilon}) \ddot{>} L \ddot{-} \varepsilon$.

When $B \in \mathbb{R}(N)_{\alpha}$, if $\delta = B \dot{-} x_{\varepsilon} \dot{>} \dot{0}$ is taken, then

$$L \ddot{-} \varepsilon \ddot{<} f(x_{\varepsilon}) \ddot{\leq} f(x) \ddot{\leq} L \ddot{<} L \ddot{+} \varepsilon,$$

namely,

$$\left| f(x) \ddot{-} L \right|_{\beta} \ddot{<} \varepsilon$$

for all $x \in \dot{[}a,B\dot{)}$ such that $B \dot{-} \delta \dot{<} x \dot{<} B$, because of the function f is *-nondecreasing. This shows that $* - \lim_{x \to B^{-}} f(x) = L$.

When $B = \dot{+}\infty$, if $\delta = x_{\varepsilon} \in \dot{[}a,B\dot{)}$ is taken,

$$L \ddot{-} \varepsilon \ddot{<} f(x_{\varepsilon}) \ddot{\leq} f(x) \ddot{\leq} L \ddot{<} L \ddot{+} \varepsilon,$$

namely,

$$\left| f(x) \ddot{-} L \right|_{\beta} \ddot{<} \varepsilon$$

for all $x \dot{>} \delta$, because of the function f is *-nondecreasing. This shows that $* - \lim_{x \to B^{-}} f(x) = L$.

b)Second, consider the function f is $\beta-$unbounded from above. We will show that $*-\lim\limits_{x\to B^-} f\left(x\right)=\ddot{+}\infty$. Any number $\varepsilon \ddot{>} \ddot{0}$ is given. There exists a $x_\varepsilon \in [\dot{a},\dot{B})$ such that $f\left(x_\varepsilon\right) \ddot{>} \varepsilon$. When $B\in\mathbb{R}(N)_\alpha$, if $\delta = B \dot{-} x_\varepsilon \dot{>} \dot{0}$ is taken, then one can find that $f\left(x\right) \ddot{\geq} f\left(x_\varepsilon\right) \ddot{>} \varepsilon$ for all $x\in[\dot{a},\dot{B})$ such that $B \dot{-} \delta \dot{<} x \dot{<} B$, because of the function f is *-nondecreasing. This shows that $*-\lim\limits_{x\to B^-} f\left(x\right)=\ddot{+}\infty$. When $B=\dot{+}\infty$, if $\delta = x_\varepsilon \in [\dot{a},\dot{B})$ is taken, then we have $f\left(x\right) \ddot{\geq} f\left(x_\varepsilon\right) \ddot{>} \varepsilon$ for all $x \dot{>} \delta$ because of the function f is *-nondecreasing. This shows that $*-\lim\limits_{x\to B^-} f\left(x\right)=\ddot{+}\infty$, hence proof is completed.

Theorem 6: (Comparison test) Let the functions $f,g:[\dot{a},\dot{B})\to\mathbb{R}(N)_\beta$ with $a\in\mathbb{R}(N)_\alpha$ and $B\in\overline{\mathbb{R}}(N)_\alpha$ be *-continuous on each $[\dot{a},\dot{b}]\subset[\dot{a},\dot{B})$, and be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_\alpha$. Let $\ddot{0}\ddot{\leq} f\left(x\right)\ddot{\leq} C\ddot{\times} g\left(x\right)$ for all $x\in[\dot{a},\dot{B})$. If ${}^*\!\int_a^B g\left(x\right)d^*x$ is convergent, then ${}^*\!\int_a^B f\left(x\right)d^*x$ is convergent and the inequality ${}^*\!\int_a^B f\left(x\right)d^*x \ddot{\leq} C\ddot{\times}\,{}^*\!\int_a^B g\left(x\right)d^*x$ holds.

Proof: Let the functions $F,G:[\dot{a},\dot{B})\to\mathbb{R}(N)_\beta$ be defined as $F\left(b\right)={}^*\!\int_a^b f\left(x\right)d^*x$ and $G\left(b\right)={}^*\!\int_a^b g\left(x\right)d^*x$ for each $b\in[\dot{a},\dot{B})$. According to $\ddot{0}\ddot{\leq} f\left(x\right)\ddot{\leq} C\ddot{\times} g\left(x\right)$ for all $x\in[\dot{a},\dot{B})$ and *-integrals property,

$${}^*\!\int_a^b \ddot{0}\,d^*x \ddot{\leq} {}^*\!\int_a^b f\left(x\right)d^*x \ddot{\leq} {}^*\!\int_a^b C\ddot{\times} g\left(x\right)d^*x$$

for each $b\in[\dot{a},\dot{B})$. Therefore

$$\left(\iota\left(b\right)\dot{-}\iota\left(a\right)\right)\ddot{\times}\ddot{0} \ddot{\leq} {}^*\!\int_a^b f\left(x\right)d^*x \ddot{\leq} C\ddot{\times}\,{}^*\!\int_a^b g\left(x\right)d^*x$$

and hence

$$\ddot{0}\ddot{\leq} F\left(b\right)\ddot{\leq} C\ddot{\times} G\left(b\right).$$

Since $\ddot{0}\ddot{<} f\left(x\right)$ for all $x\in[\dot{a},\dot{B})$, we have

$$\begin{aligned} F\left(s\right)\ddot{-}F\left(r\right) &= {}^*\!\int_a^s f\left(x\right)d^*x \ddot{-} {}^*\!\int_a^r f\left(x\right)d^*x \\ &= {}^*\!\int_r^s f\left(x\right)d^*x \ddot{+} {}^*\!\int_a^r f\left(x\right)d^*x \ddot{-} {}^*\!\int_a^r f\left(x\right)d^*x \end{aligned}$$

$$= {}^*\int_r^s f(x)\,d^*x$$

$$\ddot{\geq}\, {}^*\int_r^s \ddot{0}\,d^*x$$

for all $a \dot{\leq} r \dot{\leq} s \dot{<} B$. Hence $F(s) \ddot{\geq} F(r)$, namely, the function F is *-nondecreasing. Similarly, the function G is also *-nondecreasing. By virtue of ${}^*\int_a^B g(x)\,d^*x$ is convergent and theorem 5, the function G is $\beta-$bounded from above. In other words, there exists a number $L \ddot{\geq} \ddot{0}$ such that $G(b) \ddot{\leq} L$ for all $b \in [a,B\dot{)}$. Then the function F is $\beta-$bounded from above since $F(b) \ddot{\leq} C \ddot{\times} G(b) \ddot{\leq} C \ddot{\times} L$. So, by virtue of theorem 5, ${}^*\int_a^B f(x)\,d^*x$ is convergent. Furthermore, ${}^*\int_a^B f(x)\,d^*x \ddot{\leq} C \ddot{\times} {}^*\int_a^B g(x)\,d^*x$ if we take *-limit as $b \to B^-$ of the inequality $F(b) \ddot{\leq} C \ddot{\times} G(b)$.

Theorem 7: (Quotient test/*-limit comparison test) Let the functions $f,g : [a,B\dot{)} \to \mathbb{R}^+(N)_\beta \cup \{\ddot{0}\}$ with $a \in \mathbb{R}(N)_\alpha$ and $B \in \overline{\mathbb{R}}(N)_\alpha$ be *-continuous on each $[a,b] \subset [a,B\dot{)}$, and be $\beta-$unbounded on left of the point B if $B \in \mathbb{R}(N)_\alpha$. Let the *-limit

$$*-\lim_{x\to B^-} \frac{f(x)}{g(x)}\beta = L, \left(\ddot{0} \ddot{\leq} L \ddot{\leq} \ddot{+}\infty\right)$$

exist. In this case;

a) ${}^*\int_a^B f(x)\,d^*x$ is convergent if ${}^*\int_a^B g(x)\,d^*x$ is convergent when $\ddot{0} \ddot{\leq} L \ddot{<} \ddot{+}\infty$,

b) ${}^*\int_a^B f(x)\,d^*x$ is divergent if ${}^*\int_a^B g(x)\,d^*x$ is divergent when $\ddot{0} \ddot{<} L \ddot{\leq} \ddot{+}\infty$,

c) the improper *-integrals ${}^*\int_a^B f(x)\,d^*x$ and ${}^*\int_a^B g(x)\,d^*x$ are both convergent or both divergent when $\ddot{0} \ddot{<} L \ddot{<} \ddot{+}\infty$.

Proof: a)Since $*-\lim_{x\to B^-} \frac{f(x)}{g(x)}\beta = L$, for $\varepsilon = \ddot{1}$ there exists a number $\delta \dot{>} \dot{0}$ such that $\left|\frac{f(x)}{g(x)}\beta \ddot{-} L\right|_\beta \ddot{<} \ddot{1}$ for all $x \in [a,B\dot{)}$ with $B \dot{-} \delta \dot{<} x \dot{<} B$, for all $x \dot{>} \delta$ if $B = \dot{+}\infty$. According to this, there exists a number $c \in (B-\delta, B\dot{)}$, $c \dot{>} \delta$ if $B = \dot{+}\infty$, such that $L \ddot{-} \ddot{1} \ddot{<} \frac{f(x)}{g(x)}\beta \ddot{<} L \ddot{+} \ddot{1}$ and therefore $f(x) < (L \ddot{+} \ddot{1}) \ddot{\times} g(x)$ for every $x \in [c,B\dot{)}$. In accordance with theorem 4, ${}^{*}\int_{c}^{B} g(x)\,d^{*}x$ is convergent since ${}^{*}\int_{a}^{B} g(x)\,d^{*}x$ is convergent. By virtue of theorem 6, ${}^{*}\int_{c}^{B} f(x)\,d^{*}x$ is convergent. Again by theorem 4, ${}^{*}\int_{a}^{B} f(x)\,d^{*}x$ is also convergent.

b) If $\ddot{0} \ddot{<} L \ddot{<} \ddot{+}\infty$, for a fixed number $\varepsilon \ddot{>} \ddot{0}$ such that $\ddot{0} \ddot{<} \varepsilon \ddot{<} L$ there exists a number $\delta \dot{>} \dot{0}$ such that $\left| \frac{f(x)}{g(x)} \beta \ddot{-} L \right|_{\beta} \ddot{<} \varepsilon$ for all $x \in \dot{[}a, B\dot{)}$ with $B \dot{-} \delta \dot{<} x \dot{<} B$, for all $x \dot{>} \delta$ if $B = \dot{+}\infty$.

According to this, there exists a number $c \in \dot{(}B - \delta, B\dot{)}$, $c \dot{>} \delta$ if $B = \dot{+}\infty$, such that $L \ddot{-} \varepsilon \ddot{<} \frac{f(x)}{g(x)} \beta \ddot{<} \varepsilon \ddot{+} L$ and hence $g(x) \ddot{<} \frac{\ddot{1}}{L \ddot{-} \varepsilon} \beta \ddot{\times} f(x)$ for all $x \in \dot{[}c, B\dot{)}$. By theorem 4, ${}^{*}\int_{c}^{B} g(x)\,d^{*}x$ is divergent since ${}^{*}\int_{a}^{B} g(x)\,d^{*}x$ is divergent. In this case, by virtue of theorem 6, ${}^{*}\int_{c}^{B} f(x)\,d^{*}x$ is divergent. Then, again by theorem 4, ${}^{*}\int_{a}^{B} f(x)\,d^{*}x$ is also divergent.

If $L = \ddot{+}\infty$, for any fixed number $\varepsilon \ddot{>} \ddot{0}$, there exists a number $\delta \dot{>} \dot{0}$ such that $\varepsilon \ddot{<} \frac{f(x)}{g(x)} \beta$ for every $x \in \dot{[}a, B\dot{)}$ with $B \dot{-} \delta \dot{<} x \dot{<} B$, for all $x \dot{>} \delta$ if $B = \dot{+}\infty$. According to this, there exists a number $c \in \dot{(}B - \delta, B\dot{)}$, $c \dot{>} \delta$ if $B = \dot{+}\infty$, such that $g(x) \ddot{<} \frac{\ddot{1}}{\varepsilon} \beta \ddot{\times} f(x)$ for all $x \in \dot{[}c, B\dot{)}$. By theorem 4, ${}^{*}\int_{c}^{B} g(x)\,d^{*}x$ is divergent since ${}^{*}\int_{a}^{B} g(x)\,d^{*}x$ is divergent. In this case, by virtue of theorem 6, ${}^{*}\int_{c}^{B} f(x)\,d^{*}x$ is divergent. Then, again by theorem 4, ${}^{*}\int_{a}^{B} f(x)\,d^{*}x$ is also divergent.

c)The proof is obvious from (a) and (b).

Example 6: Show that the improper bigeometric integral $\int_{\tilde{e}}^{\tilde{+\infty}} e^{\left(\frac{1}{\ln x.\sqrt{1+(\ln x)^2}}\right)} \tilde{d}x$ is convergent.

Solution 3: We know that $\int_{\tilde{e}}^{\tilde{+\infty}} e^{\frac{1}{(\ln x)^2}} \tilde{d}x$ is convergent from example 3. Using the quotient test, we have

$$*-\lim_{x\to\infty}\left[e^{\left(\frac{1}{\ln x.\sqrt{1+(\ln x)^2}}\right)}\right]^{\frac{1}{\ln e^{\frac{1}{(\ln x)^2}}}}=\lim_{x\to\infty}e^{\frac{\frac{1}{\ln x.\sqrt{1+(\ln x)^2}}}{\frac{1}{(\ln x)^2}}}=\lim_{x\to\infty}e^{\frac{\ln x}{\sqrt{1+(\ln x)^2}}}=e.$$

Hence the improper bigeometric integral $\int_{\tilde{e}}^{\widetilde{+\infty}} e^{\left(\frac{1}{\ln x.\sqrt{1+(\ln x)^2}}\right)}\tilde{d}x$ is convergent.

Theorem 8: Let the function $f:[\dot{a},\dot{B})\to\mathbb{R}(N)_\beta$ with $a\in\mathbb{R}(N)_\alpha$ and $B\in\overline{\mathbb{R}}(N)_\alpha$ be *-continuous on each $[\dot{a},\dot{b}]\subset[\dot{a},\dot{B})$, and let the function f be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_\alpha$. If the improper *-integral ${}^*\int_a^B \left|f(x)\right|_\beta d^*x$ is convergent, then ${}^*\int_a^B f(x)\,d^*x$ is convergent.

Proof: Form the functions $f^+,f^-:[\dot{a},\dot{B})\to\mathbb{R}(N)_\beta$ as

$$f^+(x)=\frac{\left|f(x)\right|_\beta \ddot{+} f(x)}{\ddot{2}}\beta \text{ and } f^-(x)=\frac{\left|f(x)\right|_\beta \ddot{-} f(x)}{\ddot{2}}\beta$$

for given function f. By theorem 6, the improper *-integrals ${}^*\int_a^B f^+(x)\,d^*x$ and ${}^*\int_a^B f^-(x)\,d^*x$ are convergent since $\ddot{0}\ddot{\le} f^+(x)\ddot{\le}\left|f(x)\right|_\beta$ and $\ddot{0}\ddot{\le} f^-(x)\ddot{\le}\left|f(x)\right|_\beta$ for all $x\in[\dot{a},\dot{B})$. On the other hand, if we use the equality $f(x)=f^+(x)\ddot{-}f^-(x)$ and refer to theorem 3, we obtain

$${}^*\int_a^B f(x)\,d^*x={}^*\int_a^B f^+(x)\,d^*x\ddot{-}{}^*\int_a^B f^-(x)\,d^*x.$$

Then ${}^*\int_a^B f(x)\,d^*x$ is convergent.

Definition 16: If the improper *-integral ${}^*\int_a^B \left|f(x)\right|_\beta d^*x$ is convergent, then the improper *-integral ${}^*\int_a^B f(x)\,d^*x$ is said to be absolute convergent($\beta-$absolute *-convergent). However, if the improper *-integral ${}^*\int_a^B f(x)\,d^*x$ is convergent but the improper *-integral ${}^{*}\int_a^B \left|f(x)\right|_\beta d^*x$ is divergent, then the improper *-integral ${}^{*}\int_a^B f(x)\,d^*x$ is said to be conditional convergent(conditional *-convergent).

In accordance with theorem 8, every absolutely convergent integral is also convergent.

Corollary 1: (Comparison test) Let the functions $f:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_\beta$ and $g:\dot{[}a,B\dot{)}\to\mathbb{R}^+(N)_\beta\cup\{\ddot{0}\}$ with $a\in\mathbb{R}(N)_\alpha$ and $B\in\overline{\mathbb{R}}(N)_\alpha$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$, and let the functions f and g be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_\alpha$. Let $\left|f(x)\right|_\beta \ddot{\le} g(x)$ for all $x\in\dot{[}a,B\dot{)}$. If ${}^{*}\int_a^B g(x)\,d^*x$ is convergent, then ${}^{*}\int_a^B f(x)\,d^*x$ is convergent.

Example 7 We investigate convergence condition of improper bigeometric integral $\tilde{\int}_{\tilde{e}}^{\widetilde{+\infty}} e^{\frac{\cos(\ln x)}{(\ln x)^2}}\tilde{d}x$.

Solution 4: By the example 3, the improper bigeometric integral $\tilde{\int}_{\tilde{e}}^{\widetilde{+\infty}} e^{\frac{1}{(\ln x)^2}}\tilde{d}x$ is convergent. Since $\left|\frac{\cos(\ln x)}{(\ln x)^2}\right|\le\frac{1}{(\ln x)^2}$, we have

$$\left|e^{\left(\frac{\cos(\ln x)}{(\ln x)^2}\right)}\right|_\beta = e^{\left|\frac{\cos(\ln x)}{(\ln x)^2}\right|} \ddot{\le} e^{\left|\frac{1}{(\ln x)^2}\right|} = \left|e^{\left(\frac{1}{(\ln x)^2}\right)}\right|_\beta$$

for all $x\in[e,+\infty)$. Thus, in view of corollary 1, the improper bigeometric integral $\tilde{\int}_{\tilde{e}}^{\widetilde{+\infty}} e^{\frac{\cos(\ln x)}{(\ln x)^2}}\tilde{d}x$ is convergent.

Theorem 9: (Cauchy criterion) For the function $f:X\subset\mathbb{R}(N)_\alpha\to\mathbb{R}(N)_\beta$, $*-\lim_{x\to a} f(x)=A\in\mathbb{R}(N)_\beta$ iff when any number $\varepsilon\ddot{>}\ddot{0}$ is given, there exists at least a number $\delta(\varepsilon)\dot{>}\dot{0}$ such that $\left|f(x')\ddot{-}f(x'')\right|_\beta \ddot{<}\varepsilon$ for all $x',x''\in X$ with $\left|x'\dot{-}a\right|_\alpha\dot{<}\delta$ and $\left|x''\dot{-}a\right|_\alpha\dot{<}\delta$.

Proof: Suppose that $*-\lim_{x\to a} f(x)=A\in\mathbb{R}(N)_\beta$ is exists. Then, when any number $\varepsilon\ddot{>}\ddot{0}$ is given, there exists at least a number $\delta(\varepsilon)\dot{>}\dot{0}$ such that $\left|f(x')\ddot{-}A\right|_\beta\ddot{<}\frac{\varepsilon}{\ddot{2}}\beta$ and

$\left|f\left(x''\right)\ddot{-}A\right|_{\beta}\ddot{<}\frac{\varepsilon}{\ddot{2}}\beta$ for all $x',x''\in X$ with $\left|x'\dot{-}a\right|_{\alpha}\dot{<}\delta$ and $\left|x''\dot{-}a\right|_{\alpha}\dot{<}\delta$. For the same numbers x' and x'', we have

$$\begin{aligned}\left|f\left(x'\right)\ddot{-}f\left(x''\right)\right|_{\beta}&=\left|f\left(x'\right)\ddot{-}A\ddot{+}A\ddot{-}f\left(x''\right)\right|_{\beta}\\&\ddot{\leq}\left|f\left(x'\right)\ddot{-}A\right|_{\beta}\ddot{+}\left|f\left(x'\right)\ddot{-}A\right|_{\beta}\\&\ddot{<}\frac{\varepsilon}{\ddot{2}}\beta\ddot{+}\frac{\varepsilon}{\ddot{2}}\beta=\varepsilon .\end{aligned}$$

Conversely suppose that when any number $\varepsilon\ddot{>}\ddot{0}$ is given, there exists at least a number $\delta(\varepsilon)\dot{>}\dot{0}$ such that $\left|f\left(x'\right)\ddot{-}f\left(x''\right)\right|_{\beta}\ddot{<}\varepsilon$ for all $x',x''\in X$ with $\left|x'\dot{-}a\right|_{\alpha}\dot{<}\delta$ and $\left|x''\dot{-}a\right|_{\alpha}\dot{<}\delta$. Let sequence $\left(x_n\right)$, which holds $x_n\in X\setminus\{a\}$ and ${}^{\alpha}\lim_{n\to\infty}x_n=a$ for all $n\in\mathbb{N}$, be given. Since ${}^{\alpha}\lim_{n\to\infty}x_n=a$, there exist a $n_{\varepsilon}\in\mathbb{N}$ such that $x_{n'},x_{n''}\in X$, $\left|x_{n'}\dot{-}a\right|_{\alpha}\dot{<}\delta$ and $\left|x_{n''}\dot{-}a\right|_{\alpha}\dot{<}\delta$ for all $n',n''>n_{\varepsilon}$. Then, by the hypothesis, $\left|f\left(x_{n'}\right)\ddot{-}f\left(x_{n''}\right)\right|_{\beta}\ddot{<}\varepsilon$ for all numbers n' and n''. Hence the sequence $\left(f\left(x_n\right)\right)$ is $\beta-$Cauchy sequence in the space $\mathbb{R}(N)_{\beta}$. Since $\mathbb{R}(N)_{\beta}$ is complete, the sequence $\left(f\left(x_n\right)\right)$ $\beta-$converges to an element of this space. If we set ${}^{\beta}\lim_{n\to\infty}f\left(x_n\right)=A$, by the definition of *-limit, we find that $*-\lim_{x\to a}f\left(x\right)=A$. This proves the theorem.

The proof of the following theorem can easily seen from theorem 9.

Theorem 10: (Cauchy's test) Let the function $f:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_{\beta}$ with $a\in\mathbb{R}(N)_{\alpha}$ and $B\in\overline{\mathbb{R}}(N)_{\alpha}$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$, and let the function f be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_{\alpha}$. Then, the improper *-integral ${}^{*}\int_{a}^{B}f\left(x\right)d^{*}x$ is convergent iff when every number $\varepsilon\ddot{>}\ddot{0}$ is given, there exist one $b_0=b_0\left(\varepsilon\right)\in\dot{[}a,B\dot{)}$ such that

$$\left|{}^{*}\int_{b_1}^{b_2}f\left(x\right)d^{*}x\right|_{\beta}\ddot{<}\varepsilon$$

for all $b_1,b_2\in\dot{[}b_0,B\dot{)}$.

Theorem 11 Let a function $f:\dot{[}a,b\dot{]}\subset\mathbb{R}(N)_{\alpha}\to\mathbb{R}(N)_{\beta}$ be *-continuous. At this situation, the following statements are hold:

(i) The function f is $\beta-$bounded on $\dot{[}a,b\dot{]}$.

(ii) There exist numbers $x_m,x_M\in\dot{[}a,b\dot{]}$ such that $f\left(x_m\right)=m\ddot{\leq}f\left(x\right)\ddot{\leq}M=f\left(x_M\right)$ for all $x\in\dot{[}a,b\dot{]}$.

(iii) $\left| {}^{*}\int_{a}^{b} f(x)\, d^{*}x \right|_{\beta} \ddot{\leq} {}^{*}\int_{a}^{b} \left| f(x) \right|_{\beta} d^{*}x$.

Proof: Let the function f be *-continuous on $\dot{[}a,b\dot{]}$.

(i) By remark 1's first item, the function $\overline{f} = \beta^{-1} \circ f \circ \alpha$ is continuous on $\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$ since the function f is *-continuous on $\dot{[}a,b\dot{]}$. Therefore the function $\overline{f}$ is bounded on $\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$. Then, there exists a number $C > 0$ such that $\left|\beta^{-1}\left(f\left(\alpha(t)\right)\right)\right| \leq C$ for all $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$. For all $x \in \dot{[}a,b\dot{]}$, there exists one and only one $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$ such that $x = \alpha(t)$, and if we take $K = \beta(C)$, $K \ddot{>} \ddot{0}$ since $C > 0$. We infer that $\left|\beta^{-1}\left(f(x)\right)\right| \leq C$ for all $x \in \dot{[}a,b\dot{]}$ from $\left|\beta^{-1}\left(f\left(\alpha(t)\right)\right)\right| \leq C$ for all $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$, therefore we obtain that $\beta\left(\left|\beta^{-1}\left(f(x)\right)\right|\right) \leq \beta(C)$. This shows that $\left|f(x)\right|_{\beta} \ddot{\leq} K$ for all $x \in \dot{[}a,b\dot{]}$, namely, the function f is $\beta-$bounded on $\dot{[}a,b\dot{]}$.

(ii) By remark 1's first item, the function $\overline{f} = \beta^{-1} \circ f \circ \alpha$ is continuous on $\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$ since the function f is *-continuous on $\dot{[}a,b\dot{]}$. Then there exist numbers $t_m, t_M \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$ such that $\overline{f}(t_m) \ddot{\leq} \overline{f}(t) \ddot{\leq} \overline{f}(t_M)$ for all $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$. For all $x \in \dot{[}a,b\dot{]}$, there exists one and only one $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$ such that $x = \alpha(t)$, and if $\alpha(t_m) = x_m$, $\alpha(t_M) = x_M$ are taken, then $x_m, x_M \in \dot{[}a,b\dot{]}$. Since $\beta^{-1}\left(f\left(\alpha(t_m)\right)\right) \leq \beta^{-1}\left(f\left(\alpha(t)\right)\right) \leq \beta^{-1}\left(f\left(\alpha(t_M)\right)\right)$ for all $t \in \left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$, in view of the definition of $\beta-$order, we have

$$f(x_m) = f\left(\alpha(t_m)\right) \ddot{\leq} f(x) \ddot{\leq} f\left(\alpha(t_M)\right) = f(x_M)$$

for all $x \in \dot{[}a,b\dot{]}$.

(iii) By proposition 2, $\ddot{0} \ddot{-} \left|f(x)\right|_{\beta} \ddot{\leq} f(x) \ddot{\leq} \left|f(x)\right|_{\beta}$, since $\left|f(x)\right|_{\beta} \ddot{\leq} \left|f(x)\right|_{\beta}$ for all $x \in \dot{[}a,b\dot{]}$. Therefore, by *-integrals property and $\beta-$homogeneity,

$$ {}^{*}\int_{a}^{b} \left(\ddot{0} \ddot{-} \left|f(x)\right|_{\beta}\right) d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} f(x)\, d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x $$

$$ {}^{*}\int_{a}^{b} \left(\left(\ddot{0} \ddot{-} \ddot{1}\right) \ddot{\times} \left|f(x)\right|_{\beta}\right) d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} f(x)\, d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x $$

$$ \left(\ddot{0} \ddot{-} \ddot{1}\right) \ddot{\times} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} f(x)\, d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x $$

$$ \ddot{0} \ddot{-} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} f(x)\, d^{*}x \ddot{\leq} {}^{*}\int_{a}^{b} \left|f(x)\right|_{\beta} d^{*}x $$

is written. Then, by virtue of last inequality and first statement of proposition 2, $\left| {}^{*}\int_a^b f(x)\,d^*x \right|_\beta \ddot{\leq} {}^{*}\int_a^b \left|f(x)\right|_\beta d^*x$. This completes the proof.

Theorem 12: (Intermediate value theorem) Let the function $f:\mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be *-continuous on $[a,b]$ and let $f(a)\neq f(b)$. then there exist a number $c\in[a,b]$ such that $f(c)=D$ for all $D\in[f(a),f(b)]$ or $D\in[f(b),f(a)]$.

Proof: By remark 1's first item, the function $\overline{f}=\beta^{-1}\circ f\circ\alpha$ is continuous on $\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]$ since the function f is *-continuous on $[a,b]$. There exists at least a number $k\in\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]$ such that $\overline{f}(k)=\beta^{-1}\left(f\left(\alpha(k)\right)\right)=\beta^{-1}(D)$ since

$$\beta^{-1}(D)\in\left[\beta^{-1}\left(f(a)\right),\beta^{-1}\left(f(b)\right)\right]=\left[\overline{f}\left(\alpha^{-1}(a)\right),\overline{f}\left(\alpha^{-1}(b)\right)\right]$$

or

$$\beta^{-1}(D)\in\left[\overline{f}\left(\alpha^{-1}(b)\right),\overline{f}\left(\alpha^{-1}(a)\right)\right]$$

for all $D\in[f(a),f(b)]$ or $D\in[f(b),f(a)]$. If $c=\alpha(k)$ is taken, then $c=\alpha(k)\in[a,b]$. Therefore we obtain $f(c)=D$ since $\beta^{-1}\left(f\left(\alpha(k)\right)\right)=\beta^{-1}(D)$.

Theorem 13: (Second mean value theorem) Let the functions $f,g:[a,b]\subset\mathbb{R}(N)_\alpha\to\mathbb{R}(N)_\beta$ be *-continuous on $[a,b]$.

a) If the function g is *-nonincreasing and $g(x)\ddot{\geq}\ddot{0}$ for all $x\in[a,b]$, then there exist point $\xi\in[a,b]$ such that

$${}^{*}\int_a^b f(x)\ddot{\times}g(x)\,d^*x=g(a)\ddot{\times}\,{}^{*}\int_a^\xi f(x)\,d^*x.$$

b) If the function g is *-nondecreasing and $g(x)\ddot{\geq}\ddot{0}$ for all $x\in[a,b]$, then there exist point $\eta\in[a,b]$ such that

$${}^{*}\int_a^b f(x)\ddot{\times}g(x)\,d^*x=g(b)\ddot{\times}\,{}^{*}\int_\eta^b f(x)\,d^*x.$$

Proof: a) The $n-$fold $\alpha-$partition $\{x_1,x_2,...,x_n\}$ of $[a,b]$ is given. Then

$$
\begin{aligned}
{}^{*}\!\int_a^b f(x)\ddot{\times} g(x)\,d^*x &= {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} f(x)\ddot{\times} g(x)\,d^*x \right) \\
&= {}_{\beta}\sum_{k=1}^{n-1}\left(g(x_k)\ddot{\times}\, {}^{*}\!\int_{x_k}^{x_{k+1}} f(x)\,d^*x \right) \ddot{+}\, {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} \left[g(x)\ddot{-} g(x_k)\right]\ddot{\times} f(x)\,d^*x \right) \\
&= S_n^{(1)} \ddot{+} S_n^{(2)}.
\end{aligned}
$$

Now we show that $S_n^{(2)} \xrightarrow{\beta} \ddot{0}$ as $n \to \infty$. Since the function f is *-continuous on $[\dot{a}, \dot{b}]$, f is $\beta-$bounded, therefore there exist a number $K \ddot{>} \ddot{0}$ such that $\left|f(x)\right|_{\beta} \ddot{\leq} K$ for all $x \in [\dot{a}, \dot{b}]$. Then

$$
\begin{aligned}
\left|S_n^{(2)}\right|_{\beta} &= \left| {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} \left[g(x)\ddot{-} g(x_k)\right]\ddot{\times} f(x)\,d^*x \right)\right|_{\beta} \\
&\ddot{\leq}\, {}_{\beta}\sum_{k=1}^{n-1}\left| {}^{*}\!\int_{x_k}^{x_{k+1}} \left[g(x)\ddot{-} g(x_k)\right]\ddot{\times} f(x)\,d^*x \right|_{\beta} \\
&\ddot{\leq}\, {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} \left|g(x)\ddot{-} g(x_k)\right|_{\beta}\ddot{\times} \left|f(x)\right|_{\beta}\,d^*x \right) \\
&\ddot{\leq} K \ddot{\times}\, {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} \left|g(x)\ddot{-} g(x_k)\right|_{\beta}\,d^*x \right) \\
&\ddot{\leq} K \ddot{\times}\, {}_{\beta}\sum_{k=1}^{n-1}\left({}^{*}\!\int_{x_k}^{x_{k+1}} {}^{\beta}\!\sup\left\{\left|g(x)\ddot{-} g(y)\right|_{\beta} : x, y \in [\dot{x}_k, \dot{x}_{k+1}]\right\}\,d^*x \right) \\
&= K \ddot{\times}\, {}_{\beta}\sum_{k=1}^{n-1}\left[{}^{\beta}\!\sup\left\{\left|g(x)\ddot{-} g(y)\right|_{\beta} : x, y \in [\dot{x}_k, \dot{x}_{k+1}]\right\}\ddot{\times}\left(\iota(x_{k+1})\ddot{-}\iota(x_k)\right)\right]
\end{aligned}
$$

and *-continuity of the function g implies that ${}^{\beta}\!\lim_{n\to\infty} S_n^{(2)} = \ddot{0}$. According to this, we have

$$
{}^{*}\!\int_a^b f(x)\ddot{\times} g(x)\,d^*x = {}^{\beta}\!\lim_{n\to\infty}\left[{}_{\beta}\sum_{k=1}^{n-1}\left(g(x_k)\ddot{\times}\, {}^{*}\!\int_{x_k}^{x_{k+1}} f(x)\,d^*x \right)\right].
$$

Let $F(x) = {}^{*}\!\int_a^x f(t)\,d^*t$ for all $x \in [\dot{a}, \dot{b}]$. Since the function f is *-continuous, by the first fundamental theorem of *-calculus, we write $\left[\overset{*}{D} F\right](x) = f(x)$ for all $x \in [\dot{a}, \dot{b}]$. Now, let

$$
m(F) = {}^{\beta}\!\min\left\{F(x) : x \in [\dot{a}, \dot{b}]\right\}
$$

$$M(F) = {}^{\beta}\max\left\{F(x) : x \in [\dot{a}, \dot{b}]\right\}.$$

Since the function F is *-continuous, by the second fundamental theorem of *-calculus, ${}^{*}\int_{x_k}^{x_{k+1}} f(x)\,d^{*}x = F(x_{k+1}) \dot{-} F(x_k)$ is written. Therefore we obtain that

$$\begin{aligned}
S_n^{(1)} &= {}_{\beta}\sum_{k=1}^{n-1}\left(g(x_k)\ddot{\times}\,{}^{*}\int_{x_k}^{x_{k+1}} f(x)\,d^{*}x\right) \\
&= {}_{\beta}\sum_{k=1}^{n-1}\left(g(x_k)\ddot{\times}\left[F(x_{k+1})\ddot{-}F(x_k)\right]\right) \\
&= F(b)\ddot{\times}g(x_{n-1})\ddot{-}F(a)\ddot{\times}g(a)\ddot{+}\,{}_{\beta}\sum_{k=1}^{n-2}\left(F(x_{k+1})\ddot{\times}\left[g(x_k)\ddot{-}g(x_{k+1})\right]\right) \\
&= F(b)\ddot{\times}g(x_{n-1})\ddot{+}_{\beta}\sum_{k=1}^{n-2}\left(F(x_{k+1})\ddot{\times}\left[g(x_k)\ddot{-}g(x_{k+1})\right]\right).
\end{aligned}$$

Since $g(x_k)\ddot{-}g(x_{k+1})\ddot{\geq}\ddot{0}$ and $g(x_{n-1})\ddot{\geq}\ddot{0}$ for $k = 1,2,...,n-2$, by the last equality,

$$\begin{aligned}
m(F)\ddot{\times}g(a) &= m(F)\ddot{\times}g(x_{n-1})\ddot{+}\,{}_{\beta}\sum_{k=1}^{n-2}\left(m(F)\ddot{\times}\left[g(x_k)\ddot{-}g(x_{k+1})\right]\right) \\
&\ddot{\leq} F(b)\ddot{\times}g(x_{n-1})\ddot{+}\,{}_{\beta}\sum_{k=1}^{n-2}\left(F(x_{k+1})\ddot{\times}\left[g(x_k)\ddot{-}g(x_{k+1})\right]\right) \\
&\ddot{\leq} M(F)\ddot{\times}g(a).
\end{aligned}$$

If we take $\beta-$limit of this inequality as $n\to\infty$,

$$m(F)\ddot{\times}g(a)\ddot{\leq}\,{}^{\beta}\lim_{n\to\infty}S_n^{(1)} = {}^{*}\int_a^b f(x)\ddot{\times}g(x)\,d^{*}x\,\ddot{\leq}\,M(F)\ddot{\times}g(a).$$

When $g(a)=\ddot{0}$, it is obvious that desired equation holds. When $g(a)\ddot{>}\ddot{0}$, if

$$\mu = \frac{\ddot{1}}{g(a)}\beta\ddot{\times}\,{}^{*}\int_a^b f(x)\ddot{\times}g(x)\,d^{*}x$$

is taken, by the last inequality, we have $m(F)\ddot{\leq}\mu\ddot{\leq}M(F)$. Since the function F is *-continuous on $[\dot{a},\dot{b}]$, there exist a $\varepsilon\in[\dot{a},\dot{b}]$ such that $\mu = F(\varepsilon) = {}^{*}\int_a^{\varepsilon} f(x)\,d^{*}x$ or

$${}^{*}\int_a^b f(x)\ddot{\times}g(x)\,d^{*}x = g(a)\ddot{\times}\,{}^{*}\int_a^{\varepsilon} f(x)\,d^{*}x.$$

b) Proof is similar to (a).

Theorem 14: (Generalized second mean value theorem) Let the functions $f,g:\dot{[}a,b\dot{]}\to\mathbb{R}(N)_\beta$ be *-continuous and let the function g be *-nondecreasing or *-nonincreasing. Then, there exists a point $\varepsilon\in\dot{[}a,b\dot{]}$ such that

$$^{*}\int_a^b f(x)\ddot{\times}g(x)d^*x=g(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x\ddot{+}g(b)\ddot{\times}\,^{*}\int_\varepsilon^b f(x)d^*x.$$

Proof: Suppose that the function g is *-nondecreasing. The function $G(x)=g(b)\ddot{-}g(x)$ is continuous and *-nonincreasing on $\dot{[}a,b\dot{]}$ since the function g is *-continuous and *-nondecreasing on $\dot{[}a,b\dot{]}$. Further, $G(x)\ddot{\geq}\ddot{0}$ for all $x\in\dot{[}a,b\dot{]}$. Then, by the second mean value theorem, there exists a point $\varepsilon\in\dot{[}a,b\dot{]}$ such that

$$^{*}\int_a^b f(x)\ddot{\times}G(x)d^*x=G(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x.$$

Again, since

$$\begin{aligned}^{*}\int_a^b f(x)\ddot{\times}G(x)d^*x&=\,^{*}\int_a^b f(x)\ddot{\times}\left[g(b)\ddot{-}g(x)\right]d^*x\\&=g(b)\ddot{\times}\,^{*}\int_a^b f(x)d^*x\ddot{-}\,^{*}\int_a^b f(x)\ddot{\times}g(x)d^*x\end{aligned}\tag{1}$$

and

$$G(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x=g(b)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x\ddot{-}g(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x,\tag{2}$$

by the equality of right sides of equalities (1) and (2),

$$g(b)\ddot{\times}\,^{*}\int_a^b f(x)d^*x\ddot{-}\,^{*}\int_a^b f(x)\ddot{\times}g(x)d^*x=g(b)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x\ddot{-}g(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x$$

is written. Hence, we see that the equality

$$^{*}\int_a^b f(x)\ddot{\times}g(x)d^*x=g(a)\ddot{\times}\,^{*}\int_a^\varepsilon f(x)d^*x\ddot{+}g(b)\ddot{\times}\,^{*}\int_\varepsilon^b f(x)d^*x$$

holds for a point $\varepsilon\in\dot{[}a,b\dot{]}$. When the function g is *-nonincreasing, since the function $G(x)=g(x)\ddot{-}g(a)$ is *-continuous and *-nondecreasing on $\dot{[}a,b\dot{]}$, and since $G(x)\ddot{\geq}\ddot{0}$ for all $x\in\dot{[}a,b\dot{]}$, the proof is completed similar to above.

Theorem 15: (Abel's-Dirichlet's test) Let the functions $f,g:\dot{[}a,B\dot{)}\to\mathbb{R}(N)_\beta$ with $a\in\mathbb{R}(N)_\alpha$ and $B\in\overline{\mathbb{R}}(N)_\alpha$ be *-continuous on each $\dot{[}a,b\dot{]}\subset\dot{[}a,B\dot{)}$, and be $\beta-$unbounded on left of the point B if $B\in\mathbb{R}(N)_\alpha$. In this case,

a) if the improper *-integral ${}^{*}\int_a^B f(x)\,d^{*}x$ is convergent and if the function g is *-nonincreasing(or *-nondecreasing) and $\beta-$bounded, then the improper *-integral ${}^{*}\int_a^B f(x)\ddot{\times}g(x)\,d^{*}x$ is convergent(Abel's test).

b) if the function $F(b)={}^{*}\int_a^b f(x)\,d^{*}x$ with $b\in\dot{[}a,B\dot{)}$ is $\beta-$bounded on $\dot{[}a,B\dot{)}$, and if the function g is *-nonincreasing or *-nondecreasing which holds $*-\lim_{b\to B^{-}}g(x)=\ddot{0}$, then the improper *-integral ${}^{*}\int_a^B f(x)\ddot{\times}g(x)\,d^{*}x$ is convergent(Dirichlet's test).

Proof: By virtue of generalized mean value theorem, there exist at least a $\eta\in\dot{[}a,b\dot{]}$ between numbers b_1 and b_2 such that

$$ {}^{*}\int_{b_1}^{b_2} f(x)\ddot{\times}g(x)\,d^{*}x=g(b_1)\ddot{\times}\,{}^{*}\int_{b_1}^{\eta} f(x)\,d^{*}x\,\ddot{+}\,g(b_2)\ddot{\times}\,{}^{*}\int_{\eta}^{b_2} f(x)\,d^{*}x $$

for all $b_1,b_2\in\dot{[}a,B\dot{)}$.

a) Since g is $\beta-$bounded and ${}^{*}\int_a^B f(x)\,d^{*}x$ is convergent, by Cauchy's test, ${}^{*}\int_a^B f(x)\ddot{\times}g(x)\,d^{*}x$ is convergent.

b) Since the function $F(b)={}^{*}\int_a^b f(x)\,d^{*}x$ is $\beta-$bounded on $\dot{[}a,B\dot{)}$ and the *-integrals ${}^{*}\int_{b_1}^{\eta} f(x)\,d^{*}x$ and ${}^{*}\int_{\eta}^{b_2} f(x)\,d^{*}x$ are $\beta-$bounded also. Thus, by Cauchy's test, ${}^{*}\int_a^B f(x)\ddot{\times}g(x)\,d^{*}x$ is convergent since ${}^{*}\int_{b_1}^{\eta} f(x)\,d^{*}x$ and ${}^{*}\int_{\eta}^{b_2} f(x)\,d^{*}x$ are $\beta-$bounded and since $*-\lim_{b\to B^{-}}g(x)=\ddot{0}$.

Example 8: We investigate convergence condition of the improper geometric integral $\tilde{\int_1^{+\infty}} e^{\left(\frac{\cos x}{x^2}\right)}\tilde{d}x$.

Solution 5: The equality $e^{\left(\frac{\cos x}{x^2}\right)} = \left(e^{\cos x}\right)^{\ln e^{\left(\frac{1}{x^2}\right)}}$ holds. Now we take $f(x) = e^{\cos x}$ and $g(x) = e^{\left(\frac{1}{x^2}\right)}$. For all $b \in [1, +\infty)$,

$$F(b) = \tilde{\int_1^b} e^{\cos x} \tilde{d}x = e^{\left(\int_1^b \cos x dx\right)} = e^{\sin b - \sin 1}.$$

We have $\left|F(b)\right|_\beta = \left|e^{\sin b - \sin 1}\right|_\beta \ddot{\leq} \ddot{2}$ since $\left|\sin b - \sin 1\right| \leq 2$ for all $b \in [1, +\infty)$. Therefore, the function F is bounded. $* - \lim_{b \to +\infty} g(x) = \lim_{b \to +\infty} e^{\left(\frac{1}{x^2}\right)} = 1 = \ddot{0}$ holds and the function f is geometric nonincreasing since $f(x_2) = e^{\left(\frac{1}{x_2^2}\right)} \ddot{\leq} e^{\left(\frac{1}{x_1^2}\right)} = f(x_1)$ for numbers $x_1, x_2 \in [1, +\infty)$ which are $x_1 < x_2$. Then, by Dirichlet's test, the improper geometric integral $\tilde{\int_1^{+\infty}} e^{\left(\frac{\cos x}{x^2}\right)} \tilde{d}x$ is convergent.

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