ORIGINAL PAPER

A NEW APPROACH FOR ORDERING FUZZY NUMBERS AND ITS APPLICATION TO FUZZY MATRIX GAMES

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Abstract. *In this paper, a new geometric method for ordering generalized trapezoidal fuzzy numbers with different left and right heights, based on the Gergonne point of a triangle is presented. The suggested method can order not only generalized trapezoidal fuzzy numbers but also crisp numbers and generalized triangular fuzzy numbers with the same centroid point. We also give an application of the proposed ordering method to the fuzzy matrix games, whose payoffs are represented by generalized trapezoidal fuzzy numbers with different left and right heights.*

Keywords: *Generalized fuzzy numbers, Gergonne point, ranking fuzzy numbers, ordering fuzzy numbers, fuzzy matrix games.*

1. INTRODUCTION

Ordering and ranking fuzzy numbers and their comparisons play a significant role in decision-making problems in various fields such as social and economic systems, forecasting, optimization, and risk analysis [2, 5, 9, 12, 15-20, 33, 37, 39] and references therein). Various methods for the ordering fuzzy numbers have been presented since the method for ranking fuzzy numbers was initially proposed by Jain ([1, 4, 5, 8, 15-22, 25, 26, 28, 29, 32, 35, 38] and references therein).

The conventional approach for ordering fuzzy numbers is based on the centroid of a triangle, which was initially proposed by Yager [38]. Then Murakami et al. and Chen & Chen improved Yager's method [16, 35]. In [32], Lee & Chen suggested a method for fuzzy risk analysis based on fuzzy numbers with different shapes and deviations. Chen & Chen [17] proposed a method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads which considers defuzzified values. Using the areas and the centroid values of the generalized fuzzy numbers, Chen et al. [18] submitted a fuzzy ranking method for generalized fuzzy numbers with different left heights and right heights. Since this method has certain shortcomings, it is improved by Jiang et al. [29]. Bakar & Gegov propose a method for ranking fuzzy numbers that integrates the centroid point and the spread approaches to overcome the limitations and weaknesses of most existing methods [8]. By means of integral values, an another approach for the ranking of generalized fuzzy numbers with different left and right heights is given by Chutia et al. [22]. Using the center of the nine-point circle of a triangle, Düzce presented a new method [25]. Nevertheless, this method cannot rank fuzzy numbers with the same centroid.

In this paper, the centroid methods will be improved using the Gergonne point of a triangle, which is the intersection point of the cevians joining the points of tangency of the

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incircle with the side lines and the opposite vertices of the triangle [30]. In order to save most of the information of generalized trapezoidal fuzzy numbers with different left and right heights, we associate these fuzzy numbers with triplets instead of real numbers. By the lexicographical order, we will sort these triplets and consequently corresponding fuzzy numbers.

The rest of this paper is organized as follows. In Section 2, fuzzy numbers and Gergonne point of a triangle, their fundamental properties, basic definitions and theorems are reviewed. Next, in Section 3, we present a new method for ordering generalized trapezoidal fuzzy numbers with different left and right heights based on the Gergonne point of a triangle. In Section 4, numerical examples are given. We compare the results of the suggested method with other existing methods. An application to the fuzzy matrix games of the proposed method is presented in Section 5. Finally, conclusions are given in Section 6.

2. PRELIMINARIES

2.1. FUZZY NUMBERS

For the sake of completeness, we briefly recall here the certain essential concepts of fuzzy numbers and their basic properties along with this paper. For further information, see [11] and [31].

A fuzzy set $\tilde{A}$ on a set X is a function $\tilde{A}: X \to [0,1]$. Commonly, the symbol $\mu_{\tilde{A}}$ is used for the function $\tilde{A}$, and it is said that the fuzzy set $\tilde{A}$ is characterized by its membership function $\mu_{\tilde{A}}: X \to [0,1]$ which associates each $x \in X$ with a real number $\mu_{\tilde{A}}(x) \in [0,1]$. The degree to which x belongs to $\tilde{A}$ is interpreted by the value of $\mu_{\tilde{A}}(x)$.

Let $\tilde{A}$ be a fuzzy set on X. The support of $\tilde{A}$ is given as:

$$\mathrm{S}(\tilde{A}) = \{x \in X : \mu_{\tilde{A}}(x) > 0\},$$

and the height $\mathrm{h}(\tilde{A})$ of $\tilde{A}$ is defined as:

$$\mathrm{h}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x).$$

Let $\tilde{A}$ be a fuzzy set on X and $\alpha \in [0,1]$. The α–cut (α–level set) of the fuzzy set $\tilde{A}$ is given by:

$$[\tilde{A}]^{\alpha} = \begin{cases} \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\} & , \quad if\ \alpha \in (0,1] \\ \mathrm{cl}\ \mathrm{S}(\tilde{A}) & , \quad if\ \alpha = 0 \end{cases}$$

where cl denotes the closure of sets.

A fuzzy set $\tilde{A}$ on $\mathbb{R}^n$ is called a convex fuzzy set if its α–cuts $[\tilde{A}]^{\alpha}$ are convex sets for all $\alpha \in [0,1]$.

Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$, then $\tilde{A}$ is called a generalized fuzzy number if

i. $\tilde{A}$ is convex,
ii. $0 < h(\tilde{A}) \leq 1$,
iii. $\mu_{\tilde{A}}$ is upper semi-continuous,
iv. the support of $\tilde{A}$ is bounded.

Generally, certain special types of fuzzy numbers, such as LR-fuzzy numbers, trapezoidal fuzzy numbers, triangular fuzzy numbers, and so on are used for real life applications. In this study, we will focus on the generalized trapezoidal fuzzy numbers with different left and right heights.

Let $a \le b \le c \le d$ and $\omega_L, \omega_R \in (0,1]$ be real numbers. $\tilde{A}$ be a generalized fuzzy number which is characterized by membership function

$$\mu_{\tilde{A}}(\cdot): \mathbb{R} \to [0,1], \mu_{\tilde{A}}(x) = \begin{cases} \mu_1(x) & \text{if } a \le x \le b, \\ \mu_2(x) & \text{if } b \le x \le c, \\ \mu_3(x) & \text{if } c \le x \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Here,

$$\begin{aligned} &\mu_1: [a,b] \to [0, \omega_L], \\ &\mu_2: [b,c] \to [\min\{\omega_L, \omega_R\}, \max\{\omega_L, \omega_R\}] \text{ and} \\ &\mu_3: [c,d] \to [0, \omega_R] \end{aligned}$$

are continuous functions. Furthermore, μ_1 is strictly increasing, μ_3 is strictly decreasing and μ_2 is strictly increasing when $\omega_L < \omega_R$ and strictly decreasing when $\omega_L > \omega_R$. Then $\tilde{A}$ is called generalized fuzzy number with different left and right heights, where, ω_L and ω_R are the left and right heights of $\tilde{A}$.

If μ_1, μ_2 and μ_3 are also linear, that is the membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{\omega_L(x-a)}{b-a} & \text{if } a \le x \le b, \\ \dfrac{\omega_L(c-b) + (\omega_R - \omega_L)(x-b)}{c-b} & \text{if } b \le x \le c, \\ \dfrac{\omega_R(x-d)}{c-d} & \text{if } c \le x \le d, \\ 0 & \text{otherwise} \end{cases}$$

then $\tilde{A}$ is called generalized trapezoidal fuzzy number with different left and right heights and it is denoted by $\tilde{A} = (a,b,c,d;\omega_L,\omega_R)$.

If $\omega_L = \omega_R$, then $\tilde{A}$ is said generalized trapezoidal fuzzy number (see Figure 1). Furthermore, if $b = c$ then we get $\omega_L = \omega_R$ and $\tilde{A}$ is called generalized triangular fuzzy number.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and k be a real number. Then the sum of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} + \tilde{B} := \sup_{z=x+y} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}.$$

In particular, if $\tilde{A} = (a,b,c,d;\omega_L,\omega_R)$ and $\tilde{B} = (a',b',c',d';\omega'_L,\omega'_R)$ are generalized trapezoidal fuzzy numbers with different left and right heights, then we get

$$\begin{gathered} \tilde{A} + \tilde{B} := (a+a', b+b', c+c', d+d'; \\ \min\{\omega_L, \omega'_L\}, \min\{\omega_R, \omega'_R\}). \end{gathered}$$

The scalar product of k and $\tilde{A}$ is defined as

$$k\tilde{A} := \max\left\{0, \sup_{z=kx} \mu_{\tilde{A}}(x)\right\}.$$

In particular, if $\tilde{A} = (a, b, c, d; \omega_L, \omega_R)$ is generalized trapezoidal fuzzy number with different left and right heights, then one can verify that

$$k\tilde{A} = \begin{cases} (ka, kb, kc, kd; \omega_L, \omega_R) & , \quad if\ k \geq 0, \\ (kd, kc, kb, ka; \omega_R, \omega_L) & , \quad if\ k < 0. \end{cases}$$

We denote the set of all generalized trapezoidal fuzzy numbers with different left and right heights by the notation $\mathbb{F}$.

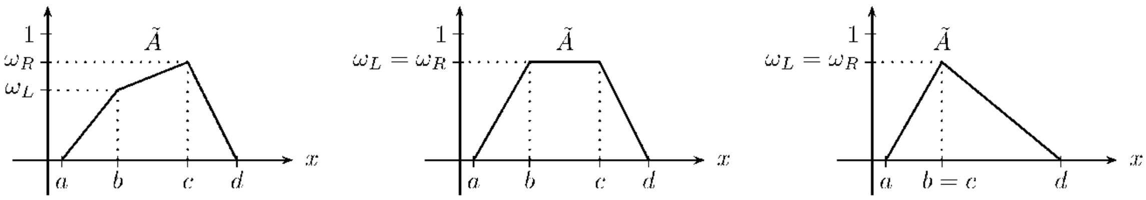

Figure 1. A generalized trapezoidal fuzzy number with different left and right heights, a generalized trapezoidal fuzzy number and a generalized triangular fuzzy number.

2.2. GERGONNE POINT OF A TRIANGLE

The Gergonne point G of a triangle ABC is defined as the point of intersection of the cevians AT_A, BT_B, and CC_T, where T_A, T_B, T_C are points where the incircle touch the sides (Fig. 2).

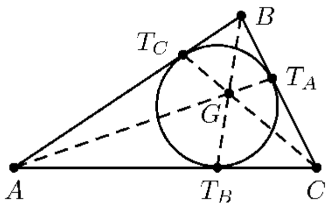

Figure 2. The Gergonne Point G of a triangle ABC.

Using Ceva's Theorem we immediately get the following result.

Theorem 2.1 [34] The three cevians joining the vertices of a triangle to the point of tangency of the opposite sides with the incircle are concurrent.

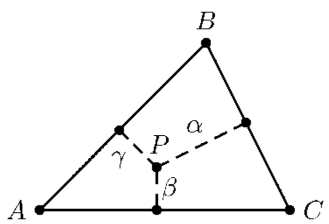

Figure 3. Trilinear coordinates α: β: γ of point P.

Given a triangle ABC, the trilinear coordinates of a point P, with respect to triangle ABC, are an ordered triple of numbers, each of which is proportional to the directed distance from P to one of the side lines. Trilinear coordinates are usually denoted by $\alpha{:}\,\beta{:}\,\gamma$ (Fig. 3). If point P has trilinear coordinates $\alpha{:}\,\beta{:}\,\gamma$, then the Cartesian coordinates of P are calculated as follows:

$$P = \left(\frac{1}{\alpha|BC|+\beta|AC|+\gamma|AB|}\right)(\alpha|BC|A + \beta|AC|B + \gamma|AB|C). \tag{2.1}$$

Trilinear coordinates of the Gergonne point are given as:

$$\sec^2\left(\frac{\widehat{BAC}}{2}\right) : \sec^2\left(\frac{\widehat{ABC}}{2}\right) : \sec^2\left(\frac{\widehat{ACB}}{2}\right)$$

or by the Law of Sines its equivalent [27]:

$$\frac{|AC|\cdot|AB|}{|AC|+|AB|-|BC|} : \frac{|BC|\cdot|AB|}{|BC|+|AB|-|AC|} : \frac{|BC|\cdot|AC|}{|BC|+|AC|-|AB|} \tag{2.2}$$

3. A NEW GEOMETRIC METHOD TO ORDER GENERALIZED TRAPEZOIDAL FUZZY NUMBERS WITH DIFFERENT LEFT AND RIGHT HEIGHTS

In this section, we present a new method for ordering generalized trapezoidal fuzzy numbers with different left and right heights which considers the Gergonne point of a triangle.

For any generalized trapezoidal fuzzy number with different left and right heights $\tilde{A} = (a, b, c, d; \omega_L, \omega_R)$, we can obtain two generalized triangular fuzzy numbers

$$\tilde{A}_L = (a, b, b, d, ; \omega_L, \omega_L)$$

and

$$\tilde{A}_R = (a, c, c, d; \omega_R, \omega_R)$$

as shown in Fig. 4.

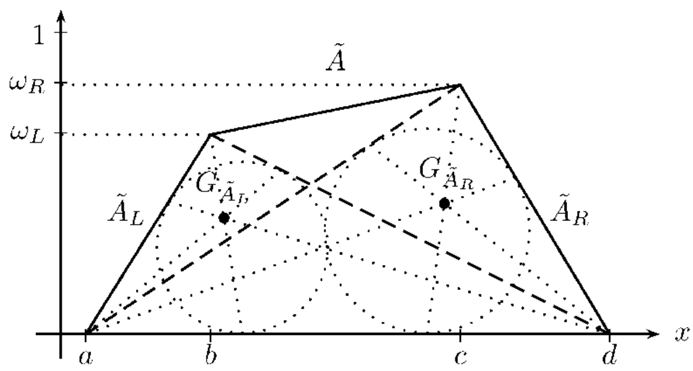

Figure 4. A generalized trapezoidal fuzzy number $\tilde{A}$ with different left and right heights and generalized triangular fuzzy numbers $\tilde{A}_L$ and $\tilde{A}_R$ with their Gergonne points $G_{\tilde{A}_L}$ and $G_{\tilde{A}_R}$.

Let us first consider generalized triangular fuzzy number $\tilde{A}_L = (a, b, b, d, ; \omega_L, \omega_L)$ then we get a triangle ABC with the coordinates of vertices are $A = (x_A, y_A) = (a, 0)$, $B = (x_B, y_B) = (b, \omega_L)$ and $C = (x_C, y_C) = (d, 0)$.

Thus we obtain

$$\begin{array}{rcl} |BC| &=& \sqrt{(b-d)^2+\omega_L^2}, \\ |AC| &=& d-a, \\ |AB| &=& \sqrt{(a-b)^2+\omega_L^2}. \end{array} \tag{3.1}$$

Using (2.2) and (3.1) we find trilinear coordinates $\alpha:\beta:\gamma$ of Gergonne point $G_{\tilde{A}_L}$ of triangle ABC as follows:

$$\alpha = \frac{(d-a)\sqrt{(a-b^2)+\omega_L^2}}{(d-a)+\sqrt{(a-b^2)+\omega_L^2}-\sqrt{(b-d^2)+\omega_L^2}},$$

$$\beta = \frac{\sqrt{(b-d^2)+\omega_L^2}\sqrt{(a-b^2)+\omega_L^2}}{\sqrt{(b-d)^2+\omega_L^2}+\sqrt{(a-b)^2+\omega_L^2}-(d-a)},$$

$$\gamma = \frac{(d-a)\sqrt{(b-d)^2+\omega_L^2}}{\sqrt{(b-d)^2+\omega_L^2}+(d-a)-\sqrt{(a-b)^2+\omega_L^2}}.$$

Using (2.1) the Cartesian coordinates of Gergonne point $G_{\tilde{A}_L}$ of left-hand side triangle ABC can be calculated as follows:

$$G_{\tilde{A}_L} := \left(x_{\tilde{A}_L}, y_{\tilde{A}_L}\right) = \left(\frac{a\alpha|BC|+b\beta|AC|+d\gamma|AB|}{\alpha|BC|+\beta|AC|+\gamma|AB|}, \frac{\beta|AC|}{\alpha|BC|+\beta|AC|+\gamma|AB|}\right). \tag{3.2}$$

Similarly, for a generalized triangular fuzzy number $\tilde{A}_R = (a, c, c, d; \omega_R, \omega_R)$ we get a triangle $A'B'C'$ with the coordinates of vertices are $A' = (x_{A'}, y_{A'}) = (a, 0)$, $B' = (x_{B'}, y_{B'}) = (c, \omega_R)$ and $C' = (x_{C'}, y_{C'}) = (d, 0)$.

Thus, we obtain

$$\begin{array}{rcl} |B'C'| &=& \sqrt{(c-d)^2+\omega_R^2}, \\ |A'C'| &=& d-a, \\ |A'B'| &=& \sqrt{(a-c)^2+\omega_R^2}. \end{array} \tag{3.3}$$

Using (2.2) and (3.3) we find trilinear coordinates $\alpha':\beta':\gamma'$ of Gergonne point $G_{\tilde{A}_R}$ as follows:

$$\alpha' = \frac{(d-a)\sqrt{(a-c^2)+\omega_R^2}}{(d-a)+\sqrt{(a-c^2)+\omega_R^2}-\sqrt{(c-d^2)+\omega_R^2}},$$

$$\beta' = \frac{\sqrt{(c-d^2)+\omega_R^2}\sqrt{(a-c^2)+\omega_R^2}}{\sqrt{(c-d)^2+\omega_R^2}+\sqrt{(a-c)^2+\omega_R^2}-(d-a)},$$

$$\gamma' = \frac{(d-a)\sqrt{(c-d)^2+\omega_R^2}}{\sqrt{(c-d)^2+\omega_R^2}+(d-a)-\sqrt{(a-b)^2+\omega_L^2}}.$$

Using (2.1) the Cartesian coordinates of Gergonne point $G_{\tilde{A}_R}$ of right-hand side triangle $A'B'C'$ can be calculated as follows:

$$G_{\tilde{A}_R} := \left(x_{\tilde{A}_R}, y_{\tilde{A}_R}\right) = \left(\frac{a\alpha'|B'C'|+c\beta'|A'C'|+d\gamma'|A'B'|}{\alpha'|B'C'|+\beta'|A'C'|+\gamma'|A'B'|}, \frac{\beta'|A'C'|}{\alpha'|B'C'|+\beta'|A'C'|+\gamma'|A'B'|}\right). \tag{3.4}$$

Using the notations given above, we define the ordering of generalized trapezoidal fuzzy numbers with different left and right heights $\tilde{A} = (a, b, c, d; \omega_L, \omega_R)$ and $\tilde{B} = (a', b', c', d'; \omega'_L, \omega'_R)$ as follows:

$$\tilde{A} \prec \tilde{B} \Leftrightarrow \left(\omega_R x_{\tilde{A}_R}, \omega_L x_{\tilde{A}_L}, 1 - y_{\tilde{A}_R}\right) <_L \left(\omega'_R x_{\tilde{B}_R}, \omega'_L x_{\tilde{B}_L}, 1 - y_{\tilde{B}_R}\right), \tag{3.5}$$

where the symbol $<_L$ denotes the lexicographical (dictionary) order, and it is defined as follows

$$(x_1, x_2, x_3) <_L (y_1, y_2, y_3) \Leftrightarrow (\exists m = 1,2,3)(\forall i < m)(x_i = y_i) \wedge (x_m < y_m).$$

Obviously, $(x_1, x_2, x_3) <_L (y_1, y_2, y_3)$ if and only if $x_1 < y_1$ or ($x_1 = y_1$ and $x_2 < y_2$) or ($x_1 = y_1$ and $x_2 = y_2$ and $x_3 < y_3$).

Let $\tilde{A}, \tilde{B} \in \mathbb{F}$. If $\tilde{A}$ and $\tilde{B}$ are the same or $\tilde{A} \prec \tilde{B}$, then we write $\tilde{A} \preccurlyeq \tilde{B}$.

If $\tilde{A}$ and $\tilde{B}$ are generalized trapezoidal fuzzy numbers with different left and right heights the precedence ordering of $\tilde{A}$ and $\tilde{B}$ can be obtained by the following algorithm.

Algorithm 3.1

Step 1: Input two generalized trapezoidal fuzzy numbers with different left and right heights $\tilde{A} = (a, b, c, d; \omega_L, \omega_R)$ and $\tilde{B} = (a', b', c', d'; \omega'_L, \omega'_R)$.

Step 2: Using (3.4) calculate the horizontal coordinates $x_{\tilde{A}_R}$ and $x_{\tilde{B}_R}$ of the Gergonne point of the right-hand side triangles. Compare $\omega_R x_{\tilde{A}_R}$ and $\omega'_R x_{\tilde{B}_R}$. If $\omega_R x_{\tilde{A}_R} < \omega'_R x_{\tilde{B}_R}$, then $\tilde{A} \prec \tilde{B}$ or $\omega'_R x_{\tilde{B}_R} < \omega_R x_{\tilde{A}_R}$ then $\tilde{B} \prec \tilde{A}$ and stop. Otherwise goto step 3.

Step 3: Using (3.2) calculate the horizontal coordinates $x_{\tilde{A}_L}$ and $x_{\tilde{B}_L}$ of the Gergonne point of the left-hand side triangles. Compare $\omega_L x_{\tilde{A}_L}$ and $\omega'_L x_{\tilde{B}_L}$. If $\omega_L x_{\tilde{A}_L} < \omega'_L x_{\tilde{B}_L}$, then $\tilde{A} \prec \tilde{B}$ or $\omega'_L x_{\tilde{B}_L} < \omega_L x_{\tilde{A}_L}$, then $\tilde{B} \prec \tilde{A}$ and stop. Otherwise goto step 4.

Step 4: Using (3.4) calculate the vertical coordinates $y_{\tilde{A}_R}$ and $y_{\tilde{B}_R}$ of the Gergonne point of the right-hand side triangles. Compare $1 - y_{\tilde{A}_R}$ and $1 - y_{\tilde{B}_R}$. If $1 - y_{\tilde{A}_R} < 1 - y_{\tilde{B}_R}$, then $\tilde{A} \prec \tilde{B}$ or $1 - y_{\tilde{B}_R} < 1 - y_{\tilde{A}_R}$, then $\tilde{B} \prec \tilde{A}$. Otherwise, $\tilde{A} \approx \tilde{B}$ that is $\tilde{A}$ and $\tilde{B}$ fuzzy numbers are the same.

Proposition 3.2 Let $0 < \omega \leq 1$ and δ_1, δ_2 be arbitrarily small positive real numbers. If coordinates of the vertices A, B, and C of a triangle ABC are $(b - \delta_1, 0)$, (b, ω) and $(b + \delta_2, 0)$ respectively, then

$$\lim_{\delta_1, \delta_2 \to 0^+} G = (b, 0).$$

Here, G denotes the Gergonne point of the triangle ABC.

Proof. If $\delta_1, \delta_2 \to 0^+$ then $|BC|, |AB| \to \omega$ and $|AC| \to 0^+$. Substituting them into (2.1) yields

$$\begin{aligned}\lim_{\delta_1,\delta_2\to 0^+} G &= \lim_{\delta_1,\delta_2\to 0^+}\left(\frac{\alpha|BC|A+\beta|AC|B+\gamma|AB|C}{\alpha|BC|+\beta|AC|+\gamma|AB|}\right)\\ &= \lim_{\delta_1,\delta_2\to 0^+}\left(\frac{(b-\delta_1)\alpha|BC|+b\beta|AC|+(b+\delta_2)\gamma|AB|}{\alpha|BC|+\beta|AC|+\gamma|AB|}, \frac{\beta\omega|AC|}{\alpha|BC|+\beta|AC|+\gamma|AB|}\right)\\ &= \left(\frac{b\alpha\omega+b\gamma\omega}{\alpha\omega+\gamma\omega}, 0\right)\\ &= (b, 0).\end{aligned}$$

This completes the proof of proposition.

Remark 3.3 In view of Proposition 3.2, if a generalized trapezoidal fuzzy number with different left and right heights $\widetilde{\mathrm{A}} = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}; \omega_{\mathrm{L}}, \omega_{\mathrm{R}})$ is a crisp (real) number, that is, $\mathrm{a} = \mathrm{b} = \mathrm{c} = \mathrm{d} = \rho$ and $\omega_{\mathrm{L}} = \omega_{\mathrm{R}} = 1$ then we can assume that $\mathrm{x}_{\widetilde{\mathrm{A}}_{\mathrm{L}}} = \mathrm{x}_{\widetilde{\mathrm{A}}_{\mathrm{R}}} = \rho$ and $\mathrm{y}_{\widetilde{\mathrm{A}}_{\mathrm{L}}} = \mathrm{y}_{\widetilde{\mathrm{A}}_{\mathrm{R}}} = 0$. Hence, for crisp number $\widetilde{\mathrm{A}}$ we get the triplet $(\rho, \rho, 1)$ and using (3.5), we can also order crisp numbers.

Let $\tilde{V} = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n)$ be any fuzzy vector, that is, all components of $\tilde{V}$ are fuzzy numbers. Then maximum and minimum of $\tilde{V}$ in the sense of proposed method are denoted as

$$\mathrm{Max}\tilde{V} \; and \; \mathrm{Min}\tilde{V},$$

respectively.

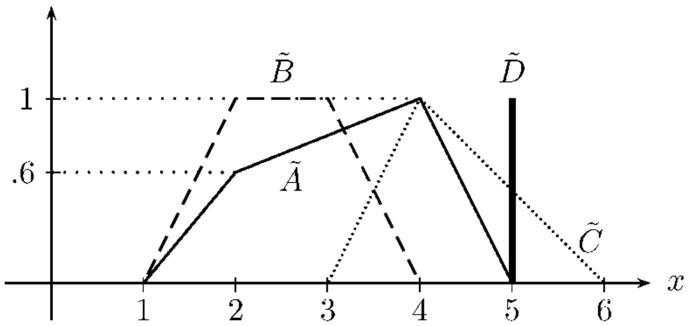

Figure 5. Fuzzy numbers $\tilde{A} = (1, 2, 4, 5; .6, 1)$, $\tilde{B} = (1, 2, 3, 4; 1, 1)$, $\tilde{C} = (3, 4, 4, 6; 1, 1)$, and $\tilde{D} = (5, 5, 5, 5; 1, 1)$ in Example 3.4.

We initially give a simple example to show that the proposed method is reliable and effective.

Example 3.4 Let $\widetilde{\mathrm{A}} = (1,2,4,5; .6,1)$, $\widetilde{\mathrm{B}} = (1,2,3,4; 1,1)$, $\tilde{\mathrm{C}} = (3,4,4,6; 1,1)$, and $\widetilde{\mathrm{D}} = (5,5,5,5; 1,1)$ be fuzzy numbers (see Figure 5). Using (3.2) and (3.4) or the Gergonne point of the given triangle is derived using the computer geometry software Maple, and Remark 3.3 we find

$$\begin{aligned} G_{\tilde{A}_L} &= (2.0069, .5239), & G_{\tilde{A}_R} &= (3.9668, .7373),\\ G_{\tilde{B}_L} &= (2.0284, .6809), & G_{\tilde{B}_R} &= (2.9716, .6809),\\ G_{\tilde{C}_L} &= (4.0284, .6809), & G_{\tilde{C}_R} &= (4.0284, .6809),\\ G_{\tilde{D}_L} &= (5,0), & G_{\tilde{D}_R} &= (5,0)\end{aligned}$$

and we have

$$\left(\omega_R x_{\tilde{A}_R}, \omega_L x_{\tilde{A}_L}, 1 - y_{\tilde{A}_R}\right) = (3.9668, 1.2041, .2627),$$
$$\left(\omega_R x_{\tilde{B}_R}, \omega_L x_{\tilde{B}_L}, 1 - y_{\tilde{B}_R}\right) = (2.9716, 2.0284, .3191),$$
$$\left(\omega_R x_{\tilde{C}_R}, \omega_L x_{\tilde{C}_L}, 1 - y_{\tilde{C}_R}\right) = (4.0284, 4.0284, .3191),$$
$$\left(\omega_R x_{\tilde{D}_R}, \omega_L x_{\tilde{D}_L}, 1 - y_{\tilde{D}_R}\right) = (5,5,1).$$

Using Algorithm 3.1 or since

$$(2.9716, 2.0284, .3191) <_L (3.9668, 1.2041, .2627) <_L (4.0284, 4.0284, .3191) <_L (5,5,1),$$

we get the ranking order

$$\tilde{B} < \tilde{A} < \tilde{C} < \tilde{D}.$$

If $\tilde{V} = \left(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\right)$ is a fuzzy vector then we also obtain

$$\mathrm{Max}\left(\tilde{V}\right) = \tilde{D} \text{ and } \mathrm{Min}\left(\tilde{V}\right) = \tilde{B},$$

which coincides with the intuition of human beings.

Example 3.5 To compare the proposed method with a known method we use three generalized trapezoidal fuzzy numbers with different left and right heights

$$\tilde{A} = (.1659, .2803, .7463, 1.154; .5, .6),$$
$$\tilde{B} = (.1611, .2475, .5696, .8187; .4, .5) \textit{ and}$$
$$\tilde{C} = (.1645, .2445, .5869, .8894; .5, .6)$$

which are adopted from [22] (see Figure 6). Using (3.2), (3.4) and Remark 3.3 we find

$$G_{\tilde{A}_L} = (.3585, .2054), \qquad G_{\tilde{A}_R} = (.7292, .2762),$$
$$G_{\tilde{B}_L} = (.3093, .1487), \qquad G_{\tilde{B}_R} = (.5468, .1857),$$
$$G_{\tilde{C}_L} = (.3298, .1657), \qquad G_{\tilde{C}_R} = (.5679, .2075)$$

and we have

$$\left(\omega_R x_{\tilde{A}_R}, \omega_L x_{\tilde{A}_L}, 1 - y_{\tilde{A}_R}\right) = (.4375, .1793, .7238),$$
$$\left(\omega_R x_{\tilde{B}_R}, \omega_L x_{\tilde{B}_L}, 1 - y_{\tilde{B}_R}\right) = (.2734, .1237, .8143),$$
$$\left(\omega_R x_{\tilde{C}_R}, \omega_L x_{\tilde{C}_L}, 1 - y_{\tilde{C}_R}\right) = (.3407, .1649, .7926).$$

In view of Algorithm 3.1 or since

$$(.2734, .1237, .8143) <_L (.3407, .1649, .7926) <_L (.4375, .1793, .7238),$$

we get the ranking order

$$\tilde{B} < \tilde{C} < \tilde{A},$$

which coincides with the result established in [22] and intuition of human beings.

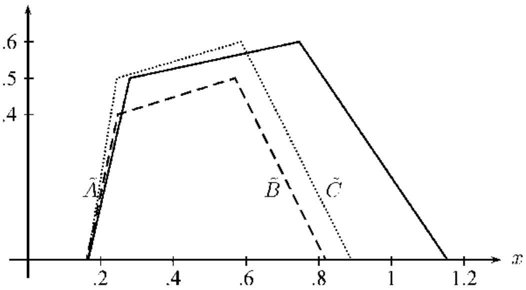

Figure 6. Generalized trapezoidal fuzzy numbers with different left and right heights $\tilde{A} = (.1659, .2803, .7463, 1.154; .5, .6)$, $\tilde{B} = (.1611, .2475, .5696, .8187; .4, .5)$ and $\tilde{C} = (.1645, .2445, .5869, .8894; .5, .6)$.

4. NUMERICAL EXAMPLES

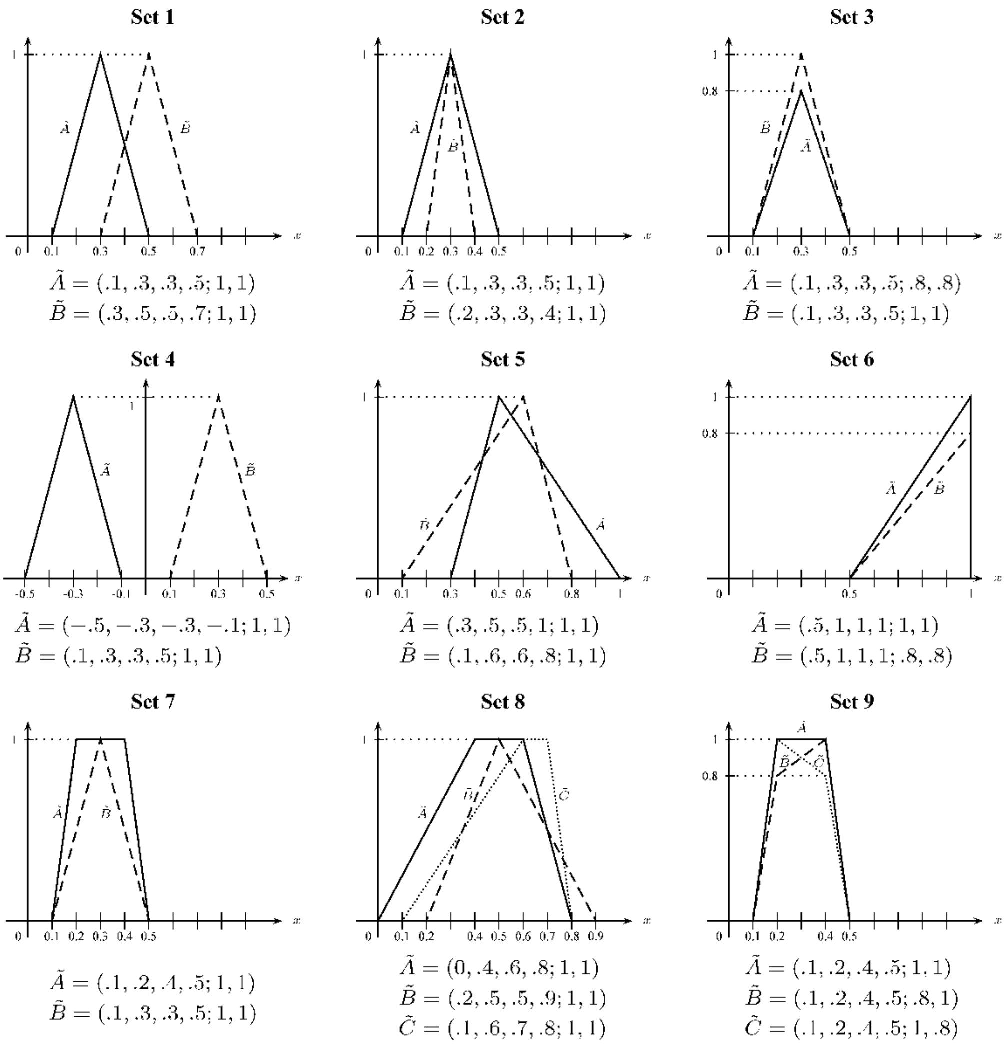

Figure 7. Sets of generalized fuzzy numbers.

Here, to compare the proposed method with certain existing methods, we use nine sets of generalized fuzzy numbers adopted from [29] as shown in Fig. 7. The comparison of the results for different methods is shown in Table 1. From Table 1, we can see that the proposed method gives the reasonable ranking order and the results are similar to listed methods. Moreover, the proposed method overcomes the shortcomings of the ranking methods given in

[8, 17, 18, 22, 25, 29, 32]. From Table 1, we can see the drawbacks of the existing methods, described as follows:

Table 1. A comparison of methods.

Method	**Set 1**	**Set 2**	**Set 3**	**Set 4**	**Set 5**
Lee & Chen (2008)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Chen & Chen (2009)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Chen et al. (2012)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Bakar & Gegov (2014)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Chutia et al., ($\alpha = .5$) (2015)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Jiang et al. (2015)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Düzce (2015)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$
Proposed	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A}$

Method	**Set 6**	**Set 7**	**Set 8**	**Set 9**
Lee & Chen (2008)	$\tilde{B} \prec \tilde{A}$	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{B}, \tilde{C} \prec \tilde{A}, \tilde{B} \sim \tilde{C}$
Chen & Chen (2009)	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$-$
Chen et al. (2012)	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{C} \prec \tilde{A} \prec \tilde{B}$
Bakar & Gegov (2014)	$\tilde{B} \prec \tilde{A}$	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Chutia et al., ($\alpha = .5$) (2015)	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \sim \tilde{B} \sim \tilde{C}$
Jiang et al. (2105)	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{C} \prec \tilde{A} \prec \tilde{B}$
Düzce (2015)	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{C} \prec \tilde{A} \prec \tilde{B}$
Proposed	$\tilde{B} \prec \tilde{A}$	$\tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{C} \prec \tilde{B} \prec \tilde{A}$

In Set 1, Set 4, Set 5 of Fig. 7, all the listed methods achieve the same order as well as the proposed method. In Set 2 of Fig. 7, the methods of Chen et al., Düzce and for $\alpha = .5$ the method of Chutia et al. is not discriminative. However the methods of Lee & Chen, Chen & Chen, Bakar & Gegov, Jiang et al., for $\alpha = 0$ the method of Chutia et al. and the proposed method obtain the same order. The method of Chutia et al. for $\alpha = 1$ achieves a counterintuitive order. In Set 3 of Fig. 7, all listed methods achieve the same order except for the method of Chutia et al. In Set 6 of Fig. 7, the methods of Chen et al. and Chutia et al. is not discriminative, while the proposed method and the other listed methods obtain the same order. In Set 7 of Fig. 7, the methods of Chen et al., Düzce and for $\alpha = .5$ the method of Chutia et al. is not discriminative. The methods of Lee & Chen, Jiang et al., for $\alpha = 0$ the method of Chutia et al. achieves a counterintuitive order. However, by the proposed method we have overcome this shortcoming and obtain the same order as the other listed methods. In Set 8 of Fig. 7, the method of Chutia et al. for $\alpha = .5$ and all the other listed methods achieve the same order as well as the proposed method. The method of Chutia et al. is not discriminative for $\alpha = 0$ and $\alpha = 1$. In Set 9 of Fig. 7, the method of Chen & Chen cannot be applicable. The methods of Lee & Chen and Chutia et al. is not discriminative. The methods of Chen et al., Jiang et al. and Düzce obtain the same order. But, listed methods with the ranking order Set 2, Set 3, Set 6, Set 7 have considered the proposed method gives more reasonable order in Set 9 of Fig. 7.

Thus, other listed methods have certain drawbacks, such as, the result order being inconsistent with human intuition, the method is not discriminative, or it cannot order fuzzy numbers with the same centroid. However, the proposed method avoids these problems for ordering generalized trapezoidal fuzzy numbers with different left and right heights.

5. APPLICATION TO TWO-PERSON ZERO–SUM GAME WITH FUZZY PAYOFFS

Fuzzy matrix game is the generation of classical matrix games. In the real world the certainty assumption is not realistic on many occasions. This lack of precision may be modeled via fuzzy numbers. In this section, we consider a zero–sum game, whose payoffs are represented by generalized trapezoidal fuzzy numbers with different left and right heights with two players. We assume that player I tries to maximize the profit and player II tries to minimize the costs.

The two–person zero–sum game with fuzzy payoffs is defined by $m \times n$ matrix $\tilde{G}$ whose entries are generalized trapezoidal fuzzy numbers with different left and right heights. A (mixed) *strategy* of player I is a probability distribution x over the rows of $\tilde{G}$, that is, an element of the set

$$X_m = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m :\ x_i \geq 0 \ for\ all\ i = 1,2,\dots,m\ and\ \textstyle\sum_{i=1}^{m} x_i = 1\}.$$

Equivalently, a strategy of player II is a probability distribution y over the columns of $\tilde{G}$, that is, an element of the set

$$Y_n = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n :\ y_i \geq 0 \ for\ all\ i = 1,2,\dots,n\ and\ \textstyle\sum_{i=1}^{n} y_i = 1\}.$$

A strategy x of player I is called *pure* if it does not involve probability, that is, $x_i = 1$ for some $i = 1,2,\dots,m$ and it is denoted by I_i. Similarly, pure strategies of player II are denoted by II_j for $j = 1,2,\dots,n$.

Let $\tilde{G}$ be a fuzzy matrix game

$$\tilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{I}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \dots & \mathrm{II}_n \end{array} \\ \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1n} \\ \tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{g}_{m1} & \tilde{g}_{m2} & \cdots & \tilde{g}_{mn} \end{pmatrix} \end{array}$$

and $x \in X_m$, $y \in Y_n$, that is, x and y are strategies for players I and II. Then the expected payoff for player I is defined by

$$\tilde{g}(x, y) = x^{\mathrm{T}} \tilde{G} y = \textstyle\sum_i \sum_j x_i y_j \tilde{g}_{ij}.$$

Table 2. $\tilde{G}$ fuzzy matrix game in Example 5.1

$$\tilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \mathrm{I}_3 \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \mathrm{II}_3 & \mathrm{II}_4 \end{array} \\ \begin{pmatrix} (0,1,2,3;.8,1) & (0,1,1,2;1,.7) & (-3,-2,-1,0;.3,1) & (0,1,2,4;1,1) \\ (-1,0,2,3;1,.8) & (1,3,5,7;1,1) & (1,3,4,5;1,.6) & (2,2,2,2;1,1) \\ (0,3,4,5;.4,.7) & (2,3,3,6;1,1) & (3,4,5,8;.8,.8) & (-2,-1,0,2;.6,1) \end{pmatrix} \end{array}$$

Example 5.1 Let $\tilde{G}$ be a fuzzy matrix game as defined in Table 2, whose entries are generalized trapezoidal fuzzy numbers with different left and right heights.

For this game, if player I plays second row ($x = (0,1,0)$) and player II plays third column ($y = (0,0,1,0)$) then player I receives and correspondingly player II pays a payoff

$\tilde{g}(\mathrm{I}_2, \mathrm{II}_3) \equiv (1,3,4,5; 1, .6)$. On the other hand, for a pair of strategies $x = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ the expected payoff for player I is $\tilde{g}(x, y) = \left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}, 4; .3, .7\right)$.

Now, we define three types of *minimax equilibrium strategies* based on the proposed ordering method. A point $(x^*, y^*) \in X_m \times Y_n$ is said to be a minimax equilibrium strategy to game $\tilde{G}$ if relations

$$x^{\mathrm{T}} \tilde{A} y^* \preccurlyeq x^{*\mathrm{T}} \tilde{A} y^*, \forall x \in X_m,$$
$$x^{*\mathrm{T}} \tilde{A} y^* \preccurlyeq x^{*\mathrm{T}} \tilde{A} y, \forall y \in Y_n$$

hold.

If $(x^*, y^*) \in X_m \times Y_n$ is the minimax equilibrium strategy to game $\tilde{G}$ then a point $\tilde{v} = x^{*\mathrm{T}} \tilde{A} y^*$ is said to be the (fuzzy) *value of game* $\tilde{G}$ and the triplet $(x^*, y^*, \tilde{v})$ is said to be a *solution of game* $\tilde{G}$ under the proposed ordering method.

5.1. BROWN–ROBINSON METHOD

The solution methods of matrix games with fuzzy payoffs have been studied by many authors. Most solution techniques are based on linear programming methods ([6, 7, 10, 14, 24] and references therein).

The Brown–Robinson Method is a common technique to approximate calculations for the value of a two-person zero-sum game. In this approach, the players choose their strategies in each step k assuming that the strategies of the other players in step k correspond to the frequency with which the various strategies were applied in the previous $k - 1$ steps. Initially, Brown [13] conjectured and then Robinson [36] proved the convergence of this method for matrix games. The Brown–Robinson Method has also been adapted to interval valued matrix games and matrix games with fuzzy payoffs [2, 3].

Let

$$\tilde{G} = \begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \vdots \\ \mathrm{I}_\mathrm{m} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{II}_1 & \mathrm{II}_2 & \dots & \mathrm{II}_\mathrm{n} \end{array} \\ \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1n} \\ \tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{g}_{m1} & \tilde{g}_{m2} & \cdots & \tilde{g}_{mn} \end{pmatrix} \end{array}$$

be a fuzzy matrix game whose entries are generalized trapezoidal fuzzy numbers with different left and right heights. Then a vector system $\left(\tilde{U}, \tilde{V}\right)$ for fuzzy matrix $\tilde{G}$ is expressed as follows:

Definition 5.2 For all $\mathrm{k} \in \mathbb{N}$ a pair $\left(\tilde{\mathrm{U}}, \tilde{\mathrm{V}}\right)$ consisting of n-dimensional fuzzy vectors $\tilde{\mathrm{U}}_\mathrm{k} = (\tilde{\mathrm{u}}_1, \tilde{\mathrm{u}}_2, \dots, \tilde{\mathrm{u}}_\mathrm{n})$ and m-dimensional fuzzy vectors $\tilde{\mathrm{V}}_\mathrm{k} = (\tilde{\mathrm{v}}_1, \tilde{\mathrm{v}}_2, \dots, \tilde{\mathrm{v}}_\mathrm{m})$ provided that

$$\mathrm{Min}\left(\tilde{U}_0\right) = \mathrm{Max}\left(\tilde{V}_0\right) \ and$$

$$\tilde{U}_{k+1} = \tilde{U}_k + \tilde{g}^r_{i(k)}, \tilde{V}_{k+1} = \tilde{V}_k + \tilde{g}^c_{j(k)}$$

is called a vector system for fuzzy matrix $\tilde{G}$. Here, $i(k)$ and $j(k)$ satisfies

$$\tilde{v}_{i(k)} = \mathrm{Max}\big(\tilde{V}_k\big), \tilde{u}_{j(k)} = \mathrm{Min}\big(\tilde{U}_k\big),$$

where $\tilde{g}_i^r$ and $\tilde{g}_j^c$ denote the i-th row and j-th column of $\tilde{G}$, respectively.

Let $\tilde{G}$ be an $m \times n$ fuzzy matrix game whose entries are generalized trapezoidal fuzzy numbers with different left and right heights and $\tilde{v}$ be the value of $\tilde{G}$. If $\big(\tilde{U}, \tilde{V}\big)$ is a vector system for $\tilde{G}$, then

$$\lim_{k\to\infty} \frac{\mathrm{Max}(\tilde{V}_k)}{k} = \lim_{k\to\infty} \frac{\mathrm{Min}(\tilde{U}_k)}{k} = \tilde{v}.$$

The convergence is with respect to the Hausdorff metric on $\mathbb{F}$ (see [2]).

5.2. A NUMERICAL EXAMPLE

The best way to demonstrate the Brown–Robinson method for fuzzy matrix game whose entries are generalized trapezoidal fuzzy numbers with different left and right heights is by the virtue of an example. Now we consider the modified example of Collins & Hu [23]. This sees an investor making a decision as to how to invest a non-divisible sum of money when the economic environment may be categorized into a finite number of states. There is no guarantee that any single value (return on the investment) can adequately model the payoff for any one of the economic states. Hence, it is more realistic to assume that each payoff is a fuzzy number. For this example, it is assumed that the decision of such an investor can be modeled under the assumption that the economic environment (or nature) is, in fact, a rational ‘player’ that will choose an optimal strategy. Assume that the options for this player are: strong economic growth, moderate economic growth, no growth or shrinkage, and negative growth. For the investor player the options are: invest in bonds, invest in stocks, and invest in a guaranteed fixed return account. In this case, clearly a single value for the payoff of either investment in bonds or stock cannot be realistically modeled by a single value representing the percent of return. Hence, a game matrix with fuzzy payoffs better represents the view of the game from both players’ perspectives. Consider then fuzzy matrix game

$$\tilde{G} = \begin{array}{l} \\ \text{Strong} \\ \text{Moderate} \\ \text{None} \\ \text{Negative} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\text{Bonds}\quad & \quad\text{Stocks}\quad & \quad\text{Fixed}\quad \end{array} \\ \begin{pmatrix} (.11, .1230, .1295, .136; 1, 1) & (.125, .1415, .1498, .158; 1, 1) & (.05, .05, .05, .05; 1, 1) \\ (.083, .1025, .1122, .122; .8, 1) & (.08, .095, .1025, .11; 1, .9) & (.05, .05, .05, .05; 1, 1) \\ (.049, .0555, .0588, .062; 1, 1) & (.002, .0220, .0320, .042; 1, 1) & (.05, .05, .05, .05; 1, 1) \\ (.022, .0260, .0280, .03; .7, 1) & (-.04, -.0125, .0013, .015; .8, .9) & (.05, .05, .05, .05; 1, 1) \end{pmatrix} \end{array}$$

whose entries are generalized trapezoidal fuzzy numbers with different left and right heights for this scenario, where the percentage of return is represented in decimal form.

We first assume that

$$\tilde{U}_0 = ((0, .1, .2, .3; 1,1), (0, .1, .2, .3; 1,1), (0, .1, .2, .3; 1,1))$$

and

$$\tilde{V}_0 = ((0, .1, .2, .3; 1,1), (0, .1, .2, .3; 1,1), (0, .1, .2, .3; 1,1), (0, .1, .2, .3; 1,1)).$$

Then $\mathrm{Min}\big(\tilde{U}_0\big) = \mathrm{Max}\big(\tilde{V}_0\big) = (0, .1, .2, .3; 1,1)$.

In the next step ($k = 1$), since all components are the same, we can choose $i(1)$ and $j(1)$ as any integer from 1 to 3 and from 1 to 4, respectively. If we choose $i(1) = 1$ and $j(1) = 1$ then we find

$$
\begin{aligned}
\widetilde{U}_1 &= \widetilde{U}_0 + \tilde{g}^r_{i(1)} \\
&= ((0,.1,.2,.3;1,1),(0,.1,.2,.3;1,1),(0,.1,.2,.3;1,1)) \\
&\quad +((.11,.1230,.1295,.136;1,1),(.125,.1415,.1498,.158;1,1),(.05,.05,.05,.05;1,1)) \\
&= ((.11,.2230,.3295,.436;1,1),(.125,.2415,.3498,.458;1,1),(.05,.15,.25,.35;1,1))
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{V}_1 &= \tilde{V}_0 + \tilde{g}^c_{j(1)} \\
&= ((0,.1,.2,.3;1,1),(0,.1,.2,.3;1,1),(0,.1,.2,.3;1,1),(0,.1,.2,.3;1,1)) \\
&\quad +((.11,.1230,.1295,.136;1,1),(.083,.1025,.1122,.122;.8,1), \\
&\quad (.049,.0555,.0588,.062;1,1),(.022,.0260,.0280,.03;.7,1)) \\
&= ((.11,.2230,.3295,.436;1,1),(.083,.2025,.3122,.422;.8,1),(.049,.1555,.2588,.362;1,1), \\
&\quad (.022,.1260,.2280,.33;.7,1))
\end{aligned}
$$

In the second step, we get

$$
\begin{aligned}
&\mathrm{Min}(\widetilde{U}_1) = (.05,.15,.25,.35;1,1), \\
&\mathrm{Max}(\tilde{V}_1) = (.11,.2230,.3295,.436;1,1).
\end{aligned}
$$

Therefore, we obtain $i(2) = 1$, $j(2) = 3$ and

$$
\begin{aligned}
\widetilde{U}_2 &= \widetilde{U}_1 + \tilde{g}^r_{i(2)} \\
&= ((.11,.2230,.3295,.436;1,1),(.125,.2415,.3498,.458;1,1),(.05,.15,.25,.35;1,1)) \\
&\quad +((.11,.1230,.1295,.136;1,1),(.125,.1415,.1498,.158;1,1),(.05,.05,.05,.05;1,1)) \\
&= ((.22,.3460,.4590,.572;1,1),(.250,.3830,.4996,.616;1,1),(.10,.20,.30,.40;1,1))
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{V}_2 &= \tilde{V}_1 + \tilde{g}^c_{j(2)} \\
&= ((.11,.2230,.3295,.436;1,1),(.083,.2025,.3122,.422;.8,1), \\
&\quad (.049,.1555,.2588,.362;1,1),(.022,.1260,.2280,.33;.7,1)) \\
&\quad +((.05,.05,.05,.05;1,1),(.05,.05,.05,.05;1,1), \\
&\quad (.05,.05,.05,.05;1,1),(.05,.05,.05,.05;1,1)) \\
&= ((.16,.2730,.3795,.486;1,1),(.133,.2525,.3622,.472;.8,1), \\
&\quad (.099,.2055,.3088,.412;1,1),(.072,.1760,.2780,.38;.7,1))
\end{aligned}
$$

Progressing in this way, and using the Maple computer algebra system, we build up Table 3. By virtue of Table 3, we can see that the value of the game approaches the crisp number $(.05,.05,.05,.05;1,1)$. This corresponds to the movements of investors who don't have any intuition into what the economy may do and who cannot take high risks.

Table 3. The Brown–Robinson method for solving the example.

k	$i(k)$	$j(k)$	$\frac{\mathrm{Min}(\tilde{U}_k)}{k}$	$\frac{\mathrm{Max}(\tilde{V}_k)}{k}$
1	1	1	$[.05,.15,.25,.35;1,1]$	$[.11,.2230,.3295,.436;1,1]$
2	1	3	$[.05,.1,.15,.2;1,1]$	$[.08,.1365,.1898,.243;1,1]$
3	1	3	$[.05,.0833,.1167,.15;1,1]$	$[.07,.1077,.1432,.1787;1,1]$
4	1	3	$[.05,.075,.1,.125;1,1]$	$[.065,.0933,.1199,.1465;1,1]$
5	1	3	$[.05,.07,.09,.11;1,1]$	$[.062,.0846,.1059,.1272;1,1]$
6	1	3	$[.05,.0667,.0833,.1;1,1]$	$[.06,.0788,.0966,.1143;1,1]$
7	1	3	$[.05,.0643,.0786,.0929;1,1]$	$[.0586,.0748,.09,.0105;1,1]$
8	1	3	$[.05,.0625,.075,.0875;1,1]$	$[.0575,.0716,.0849,.0983;1,1]$
9	1	3	$[.05,.0611,.0722,.0833;1,1]$	$[.0567,.0692,.0811,.0929;1,1]$
10	1	3	$[.05,.06,.07,.08;1,1]$	$[.056,.0673,.07796,.0886;1,1]$
⋮			⋮	⋮
100	1	3	$[.05,.051,.052,.053;1,1]$	$[.0506,.0517,.0528,.0539;1,1]$
⋮			⋮	⋮
1000	1	3	$[.05,.0501,.0502,.0503;1,1]$	$[.0501,.0502,.0503,.0504;1,1]$
⋮			⋮	⋮
10000	1	3	$[.05,.05,.05002,.05003;1,1]$	$[.05,.05002,.05002,.05004;1,1]$
⋮			⋮	⋮
100000	1	3	$[.05,.05,.050002,.050003;1,1]$	$[.05,.050002,.050003,.050004;1,1]$
⋮			⋮	⋮

6. CONCLUSIONS

In this paper, we present a new method for ordering generalized trapezoidal fuzzy numbers with different left and right heights, based on the Gergonne point of a triangle. We show that the proposed method can overcome the drawbacks of certain well-known ordering and ranking methods. The advantages of the proposed method are presented with comparative examples. We also have adapted Brown–Robinson method to fuzzy matrix game whose entries are generalized trapezoidal fuzzy numbers with different left and right heights. It is showed that by means of this method, value of fuzzy matrix game can be easily calculated. Further, Brown–Robinson method can be used even if fuzzy matrix dimension is too high.

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