

ON THE NATURAL LIFT CURVE AND THE BERTRAND MATE

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Abstract. *In this study, the relation between Frenet vectors of the natural lift curve $\bar{\alpha}$ of the curve α and Frenet vectors of a β bertrand mate of the curve is given in \mathbb{R}^3 and \mathbb{R}_1^3 .*

Keywords: *Natural Lift Curve, Bertrand Mate, Frenet Vectors.*

1. INTRODUCTION

The concepts of the natural lift curve and geodesic sprays have first been given by Thorpe in [24]. Thorpe provided the natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M . Çalışkan et al. studied the natural lift curves of the spherical indicatrices of tangent, principal normal, binormal vectors and fixed centrode of a curve in [19]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$. Some properties of \bar{M} -vector field Z defined on a hypersurface M of \bar{M} were studied by Agashe in [17]. \bar{M} -integral curve of Z and \bar{M} -geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\bar{\alpha}$ of the curve α (in \bar{M}) is an \bar{M} -integral curve of the geodesic spray Z iff α is an \bar{M} -geodesic in [23]. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [21]. The analogue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [21]. Çalışkan and Ergün defined \bar{M} -vector field Z , \bar{M} -geodesic spray, \bar{M} -integral curve of Z , \bar{M} -geodesic in [20]. The analogue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [20]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [25]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [22]. Bertrand curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal

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normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. It is proved in most texts on the subject that the characteristic property of such a curve is the existence of a linear relation between the curvature and the torsion; the discussion appears as an application of the Frenet-Serret formulas. So, a circular helix is a Bertrand curve. Bertrand mates represent particular examples of offset curves [7], which are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). For classical and basic treatments of Bertrand curves, we refer to [2, 4, 8].

There are recent works about the Bertrand curves. Ekmekçi and İlarslan studied Nonnull Bertrand curves in the n -dimensional Lorentzian space. Straightforward modification of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Izumiya and Takeuchi [12] have shown that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Also, the representation formulae for Bertrand curves were given by [5].

Let α be a regular curve with $\alpha''(s) \neq 0$. For every point of $\alpha(s)$, the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where

$$T(s) = \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad B(s) = T(s) \times N(s)$$

are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s) \rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [2].

Let M be a hypersurface in \mathbb{R}^3 and let $\alpha : I \rightarrow M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \quad (\text{for all } t \in I)$$

where X is a smooth tangent vector field on M . We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M .

For any parametrized curve $\alpha : I \rightarrow M$, $\bar{\alpha} : I \rightarrow TM$ given by

$$\bar{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TM . Thus, we can write

$$\frac{d\bar{\alpha}}{ds} = \frac{d}{ds}(\alpha'(s)|_{\alpha(s)}) = D_{\alpha'(s)}\alpha'(s)$$

where D is the Levi-Civita connection on \mathbb{R}^3 [24].

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α . The rotation is determined by an angular velocity vector W which satisfies $T' = W \times T$, $N' = W \times N$ and $B' = W \times B$. The vector W is called the *Darboux vector*. In terms of Frenet vectors T , N and B , Darboux vector is given by $W = \tau T + \kappa B$. Also, we have $\kappa = \|W\| \cos \varphi$, $\tau = \|W\| \sin \varphi$, where φ is the angle between the Darboux vector of α and binormal vector $B(s)$.

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length curve and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be natural lift of α . Then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos \varphi & 0 & \sin \varphi \\ \sin \varphi & 0 & \cos \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are Frenet frames of the curves α and $\bar{\alpha}$, respectively, and φ is the angle between the Darboux vector and binormal vector.

Definition 1: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be two regular curves in \mathbb{R}^3 . $\alpha(s)$ and $\beta(s^*)$ are called *Bertrand curves* if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation, (α, β) is called a *Bertrand couple*.

Lemma 1: Let β be a curve Bertrand mate of α . The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Minkowski 3-space \mathbb{R}_1^3 is the vector space \mathbb{R}^3 equipped with the Lorentzian inner product g given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ or $X = 0$ and lightlike (or null) if $g(X, X) = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in \mathbb{R}_1^3 can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector X is

said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector X is defined by $\|X\|_L = \sqrt{g(X, X)}$ [6].

The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$ are orthogonal if and only if $g(X, Y) = 0$ [16].

Now let X and Y be two vectors in \mathbb{R}_1^3 , then the Lorentzian cross product is given by [26]

$$\begin{aligned} X \times Y &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2). \end{aligned}$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α . Then T , N and B are the tangent, the principal normal and the binormal vector of the curve α , respectively.

Let α be a unit speed timelike curve with curvature κ and torsion τ . So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where \times is the Lorentzian cross product in \mathbb{R}_1^3 [25]. The binormal vector field $B(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T, N, B\}$ has the same orientation as \mathbb{R}_1^3 . Then, Frenet formulas are given by [25]

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N.$$

Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

The binormal vector field $B(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T, N, B\}$ has the same orientation as \mathbb{R}_1^3 . Then, Frenet formulas are given by [25]

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N.$$

Let α be a unit speed spacelike curve with timelike binormal. Now, T and N are spacelike vector fields and B is a timelike vector field. In this situation,

$$T \times N = B, \quad N \times B = -T, \quad B \times T = -N,$$

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to the spacelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T, N, B\}$ has the same orientation as \mathbb{R}_1^3 . Then, Frenet formulas are given by [25]

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

Lemma 2: Let X and Y be nonzero Lorentz orthogonal vectors in \mathbb{R}_1^3 . If X is timelike, then Y is spacelike [16].

Lemma 3: Let X and Y be positive (negative) timelike vectors in \mathbb{R}_1^3 . Then $g(X, Y) \leq \|X\| \|Y\|$ whith equality if and only if X and Y are linearly dependent [16].

Lemma 4:

i) Let X and Y be positive (negative) timelike vectors in \mathbb{R}_1^3 . By Lemma 2, there is a unique nonnegative real number $\phi(X, Y)$ such that

$$g(X, Y) = \|X\| \|Y\| \cosh \phi(X, Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\phi(X, Y)$ [16].

ii) Let X and Y be spacelike vectors in \mathbb{R}_1^3 that span a spacelike vector subspace. Then we have

$$|g(X, Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number $\phi(X, Y)$ between 0 and π such that

$$g(X, Y) = \|X\| \|Y\| \cos \phi(X, Y).$$

$\phi(X, Y)$ is defined to be the Lorentzian spacelike angle between X and Y [16].

iii) Let X and Y be spacelike vectors in \mathbb{R}_1^3 that span a timelike vector subspace. Then, we have

$$g(X, Y) > \|X\| \|Y\|.$$

Hence, there is a unique positive real number $\phi(X, Y)$ between 0 and π such that

$$|g(X, Y)| = \|X\| \|Y\| \cosh \phi(X, Y).$$

$\phi(X, Y)$ is defined to be the Lorentzian timelike angle between X and Y [16].

iv) Let X be a spacelike vector and Y be a positive timelike vector in \mathbb{R}_1^3 . Then there is a unique nonnegative real number $\phi(X, Y)$ such that

$$|g(X, Y)| = \|X\| \|Y\| \sinh \phi(X, Y).$$

$\phi(X, Y)$ is defined to be the Lorentzian timelike angle between X and Y [16].

Let P be a surface in \mathbb{R}_1^3 and let $\alpha : I \rightarrow P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \text{ (for all } t \in I),$$

where X is a smooth tangent vector field on P [6]. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where $T_p P$ is the tangent space of P at p and $\chi(P)$ is the space of tangent vector fields on P .

For any parametrized curve $\alpha : I \rightarrow P$, $\bar{\alpha} : I \rightarrow TP$ is given by

$$\bar{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TP [21].

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cosh \varphi & 0 & \sinh \varphi \\ \sinh \varphi & 0 & \cosh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If W is a timelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh \varphi & 0 & \cosh \varphi \\ \cosh \varphi & 0 & \sinh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal.

i) If W is a spacelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cosh \varphi & 0 & \sinh \varphi \\ -\sinh \varphi & 0 & -\cosh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If W is a timelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh \varphi & 0 & \cosh \varphi \\ -\cosh \varphi & 0 & -\sinh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length spacelike curve with spacelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a timelike curve. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

Let the natural lift $\bar{\alpha}$ of α is a timelike curve.

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos \theta & 0 & \sin \theta \\ \sin \theta & 0 & -\cos \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc spacelike curve with timelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a timelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh \varphi & 0 & \sinh \varphi \\ -\sinh \varphi & 0 & \cosh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If W is a spacelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sinh \varphi & 0 & \cosh \varphi \\ -\cosh \varphi & 0 & \sinh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal.

i) If W is a timelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh \varphi & 0 & \sinh \varphi \\ \sinh \varphi & 0 & -\cosh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If W is a spacelike vector, then we have

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sinh \varphi & 0 & \cosh \varphi \\ \cosh \varphi & 0 & -\sinh \varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Definition 2: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ be a regular timelike curve and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be a regular spacelike curve in \mathbb{R}_1^3 . $\alpha(s)$ and $\beta(s^*)$ are called Bertrand curves if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation, (α, β) is called a timelike-spacelike Bertrand couple, [1].

Lemma 5: Let (α, β) be a timelike-spacelike Bertrand couple. The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \sinh \theta & 0 & \cosh \theta \\ 0 & 1 & 0 \\ \cosh \theta & 0 & \sinh \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

where θ is the hyperbolic timelike angle between T and T^* , [5].

Definition 3: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be two regular timelike curves in \mathbb{R}_1^3 . $\alpha(s)$ and $\beta(s^*)$ are called Bertrand curves if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation, (α, β) is called a timelike Bertrand couple [1].

Lemma 6: Let (α, β) be a timelike Bertrand couple. The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \cosh \theta & 0 & \sinh \theta \\ 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

where θ is the hyperbolic timelike angle between T and T^* , [5].

Lemma 7: Let β be a spacelike curve with spacelike binormal Bertrand mate of a spacelike curve with spacelike binormal α . The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

$$g(T, T^*) = \cos \theta = \text{constant}, [5].$$

Definition 4: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be two regular non-null curves in \mathbb{R}_1^3 . $\alpha(s)$ and $\beta(s^*)$ are called *Bertrand curves* if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation, (α, β) is called a *Bertrand couple* \mathbb{R}_1^3 . [1].

Lemma 8: Let α be a spacelike curve with timelike binormal. In this situation, β is a spacelike curve with timelike binormal Bertrand mate of α . The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \cosh \theta & 0 & \sinh \theta \\ 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

$$g(T, T^*) = \cosh \theta = \text{constant}, [5].$$

Lemma 9: Let α be a spacelike curve with timelike binormal. In this situation, β is a timelike Bertrand mate of α . The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} \sinh \theta & 0 & \cosh \theta \\ 0 & 1 & 0 \\ \cosh \theta & 0 & \sinh \theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

$$g(T, T^*) = \sinh \theta = \text{constant}, [5].$$

2. THE NATURAL LIFT CURVE and THE BERTRAND MATE

Let $\bar{\alpha}$ be the natural lift of α and β be a curve Bertrand mate of α in \mathbb{R}^3 .

$$\begin{aligned} T^* &= \cos \theta T - \sin \theta B \\ &= \cos \theta (-\cos \varphi \bar{N} + \sin \varphi \bar{B}) - \sin \theta (\sin \varphi \bar{N} + \cos \varphi \bar{B}) \\ &= (-\cos \theta \cos \varphi - \sin \theta \sin \varphi) \bar{N} + (\cos \theta \sin \varphi - \sin \theta \cos \varphi) \bar{B} \\ &= -\cos(\theta - \varphi) \bar{N} - \sin(\theta - \varphi) \bar{B} \end{aligned}$$

$$\begin{aligned}
N^* &= N = \bar{T} \\
B^* &= \sin \theta T + \cos \theta B \\
&= \sin \theta (-\cos \varphi \bar{N} + \sin \varphi \bar{B}) + \cos \theta (\sin \varphi \bar{N} + \cos \varphi \bar{B}) \\
&= -(\sin \theta \cos \varphi - \cos \theta \sin \varphi) \bar{N} + (\sin \theta \sin \varphi + \cos \theta \cos \varphi) \bar{B} \\
&= -\sin(\theta - \varphi) \bar{N} + \cos(\theta - \varphi) \bar{B}
\end{aligned}$$

where $\{T, N, B\}$, $\{\bar{T}, \bar{N}, \bar{B}\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of the curves α , $\bar{\alpha}$ and β , respectively.

So from the above calculations we can give the following propositions.

Proposition 1: If $\bar{\alpha}$ is natural lift curve of α and β is a Bertrand mate of α in \mathbb{R}^3 , then $\{\bar{T}, N^*\}$ is linearly dependent.

Proposition 2: Let $\bar{\alpha}$ be the natural lift of α and β be a curve Bertrand mate of α in \mathbb{R}^3 . Then

$$\begin{aligned}
T^* &= -\cos(\theta - \varphi) \bar{N} - \sin(\theta - \varphi) \bar{B} \\
N^* &= \bar{T} \\
B^* &= -\sin(\theta - \varphi) \bar{N} + \cos(\theta - \varphi) \bar{B}
\end{aligned}$$

Let α be an arclength spacelike curve with spacelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a timelike curve and Bertrand mate β of α is a spacelike curve with spacelike binormal in \mathbb{R}_1^3 .

$$\begin{aligned}
T^* &= \cos \theta T + \sin \theta B \\
&= \cos \theta (\cos \varphi \bar{N} + \sin \varphi \bar{B}) + \sin \theta (\sin \varphi \bar{N} - \cos \varphi \bar{B}) \\
&= (\cos \theta \cos \varphi + \sin \theta \sin \varphi) \bar{N} + (\cos \theta \sin \varphi - \sin \theta \cos \varphi) \bar{B} \\
&= \cos(\theta + \varphi) \bar{N} - \sin(\theta + \varphi) \bar{B} \\
N^* &= N = \bar{T} \\
B^* &= \sin \theta T + \cos \theta B \\
&= \sin \theta (\cos \varphi \bar{N} + \sin \varphi \bar{B}) + \cos \theta (\sin \varphi \bar{N} - \cos \varphi \bar{B}) \\
&= (\sin \theta \cos \varphi + \cos \theta \sin \varphi) \bar{N} + (\sin \theta \sin \varphi - \cos \theta \cos \varphi) \bar{B} \\
&= \sin(\theta + \varphi) \bar{N} - \cos(\theta + \varphi) \bar{B}
\end{aligned}$$

where $\{T, N, B\}$, $\{\bar{T}, \bar{N}, \bar{B}\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of the curves α , $\bar{\alpha}$ and β , respectively.

So from the above calculations we can give the following proposition.

Proposition 3: Let α be an arclength spacelike curve with spacelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a timelike curve and Bertrand mate β of α is a spacelike curve with spacelike binormal.

$$T^* = \cos(\theta - \varphi)\bar{N} - \sin(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sin(\theta + \varphi)\bar{N} - \cos(\theta + \varphi)\bar{B}$$

We can give the following propositions when necessary calculations are made in \mathbb{R}_1^3 .

Proposition 4: *Let α be an arclength spacelike curve with timelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal and Bertrand mate β of α is a spacelike curve with timelike binormal*

(i) If W is a spacelike vector field, then

$$T^* = \sinh(\theta + \varphi)\bar{N} - \cosh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta + \varphi)\bar{N} - \sinh(\theta + \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = -\cosh(\theta + \varphi)\bar{N} + \sinh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = -\sinh(\theta + \varphi)\bar{N} + \cosh(\theta + \varphi)\bar{B}$$

Proposition 5: *Let α be an arclength spacelike curve with timelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal and Bertrand mate β of α is a spacelike curve with timelike binormal*

(i) If W is a spacelike vector field, then

$$T^* = \sinh(\theta + \varphi)\bar{N} + \cosh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta + \varphi)\bar{N} + \sinh(\theta + \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = -\cosh(\theta + \varphi)\bar{N} - \sinh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = -\sinh(\theta + \varphi)\bar{N} - \cosh(\theta + \varphi)\bar{B}$$

Proposition 6: *Let α be an arclength spacelike curve with timelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal and Bertrand mate β of α is a timelike curve*

(i) If W is a spacelike vector field, then

$$T^* = \cosh(\theta + \varphi)\bar{N} - \sinh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta + \varphi)\bar{N} - \cosh(\theta + \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = -\sinh(\theta + \varphi)\bar{N} + \cosh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = -\cosh(\theta + \varphi)\bar{N} + \sinh(\theta + \varphi)\bar{B}$$

Proposition 7: Let α be an arclength spacelike curve with timelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal and Bertrand mate β of α is a timelike curve

(i) If W is a spacelike vector field, then

$$T^* = \cosh(\theta + \varphi)\bar{N} + \sinh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta + \varphi)\bar{N} + \cosh(\theta + \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = -\sinh(\theta + \varphi)\bar{N} - \cosh(\theta + \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = -\cosh(\theta + \varphi)\bar{N} - \sinh(\theta + \varphi)\bar{B}$$

Proposition 8: Let α be an arclength timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal and Bertrand mate β of α is a timelike curve

(i) If W is a spacelike vector field, then

$$T^* = \cosh(\theta - \varphi)\bar{N} + \sinh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta - \varphi)\bar{N} + \cosh(\theta - \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = \sinh(\theta - \varphi)\bar{N} + \cosh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta - \varphi)\bar{N} + \sinh(\theta - \varphi)\bar{B}$$

Proposition 9: Let α be an arclength timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal and Bertrand mate β of α is a timelike curve

(i) If W is a spacelike vector field, then

$$T^* = \cosh(\theta - \varphi)\bar{N} - \sinh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta - \varphi)\bar{N} - \cosh(\theta - \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = \sinh(\theta - \varphi)\bar{N} - \cosh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta - \varphi)\bar{N} - \sinh(\theta - \varphi)\bar{B}$$

Proposition 10: Let α be an arc length timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal and Bertrand mate β of α is a spacelike curve

(i) If W is a spacelike vector field, then

$$T^* = -\sinh(\theta - \varphi)\bar{N} + \cosh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta - \varphi)\bar{N} - \sinh(\theta - \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = \cosh(\theta - \varphi)\bar{N} + \sinh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta - \varphi)\bar{N} + \cosh(\theta - \varphi)\bar{B}$$

Proposition 11: Let α be an arclength timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal and Bertrand mate β of α is a spacelike curve

(i) If W is a spacelike vector field, then

$$T^* = \sinh(\theta - \varphi)\bar{N} - \cosh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \cosh(\theta - \varphi)\bar{N} - \sinh(\theta - \varphi)\bar{B}$$

(ii) If W is a timelike vector field, then

$$T^* = \cosh(\theta - \varphi)\bar{N} - \sinh(\theta - \varphi)\bar{B}$$

$$N^* = \bar{T}$$

$$B^* = \sinh(\theta - \varphi)\bar{N} - \cosh(\theta - \varphi)\bar{B}$$

Proposition 12: If $\bar{\alpha}$ is natural lift curve of α and β is a Bertrand mate of α in \mathbb{R}_1^3 , then $\{\bar{T}, N^*\}$ is linearly dependent.

3. EXAMPLES

Example1: Given the arclength curve

$$\alpha(s) = \left(-\frac{\sqrt{2}}{2} \sin\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2} \cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2} \right)$$

its Frenet apparatus are

$$T(s) = \left(-\cos\left(\frac{\sqrt{2}}{2}s\right), -\sin\left(\frac{\sqrt{2}}{2}s\right), 0 \right),$$

$$N(s) = \left(\sin\left(\frac{\sqrt{2}}{2}s\right), -\cos\left(\frac{\sqrt{2}}{2}s\right), 0 \right),$$

$$B(s) = (0, 0, 1),$$

The natural lift of the curve α is

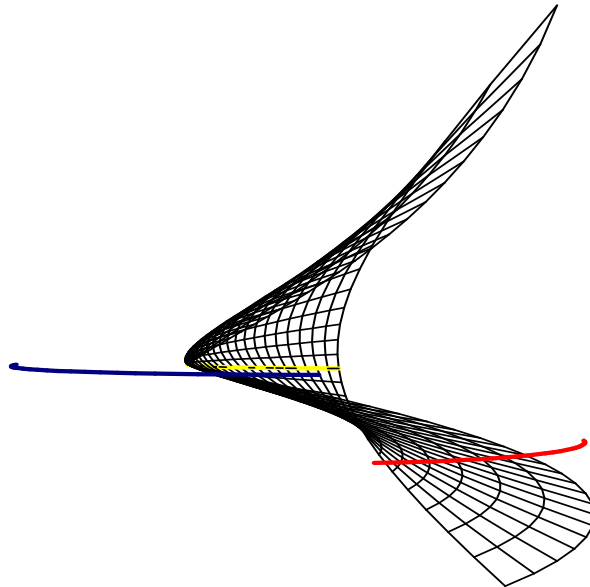
$$\bar{\alpha}(s) = \left(-\cos\left(\frac{\sqrt{2}}{2}s\right), -\sin\left(\frac{\sqrt{2}}{2}s\right), 0 \right)$$

and Bertrand mate for can be given by the equation

$$\begin{aligned} \beta(s) &= \left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2} \right) + \lambda \left(\sin\left(\frac{\sqrt{2}}{2}s\right), -\cos\left(\frac{\sqrt{2}}{2}s\right), 0 \right) \\ \beta(s) &= \left(\left(-\frac{\sqrt{2}}{2} + \lambda \right) \sin\left(\frac{\sqrt{2}}{2}s\right), \left(\frac{\sqrt{2}}{2} - \lambda \right) \cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2} \right), \lambda \in \mathbb{R} \end{aligned}$$

for $\lambda = -\frac{\sqrt{2}}{2}$, we have

$$\beta(s) = \left(-\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}s\right), \sqrt{2}\cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2} \right).$$



Example2: Given the arclength curve

$$\alpha(s) = \left(\frac{\sqrt{3}}{2}\sin s, \frac{1}{2}s, \frac{\sqrt{3}}{2}\cos s \right)$$

its Frenet apparatus are

$$\begin{aligned}
 T(s) &= \left(\frac{\sqrt{3}}{2} \cos s, \frac{1}{2} - \frac{\sqrt{3}}{2} \sin s \right), \\
 N(s) &= (-\sin s, 0, -\cos s), \\
 B(s) &= \left(-\frac{1}{2} \cos s, \frac{\sqrt{3}}{2}, \frac{1}{2} \sin s \right),
 \end{aligned}$$

The natural lift of the curve α is

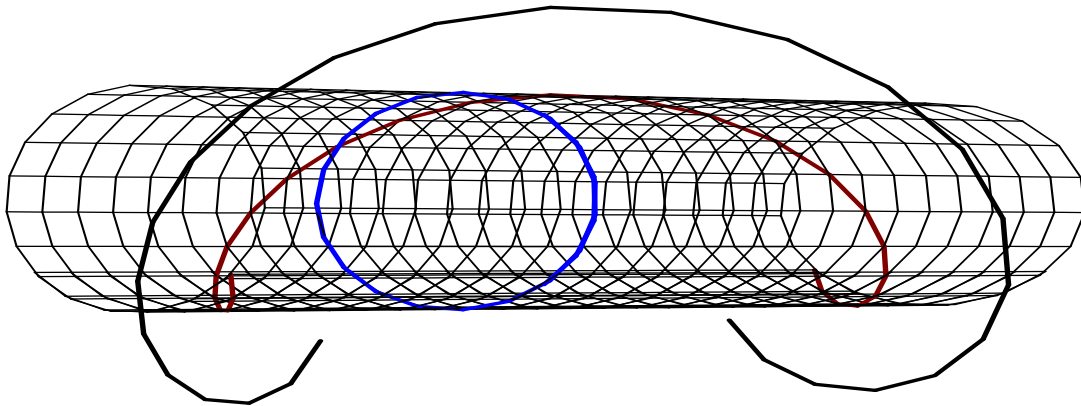
$$\bar{\alpha}(s) = \left(\frac{\sqrt{3}}{2} \cos s, \frac{1}{2} - \frac{\sqrt{3}}{2} \sin s \right)$$

and Bertrand mate for can be given by the equation

$$\beta(s) = \left(\left(\frac{\sqrt{3}}{2} - \lambda \right) \sin s, \frac{1}{2} s, \left(\frac{\sqrt{3}}{2} - \lambda \right) \cos s \right), \lambda \in \mathbb{R},$$

for $\lambda = -\frac{\sqrt{2}}{2}$, we have

$$\beta(s) = \left(\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \right) \sin s, \frac{1}{2} s, \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \right) \cos s \right)$$



Example3: Let

$$\alpha(s) = (\cosh s, 0, \sinh s)$$

be a timelike curve. It is easy to show that

$$\begin{aligned}
 T(s) &= (\sinh s, 0, \cosh s), \\
 N(s) &= (\cosh s, 0, \sinh s), \\
 B(s) &= (0, 1, 0),
 \end{aligned}$$

In this situation, the natural lift of α and l Bertrand mate for can be given by the equations

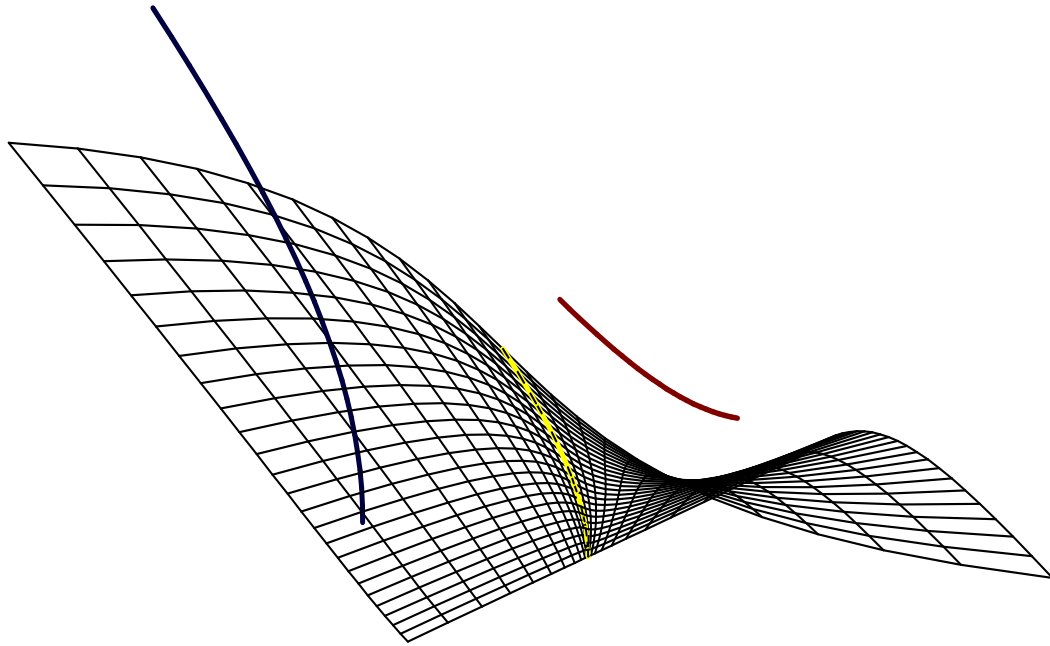
$$\bar{\alpha}(s) = (\sinh s, 0, \cosh s)$$

and

$$\beta(s) = ((\lambda + 1)\cosh s, 0, (\lambda + 1)\sinh s), \lambda \in \mathbb{R}$$

for $\lambda = \frac{\sqrt{7}-1}{3}$, we have

$$\beta(s) = ((\sqrt{2} + 1)\cosh s, 0, (\sqrt{2} + 1)\sinh s)$$



Example4:

Given the arclength timelike curve

$$\alpha(s) = \left(\frac{5}{3}s, \frac{4}{9}\cos 3s, \frac{4}{9}\sin 3s \right)$$

its Frenet apparatus are

$$T(s) = \left(\frac{5}{3}, -\frac{4}{3}\sin 3s, \frac{4}{3}\cos 3s \right),$$

$$N(s) = (0, -\cos 3s, -\sin 3s),$$

$$B(s) = \left(-\frac{4}{3}, \frac{5}{3}\sin 3s, -\frac{5}{3}\cos 3s \right),$$

The natural lift of the curve α is

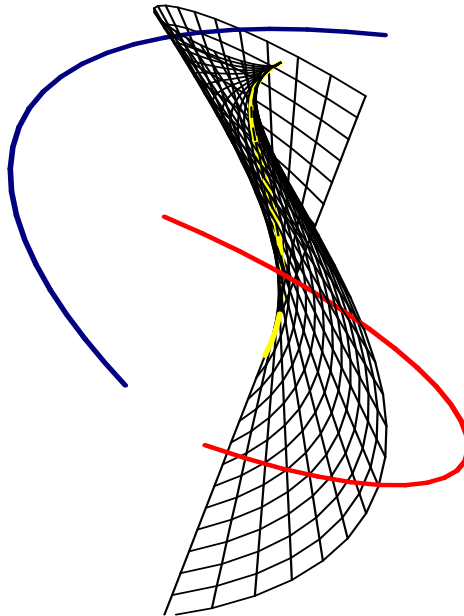
$$\bar{\alpha}(s) = \left(\frac{5}{3}s, -\frac{4}{3}\sin 3s, \frac{4}{3}\cos 3s \right)$$

and Bertrand mate for can be given by the equation

$$\beta(s) = \left(\frac{5}{3}s, \left(\lambda + \frac{4}{9} \right) \sin(3s), \left(\lambda + \frac{4}{9} \right) \cos(3s) \right), \lambda \in \mathbb{R}$$

for $\lambda = \frac{5}{9}$, we have

$$\beta(s) = \left(\frac{5}{3}s, \sin(3s), \cos(3s) \right).$$



Example5: Let

$$\alpha(s) = \left(\frac{1}{3} \sinh(\sqrt{3}s), \frac{2}{3} \sqrt{3}s, \frac{1}{3} \cosh(\sqrt{3}s) \right)$$

be a unit speed spacelike hyperbolic helix with

$$\begin{aligned} T &= \frac{\sqrt{3}}{3} (\cosh(\sqrt{3}s), 2, \sinh(\sqrt{3}s)) \\ N &= (\sinh(\sqrt{3}s), 0, \cosh(\sqrt{3}s)), \quad \kappa = 1 \text{ and } \tau = 2 \\ B &= \frac{\sqrt{3}}{3} (2 \cosh(\sqrt{3}s), 1, 2 \sinh(\sqrt{3}s)) \end{aligned}$$

In this situation, the natural lift of α and spacelike with spacelike binormal Bertrand mate for can be given by the equations

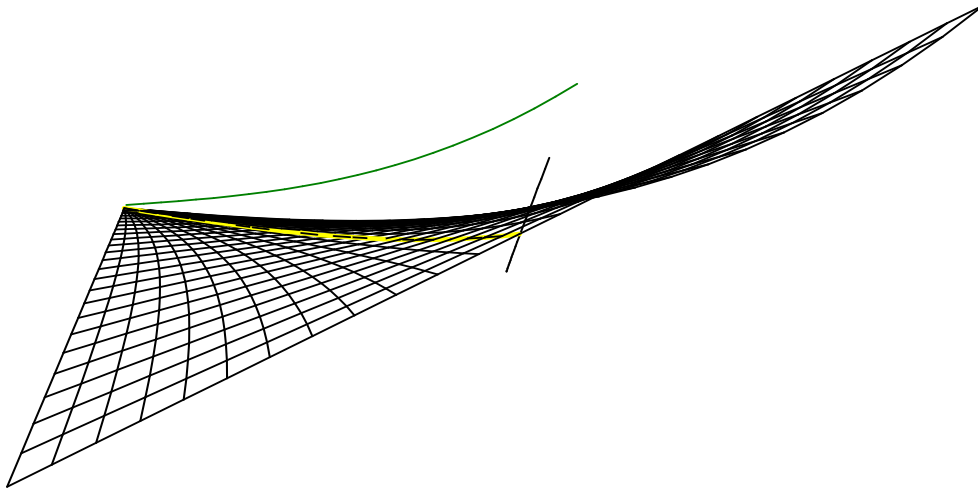
$$\bar{\alpha}(s) = \left(\frac{\sqrt{3}}{3} \cosh(\sqrt{3}s), \frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \sinh(\sqrt{3}s) \right)$$

and

$$\beta(s) = \left(\left(\lambda + \frac{1}{3} \right) \sinh(\sqrt{3}s), \frac{2\sqrt{3}}{3}s, \left(\lambda + \frac{1}{3} \right) \cosh(\sqrt{3}s) \right), \lambda \in \mathbb{R}$$

for $\lambda = \frac{\sqrt{7}-1}{3}$, we have

$$\beta(s) = \left(\frac{\sqrt{7}}{3} \sinh(\sqrt{3}s), \frac{2\sqrt{3}}{3}s, \frac{\sqrt{7}}{3} \cosh(\sqrt{3}s) \right).$$



Example6: Let

$$\alpha(s) = (\sinh s, 0, \cosh s)$$

be a timelike curve. It is easy to show that

$$\begin{aligned} T(s) &= (\cosh s, 0, \sinh s), \\ N(s) &= (\sinh s, 0, \cosh s), \\ B(s) &= (0, -1, 0), \\ \kappa(s) &= 1, \quad \tau(s) = 0. \end{aligned}$$

In this situation, the natural lift of α and Bertrand mate for can be given by the equations

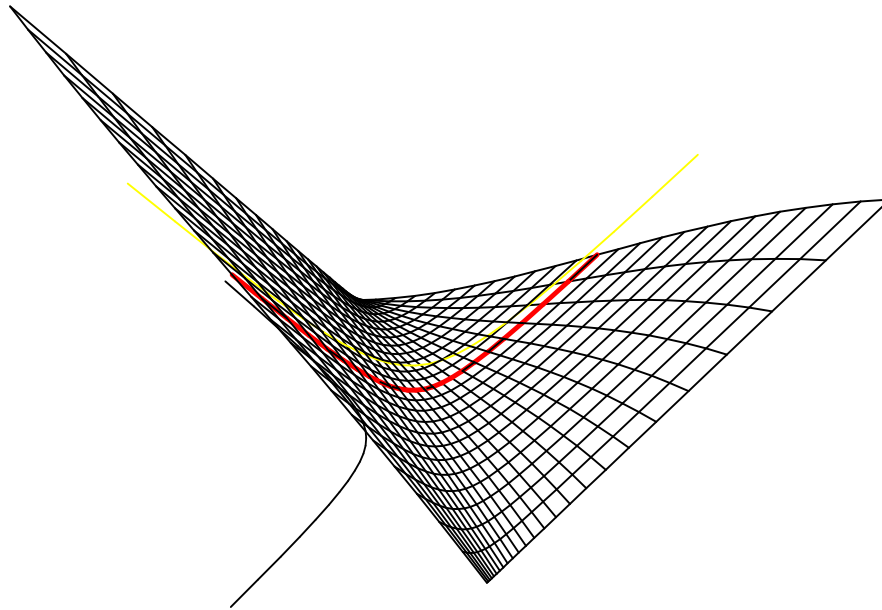
$$\bar{\alpha}(s) = (\cosh s, 0, \sinh s)$$

and

$$\beta(s) = ((\lambda + 1) \sinh s, 0, (\lambda + 1) \cosh s), \lambda \in \mathbb{R}$$

for $\lambda = \frac{\sqrt{7}-1}{3}$, we have

$$\beta(s) = \left(\frac{\sqrt{7}+2}{3} \sinh s, 0, \frac{\sqrt{7}+2}{3} \cosh s \right).$$



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