ORIGINAL PAPER

ON AN EFFICIENT TECHNIQUE TO SOLVE NONLINEAR FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *In this paper an efficient transformation, in combination with Exp-function method has been applied to construct generalized solitary and periodic wave solutions of the nonlinear Lax equation of fractional-order. First the nonlinear partial differential equation is converted into ordinary differential equation by a suitable transformation. Then desired solitary wave solutions has been obtained. Computational work and subsequent results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.*

Keywords: *Lax equation; fractional calculus; Exp-function method; solitary wave solutions*

1. INTRODUCTION

The subject of factional calculus [1, 2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3, 4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences and workshops, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5-8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics, signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical physics and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1–8] and the references therein. Recently, He and Wu [9] developed a very efficient technique which is called Exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The Exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et. al. [10-11] used this scheme to

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find periodic solutions of evolution equations; Mohyud-Din [12-15] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et. al. [17, 18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolution equations; Zhu [21, 22] for the Hybrid-Lattice system and discrete modified KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation; Ebaid [25] for the improvement on the Exp-function method when balancing the highest order linear and nonlinear terms. The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation**,** Exp-function method using Jumarie's derivative approach [26-31] and its subsequent application to construct generalized solitary wave solutions of the nonlinear Lax equation of fractional-order. It is to be highlighted that Ebaid [25] proved that $c=d$ and $p=q$ are the only relations that can be obtained by applying Exp-function method to any nonlinear ordinary differential equation. The Lax equation appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics, applied and physical sciences.

2. PRELIMINARY DEFINITIONS AND THEOREMS

In this section, basic definitions of fractional calculus and some theorems are given to find positive integers p, q, c and d involved in trial solution of Exp-function method.

Theorem 1. Suppose that $u^{(r)}$ and u^{γ} are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz;

$$U(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)},$$

leads to $c=d$ and $p=q$, $\forall\ r\geq 1, \gamma\geq 2$.

Theorem 2. Suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and k are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r, s, k\geq 1$.

Theorem 3. Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r, s\geq 1, \forall \Omega\geq 2$.

Theorem 4. Suppose that $\mathrm{u}^{(r)}$ and $(\mathrm{u}^{(s)})^{\Omega}u^{\lambda}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall\, r, s, \Omega, \lambda \geq 1$.

Jumarie's Fractional Derivative

Jumarie's fractional derivative is a modified Riemann-Liouville derivative defined as [27-30]

$$D_t^{\alpha} f(x) = \begin{Bmatrix} \dfrac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} (f(t) - f(0))dt, \alpha \leq 0, \\ \dfrac{1}{\Gamma(-\alpha)} \dfrac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0))dt, 0 \leq \alpha \leq 1 \\ \left[f^{\alpha-n}(x)^n \right]^n, n \leq \alpha \leq n+1, n \geq 1 \end{Bmatrix} \tag{1}$$

Where f: R → R, x → f(x) denotes a continuous (but not necessarily differentiable) function.

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative were summarized in Refs. [27-30].

$$D_x^{\alpha} c = 0, \alpha \geq 0,\ c = \text{constant} \tag{2}$$

$$D_x^{\alpha} [cf(x)] = cD_x^{\alpha} f(x) \alpha \geq 0,\ c = \text{constant} \tag{3}$$

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \beta \geq \alpha \geq 0. \tag{4}$$

$$D_x^{\alpha} [f(x) g(x)] = [D_x^{\alpha} f(x) g(x) + f(x) [D_x^{\alpha} g(x)]]. \tag{5}$$

$$D_x^{\alpha} f(x(t)) = f_x'(x).x^{\alpha}(t). \tag{6}$$

3. EXP-FUNCTION METHOD

We consider the general nonlinear FPDE of the type

$$P(u, u_t, u_x, u_{xx}\ u_{xxx}, \ldots, D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u, \ldots) = 0, \quad 0 < \alpha \leq 1. \tag{7}$$

Where $D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx respectively.

Using a transformation

$$\eta = kx + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0,\ k, \omega, \eta_0 \text{ are all constants with } k, \omega, \neq 0. \tag{8}$$

We can rewrite equation (7) in the following nonlinear ODE:

$$Q(u,u^{'},u^{''},u^{'''},u^{iv},\ldots)=0. \tag{9}$$

Where the prime denotes derivative with respect to η.

According to Exp-function method, we assume that the wave solution can be expressed in the following form

$$u(\eta)=\frac{\sum_{n-c}^{d} a_n \exp[n\eta]}{\sum_{m-p}^{q} b_m \exp[m\eta]} \tag{10}$$

Where p,q,c and d are positive integers which are known to be further determined, a_n and b_n are unknown constants. We can rewrite Eq.(10) in the following equivalent form

$$u(\eta)=\frac{a_c \exp(c\eta)+...+a_{-d} \exp(-d\eta)}{b_p \exp(p\eta)+...+b_{-q} \exp(-q\eta)}. \tag{11}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and q by using [25], we have

$$p=c, q=d. \tag{12}$$

4. SOLUTION PROCEDURE

In this section, we apply Exp-function method for fractional order nonlinear Lax equation.

Consider the general form of the Lax equation

$$D_t^{\alpha}u+u_t+\frac{3}{10}\alpha^2u^2u_x+2\alpha u_x u_{2x}+\alpha u u_{3x}+u_{5x}=0,\ \ 0<\alpha\le 1. \tag{13}$$

Using (8) equation (13) can be converted to an ordinary differential equation

$$10\omega u'+3k\alpha^2u^2u'+20\alpha k^3u'u''+10\alpha k^3uu'''+10k^5u^{(5)}=0. \tag{14}$$

Where the prime denotes the derivative with respect to η. The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and p, by using referance [25] we have

$$p=c, q=d. \tag{15}$$

Case I. We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p=c=1$ and $q=d=1$ equation (11) reduces to

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]}. \quad (16)$$

Substituting equation (16) into equation (14), we have

$$\frac{1}{A}\begin{bmatrix}c_5\exp(5\eta)+c_4\exp(4\eta)+c_3\exp(3\eta)+c_2\exp(2\eta)+c_1\exp(\eta)+c_0+c_{-1}\exp(-\eta)\\ +c_{-2}\exp(-2\eta)+c_{-3}\exp(-3\eta)+c_{-4}\exp(-4\eta)+c_{-5}\exp(-5\eta)\end{bmatrix}=0 \quad (17)$$

Where $A=(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta))^6$, $c_i(i=-5,-4,......,4,5)$ are constants obtained by using Maple 17. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$(c_{-5}=0,c_{-4}=0,c_{-3}=0,c_{-2}=0,c_{-1}=0,c_0=0,c_1=0,c_2=0,c_3=0,c_4=0,c_5=0) \quad (18)$$

Solution will yield

$$a_{-1}=\frac{1}{4}\frac{a_1b_0^2}{b_1^2},\qquad b_0=b_0,\qquad b_1=b_1,\quad b_{-1}=\frac{1}{4}\frac{b_0^2}{b_1},\quad a_1=a_1,$$
$$\omega=-\frac{1}{10}\frac{k\left(10k^4b_1^2+10k^2\alpha a_1b_1+3\alpha^2a_1^2\right)}{b_1^2},\qquad a_0=\frac{b_0\left(\alpha a_1+10k^2b_1\right)}{\alpha b_1}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of Lax equation

$$u(x,t)=\frac{a_1}{b_1}+\frac{1}{\alpha}\left(\frac{10k^2b_0}{b_1e^{(kx+\omega t)}+b_0+\frac{b_0^2}{4b_1}e^{(-kx-\omega t)}}\right), \quad (19)$$

Where $\omega=-\frac{k\left(10k^4b_1^2+10k^2\alpha a_1b_1+3\alpha^2a_1^2\right)}{10b_1^2}$ and a_1,b_1,α and k are real numbers.

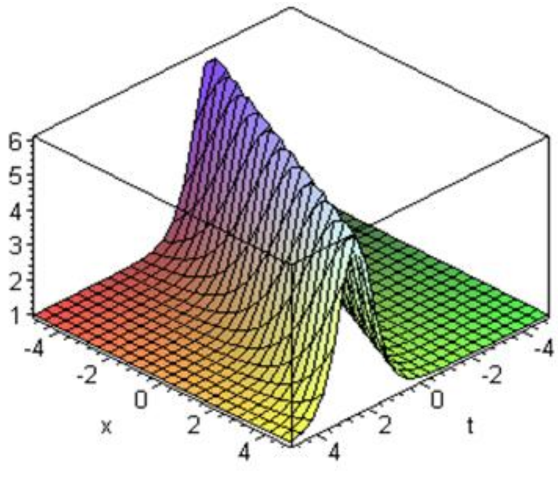

Figure 1. Solitary Wave Solution.

Fig. 1 depicts soliton solutions of Lax equation, when $a_1 = b_0 = b_1 = k = \alpha = 1$. In case k is an imaginary number, the obtained soliton solutions can be converted into periodic solutions or compact-like solutions. Therefore, we write $k = iK$ consequently, solution (19) becomes

$$u(x,t) = \frac{a_1}{b_1} - \frac{1}{\alpha}\left(\frac{10K^2 b_0}{b_1 e^{(iKx+\omega t)} + b_0 + \frac{b_0^{\ 2}}{4b_1} e^{(-iKx-\omega t)}}\right), \tag{20}$$

Where $\omega = -\frac{iK\left(10K^4 b_1^2 - 10K^2 \alpha a_1 b_1 + 3\alpha^2 a_1^2\right)}{10 b_1^2}$ and a_1, b_1, α and K are real numbers. If we search for periodic solutions or compact-like solutions than the imaginary part of equation (20) must be zero, consequently

$$u(x,t) = \frac{\left(\begin{array}{l} 8\alpha a_1 b_0^{\ 2} b_1^{\ 2} + 16\alpha a_1 b_1^{\ 4} + \alpha a_1 b_0^{\ 4} - 160K^2 b_0^{\ 2} b_1^{\ 3} \\ + \cos(-Kx + \lambda t)\left[32\alpha a_1 b_1^{\ 3} b_0 + 16\alpha a_1 b_1^{\ 2} b_0^{\ 2} \cos(-Kx + \lambda t) + 8\alpha a_1 b_1 b_0^{\ 3} - 160K^2 b_0 b_1^{\ 4} - 40K^2 b_0^{\ 3} b_1^{\ 2}\right]\end{array}\right)}{\alpha b_1\left[\cos(-Kx + \lambda t)\left[32 b_1^{\ 3} b_0 + 16 b_1^{\ 2} b_0^{\ 2} \cos(-Kx + \lambda t) + 8 b_1 b_0^{\ 3}\right] + 8 b_1^{\ 2} b_0^{\ 2} + 16 b_1^{\ 4} + b_0^{\ 4}\right]},$$

Where $\lambda = \frac{K\left(10K^4 b_1^2 - 10K^2 \alpha a_1 b_1 + 3\alpha^2 a_1^2\right)}{10 b_1^2}$, which is the periodic solution of Lax equation

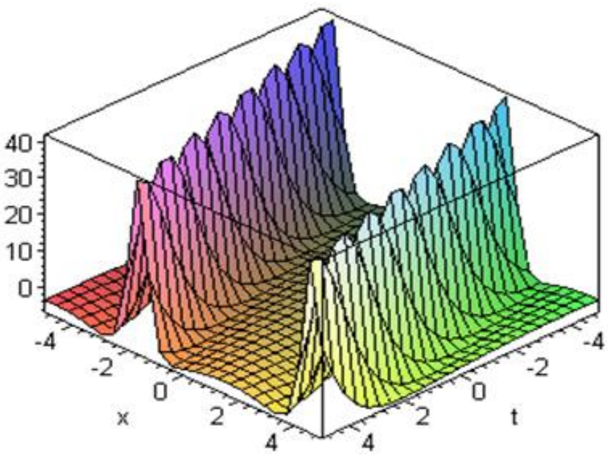

Figure 2. Periodic Wave Solution.

Fig. 2 depicts periodic solutions, when $a_1 = b_0 = b_1 = K = \alpha = 1$.

Case II. If $p = c = 2,$ and $q = d = 2$ then equation (11) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta] + a_{-2} \exp[-2\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta] + b_{-2} \exp[-2\eta]}. \tag{21}$$

and contains some free parameters, we set $b_1 = b_{-1} = 0$, for simplicity , the trial-function (21) is simplified as follows:

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta] + a_{-2} \exp[-2\eta]}{b_2 \exp[2\eta] + b_0 + b_{-2} \exp[-2\eta]}.$$

Proceeding as before, we obtain

$$a_{-1} = 0, \quad a_2 = a_2, \quad b_2 = b_2, \quad b_0 = b_0, \quad a_{-2} = \frac{1}{4}\frac{a_2 b_0^{\ 2}}{b_2^{\ 2}}, \quad a_1 = 0$$

$$b_{-2} = \frac{1}{4}\frac{b_0^{\ 2}}{b_2}, \quad \omega = -\frac{1}{10}\frac{k\left(40k^2\alpha a_2 b_2 + 3\alpha^2 a_2^{\ 2} + 160k^4 b_2^{\ 2}\right)}{b_2^{\ 2}}, \quad a_0 = \frac{b_0\left(\alpha a_2 + 40k^2 b_2\right)}{\alpha b_2}.$$

Hence we get the generalized solitary wave solution of Lax equation as follows

$$u(x,t) = \frac{a_2}{b_2} + \frac{1}{\alpha}\left(\frac{60K^2 b_0}{b_2 e^{2(kx+\omega t)} + b_0 + \frac{b_0^{\ 2}}{4b_2} e^{2(-kx-\omega t)}}\right),$$

Where $\omega = -\frac{k\left(160k^4 b_2^2 + 40k^2\alpha a_2 b_2 + 3\alpha^2 a_2^2\right)}{10b_2^2}$ and a_2, b_2, α and k are real numbers.

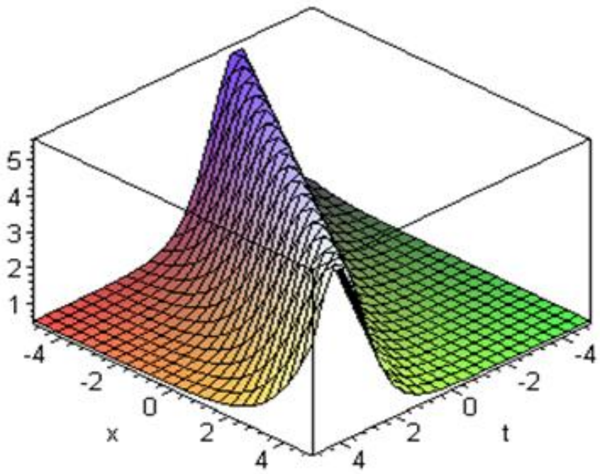

Figure 3. Solitary Wave Solution.

Fig. 3 depicts soliton solutions of Lax equation when $a_2 = b_0 = b_2 = \alpha = 1$ and $k = \frac{1}{2}$.

5. CONCLUSION

In this paper, Exp-function method is applied to construct generalized solitary and periodic wave solutions of the nonlinear fractional order Lax equation. We attain desired solutions through exponential functions. It is guaranteed the accuracy of the attain results by backward substitution into the original equation with Maple software. The scheming procedure of this method is simplest, straight and productive. It is concluded that the under study technique is more reliable and have minimum computational task, so widely applicable. Solutions clearly depict solitary and periodic wave solutions.

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