ORIGINAL PAPER

MODULES THAT HAVE A WEAK δ-SUPPLEMENT IN EVERY COFINITE EXTENSION

ESRA OZTURK SOZEN[1], SENOL EREN[2]

Manuscript received: 31.10.2017; Accepted paper: 02.01.2018;
Published online: 30.03.2018.

Abstract. *In this paper, we study on modules that have a weak (ample) δ-supplement in every extension which are adapted Zöschinger's modules with the properties (E) and (EE). It is shown that: (1) Direct summands of modules with the property δ-(CWE) have the property δ-(CWE); (2) For a module M, if every submodule of M has the property δ-(CWE) then so does M; (3) For a ring R, R is δ-semilocal iff every R-module has the property δ-(CWE); (4) Every factor module of a finitely generated module that has the property δ-(CWE) also has the property δ-(CWE) under a special condition; (5) Let M be a module and L be a submodule of M such that $L \ll_\delta M$. If the factor module M/L has the property δ-(CWE), then so does M; (6) On a semisimple module the concepts of modules that have the property δ-(CE) and δ-(CWE) coincide with each other.*

Keywords: *cofinite extension; δ-supplement; weak δ-supplement; δ-semilocal ring.*
2010 Mathematics Subject Classification: *16D10.*

1. INTRODUCTION

Throughout this paper, we assume that all rings are associative with identity and all modules are unital left modules. By $X \leq M$, we mean X is a *submodule* of M or M is an *extension* of X. A submodule $K \leq M$ is called small in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. Dually, a submodule $L \leq M$ is called *essential* in M (denoted by $L \trianglelefteq M$) if $L \cap X \neq 0$ for every nonzero submodule X of M. Let U and V be submodules of M. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$ [13]. A submodule S of a module M has *ample supplements* in M if every submodule T such that $M = S + T$ containing submodule has a supplement in M and it is called *amply supplemented* if every submodule has ample supplements in M. If $M = U + V$ and $U \cap V \ll M$, then V is called a *weak supplement* of U in M, and M is a *weakly supplemented* module if every submodule of M has a weak supplement in M.

Recall that a submodule N of a module M is said to be δ-small in M, written $N \ll_\delta M$, provided $M \neq N + X$ for any proper submodule X of M with M/X singular [14]. Let L be a submodule of a module M. A submodule K of M is called a δ-supplement of L in M provided $M = L + K$ and $M \neq L + X$ for any proper submodule X of K with K/X singular, equivalently, $M = L + K$ and $L \cap K \ll_\delta K$. The module M is called δ-*supplemented* if every submodule of M has a δ-supplement in M [4]. On the other hand the submodule N is said to have *ample δ-supplement* in M if every submodule L of M with $M = N + L$ contains a

[1] Ondokuz Mayıs University, Faculty of Sciences and Arts, Department of Mathematics, Samsun, Turkey. E-mail: esraozturk55@hotmail.com; seren@omu.edu.tr.

δ-supplement of N in M. The module M is called *amply δ-supplemented* if every submodule of M has ample δ-supplements in M [11]. Let P be the class of all singular simple modules and M be a module. Then $\delta(M) = \cap \{N \leq M \mid M/N \in P\} = \sum\{N \leq M \mid N \ll_{\delta} M\}$.

Zöschinger generalized injective modules to modules with the property (E). He said that a module M has the property (E) if M has a supplement in every extension. He also said that a moodule M has the property (EE) if M has ample supplements in every extension [15]. In [4], a submodule M of a module N is called cofinite if the factor module N/M is finitely generated. Adapting Zöschinger's module with the properties (E) and (EE), Çalışıcı and Türkmen say that a module M has the property (CE) $((CEE))$ if M has a supplement (ample supplements) in every cofinite extension. Following this, in [9] the authors introduced modules with the properties (CWE) and $(CWEE)$.

Generalizing Zöschinger's module with the properties (E) and (EE) in [7] the authors introduced the concepts of modules with the properties δ-(CE) and δ-(CEE) and investigate basic properties of them. In conclusion, we show that if every submodule of a module M has the property δ-(CWE), then M has the property δ-$(CWEE)$. Moreover, if M has the property δ-(CWE), then every direct summand of M has the property δ-(CWE). We prove that over a left hereditary ring every factor module of a finitely generated module that has the property δ-(CWE) also has the property δ-(CWE). In addition, we give a characterization for δ-semilocal rings by using the property δ-(CWE) and over a δ-V-ring the concepts of modules with the properties δ-(CWE) and δ-(CE) coincide.

2. MAIN RESULTS

Definition: Let M be a module. We say that M has the property δ-(CE) if M has a δ-supplement in every cofinite extension.

Definition: Let M be a module. We say that M has the property δ-(CWE) if M has a weak δ-supplement in every cofinite extension and M has the property δ-$(CWEE)$ if M has weak ample δ-supplement in every cofinite extension.

Proposition: Every simple module has the property δ-(CWE).

Proof: Let S be a simple module and N be any cofinite extension of S. Then S is either a direct summand of N or δ-small in M. In the first case $S \oplus S' = N$ for a submodule $S' \leq N$ and so S' is a weak δ-supplement of S in N. In the second case, N is a weak δ-supplement of S in N. So in each case S has a weak δ-supplement in N. Finally S has the property δ-(CWE).

It is easy to see that every module with the property (CWE) and δ-(CE) has the property δ-(CWE). Let consider the $\mathbb{Z}$-module $\mathbb{Z}$ and $\mathbb{Z}$-module Q. Each of them is an example of a module that has the property $\mathbb{Z}$-module. It is natural to pose the question whether there exists similar result fort he properties of δ-(CE) and δ-(CE). To answer this, at the end of this section we shall give an example of a module which has the property δ-(CWE) but not δ-(CE).

Zöschinger proved in [15] that a module has the property (EE) if and only if every submodule has the property (E). Now we adopt only one side of this fact for our modules.

Theorem: Let M be a module. If every submodule of M has the property δ-(CWE), then M has the property δ-$(CWEE)$.

Proof: Suppose that every submodule of M has the property δ-(CWE). For a cofinite extension N of M, let $N = M + K$ for some submodule $K$of N. Then $N/M \cong K/(M \cap K)$ is finitely generated and so $M \cap K$ is a cofinite submodule of K. By the hypothesis, there exists a submodule V of K such that $K = (M \cap K) + V$ and $(M \cap K) \cap V = M \cap V \ll_\delta K$. Note that $N = M + V$. It follows that V is a weak δ-supplement of M in N. So M has the property δ-$(CWEE)$.

In the following proposition we show that the property δ-(CWE) is preserved by direct summands.

Proposition: Every direct summand of a module with the property δ-(CWE) has the property δ-(CWE).

Proof: Let N be a direct summand of M. Then there exists a submodule K of M such that $M = N \oplus K$. Let L be a cofinite extension of N, T be the external direct sum $L \oplus K$ and $\gamma: M \longrightarrow T$ be the canonical embedding. Then $M \cong \gamma(M)$ has the property δ-(CWE). We have $L/N \cong (L \oplus K)/\gamma(M)$ is finitely generated. Since $\gamma(M)$ has the property δ-(CWE), then there exists a submodule U of T such that $T = \gamma(M) + U$ and $\gamma(M) \cap U \ll_\delta T$. Consider the projection $\pi: T \longrightarrow L$. By this way, we have $N + \pi(U) = L$. Also $(\pi) \leq \gamma(M)$, $\pi(\gamma(M) \cap U) \leq \pi(\gamma(M)) \cap \pi(U) = N \cap \pi(U) \ll_\delta \pi(T) = L$. Therefore $\pi(U)$ is a weak δ-supplement of N in L.

Now by using the property δ-(CWE) we give a characterization for δ-semilocal rings which is related to cofinitely weak δ-supplemented modules investigated in [3, 8].

Theorem: Let R be a ring. Then the following statements are equivalent:

a) R is a δ-semilocal ring.

b) Every R-module has the property *δ-(CWE).*

Proof: Let R be a δ-semilocal ring, M be an R-module and N be a cofinite extension of M. Since R is δ-semilocal, N is a cofinitely weak δ-supplemented module from [3]. Therefore M has a weak δ-supplement in N as a submodule of M. Conversely, let M be an R-module and U be any cofinite submodule of M. By hypothesis, U has the property δ-(CWE). Then U has a weak δ-supplement in M, so that M is cofinitely weak δ-supplemented. Hence R is δ-semilocal by [3].

Corollary: Let R be a ring. Then every R-module is cofinitely weak δ-supplemented if and only if every R-module has the property δ-(CWE).

Let M be a module and U be a submodule of M. If the factor module M/U has the property δ-(CWE) M does not need to have the property δ-(CWE). For example, fort he ring $R = \mathbb{Z}$, the R-module $M = 2\mathbb{Z}/4\mathbb{Z}$ has a weak δ-supplement in every cofinite extension since it is simple. But $2\mathbb{Z}$ does not have a weak δ-supplement in its cofinite extension $\mathbb{Z}$.

Now we show that the statement mentioned above is true under a special condition.

Proposition: Let M be a module and U be a submodule of M. If $U \ll_\delta M$ and the factor module M/U has the property δ-(CWE), then M has the property δ-(CWE).

Proof: Let N be any extension of M. Since M/U has the property δ-(CWE), there exists a submodule V/U of N/U such that $M/U + V/U = N/U$ and $(M \cap V)/U \ll_\delta N/U$. Note that

$M+V=N$. Suppose that $(M\cap V)+S=N$ for a submodule S of N with N/S singular. Then we obtain $((M\cap V)/U)+((S+U)/U)=N/U$. Since $(M\cap V)/U \ll_{\delta} N/U$ and $N/(S+U)\cong (N/S)/(S+U)/S$ is singular, we have that $(S+U)/U=N/U$. It follows that $N=S+U=S$ and so $M\cap V \ll_{\delta} N$ is obtained.

Corollary: Every δ-local module has the property δ-(CWE).

Corollary: Let M be a module. If M has the property δ-(CWE), then so does every δ-small cover of M.

In [2], Çalışıcı and Türkmen defined cofinitely injective modules, that is, a module M is called cofinitely injective if M is a direct summand of every cofinite extension.

Recall that a ring R is called left δ-V-ring if $\delta(M)=0$ for every left R-module M [12].

Proposition: Let R be a left δ-V-ring. An R-module M has the property δ-(CWE) if and only if M is cofinitely injective.

Proof: Let M has the property δ-(CWE) and N be any extension of M. Then M has a weak δ-supplement V in N. We have $M+V=N$ and $M\cap V \ll_{\delta} N$. Hence $M\cap V\le \delta(N)=0$ and so $N=M\oplus V$. Conversely, let M be injective and N be any extension of M. Then there exists a submodule K of N such that $N=M\oplus K$. Hence K is a weak δ-supplement of M in N.

Corollary: Let R be a left δ-V-ring. An R-module M has the property δ-(CWE) if and only if M has the property δ-(CE).

Since every submodule of a δ-hollow module is δ-small we can give the following proposition fort he completeness.

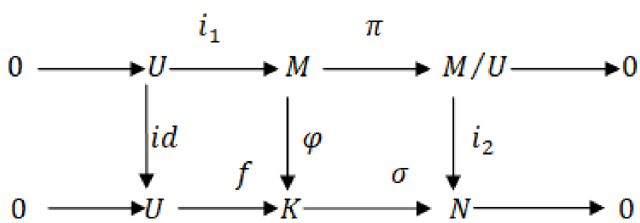
Proposition: If M is a δ-hollow module, then M has the property δ-(CWE).

Proof: Clear

Recall that over a left hereditary ring every factor module of an injective module is injective. In the following proposition, we show that every factor module of a module that has the property δ-(CWE) over a left hereditary ring has the property δ-(CWE).

Proposition: Let R be a left hereditary ring and M be a finitely generated module. If M has the property δ-(CWE),then so does every factor module of M.

Proof: For any submodule U of M, let N be a cofinite extension of M/U. Then N is finitely generated. By $E(M)$, we denote the injective hull of M. Since R is left hereditary, $E(M)/U$ is injective, and so there exists a commutative diagram with exact rows in the following:



i.e., $=\varphi i_1$, $\sigma\varphi = i_2\pi$, where $\varphi: M \longrightarrow K$ is a monomorphism. It follows that $K/\varphi(M) \cong K/Çek(\sigma) \cong N$. Since M has the property δ-(CWE), $\varphi(M)$ has a weak δ-supplement V in K. So we obtain that $\sigma(V)$ is a weak δ-supplement of M/U in N. Hence M/U has the property δ-(CWE).

It is easy to see that every module that has the property δ-(CE) also has the property δ-(CWE). Now we give the following example to show that the converse statement may not be true in general.

Example *(see in [1]):* For primes p and q, consider the ring $R \coloneqq \mathbb{Z}_{p,q} \coloneqq \left\{\frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0, (p,b) = (q,b) = 1\right\}$. R is a δ-semilocal ring that is not δ-semiperfect. Then there exists an R-module M that does not have the property δ-(CE). But since R is a δ-semilocal ring, M has the property δ-(CWE).

In the following theorem we see a kind of a module that coincide the concepts of properties δ-(CE) and δ-(CWE) over it.

Theorem: Let M be a semisimple module. Then the following statements are equivalent:
a) M has the property *δ-(CE).*
b) M has a δ-supplement in every cofinite extension N that is a direct summand of N.
c) M has the property *δ-(CWE).*

Proof: $(a \Longrightarrow b)$: Let N be any cofinite extension of M. By (a), we have $N = M + K$ and $M \cap K \ll_\delta K$ for some submodule $K \leq N$. Since M is a semisimple module, then there exists a submodule X of M such that $M = (M \cap K) \oplus X$. So $(M \cap K) \cap X = K \cap X = 0$. Therefore $N = M + K = [(M \cap K) \oplus X] = K \oplus X$. This means K is a δ-supplement of M that is a direct summand in N.

$(b \Longrightarrow c)$: Clear

$(c \Longrightarrow a)$: Let N be any cofinite extension of M. By (c), there exists a submodule K of N provided $N = M + K$ and $M \cap K \ll_\delta N$. Since $M \cap K \leq M$ and M is semisimple there exists a submodule T of M such that $(M \cap K) \oplus T = M$.

So, $N = M + K = (M \cap K) \oplus T + K = K \oplus T$ is obtained. Since K is a direct sum of N and $M \cap K \ll_\delta N$, it is obtained that $M \cap K \ll_\delta K$.

REFERENCES

[1] Anderson, F., Fuller, K.R., *Rings and categories of modules.* Springer-Verlag, 376s, New-York, 1992.
[2] Çalisici, H., Türkmen, E., *Georgian Mathematical Journal*, **19**(2), 209, 2012.
[3] Eryilmaz, F., *Miskolc Mathematical Notes*, **18**(2), 731, 2017.
[4] Kosan, M.T., *Algebra Colloquium*, **14**(1), 53, 2007.
[5] Nişancı, T.B., *Miskolc Mathematical Notes*, **14**(3), 1059, 2013.
[6] Özdemir, S., *Journal of the Korean Mathematical Society*, **53**(2), 403, 2016.
[7] Öztürk Sözen, E., Eren, Ş., *Europen Journal of Pure and Applied Mathematics*, **10**(4), 730, 2017.
[8] Öztürk Sözen, E., Eryımaz, F., Eren, Ş., *Journal of Science and Arts*, **2**(39), 269, 2017.
[9] Polat, M.N., Çalışıcı, H., Önal, E., *Palestine Journal of Mathematics*, **4**(1), 553, 2015.

[10] Özdemir, S., *Journal of the Korean Mathematical Society*, **53**(2), 403, 2016.
[11] Tribak, R., *Journal of Algebra and Its Applications*, **12**(2), 1250144, 2013.
[12] Ungor, B., Halıcıoğlu, S., Harmancı, A., *Deformation Theory of Algebras and Their Diagrams*, **116**, 123, 2012.
[13] Wisbauer, R., *Foundations of module and ring theory*, Revised and Updated English edition, Gordon and Breach, Philedelphia, 1991.
[14] Zhou, Y., *Algebra Colloquium*, **7**(3), 305, 2000.
[15] Zöschinger, H., *Mathematica Scandinavica*, **35**, 267, 1975.