ORIGINAL PAPER

A NEW PERSPECTIVE ON THE INVOLUTES OF THE SPACELIKE CURVE WITH A SPACELIKE BINORMAL IN MINKOWSKI 3-SPACE

MUSTAFA BILICI[1], MUSTAFA CALISKAN[2]

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Abstract. *In this paper, we have expressed the transformation matrix between the Frenet frames of the spacelike curve with a spacelike binormal and its timelike involute curve as depend on lorentzian spacelike angle between the unit binormal vector and the Darboux vector of the spacelike evolute curve. Furthermore, some new characterizations with relation to the involute-evolute curve couple have been found.*

Keywords: *Involute curve, Minkowski space, Frenet frames, Darboux vector*
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1. INTRODUCTION

C. Huygens, who is also known for his works in optics, discovered involutes while trying to build a more accurate clock. The involute of a given curve is a well-known concept in the Euclidean space [1-4]. In [5], authors have shown that the length between the spacelike curve η with a spacelike binormal and the timelike involute curve η^* is constant. Furthermore, the curvature and the torsion of the involute curves η^* have been found as depend on the curvature and the torsion of the evolute curve η. Also transformation matrix between the Frenet frames of the curve couple (η^*, η) have been found as depend on curvatures of the evolute curve η. In this study, we have obtained the relationships between the Frenet frames of the spacelike curve η and the timelike involute curve η^* as depend on lorentzian spacelike angle θ $(0 < \theta < \pi)$ between the unit binormal vector and the Darboux vector of spacelike curve η in the Minkowski 3-space. Also, some new results and relations between Darboux vectors and unit Darboux vectors of the involute-evolute curve couple have been given. During our study, we assume that the Frenet vectors of the curve couple (η^*, η) are non-null.

[1] Ondokuz Mayıs University, Education Faculty, Mathematics and Science Education, 55200 Samsun,Turkey.
E-mail: mbilici@omu.edu.tr.
[2] Gazi University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey.
E-mail: mustafacaliskan@gazi.edu.tr.

2. PRELIMINARIES

Let Minkowski 3-space IR_1^3 be the vector space IR^3 provide with the Lorentzian inner product g given by $g(X,X)=-x_1^2+x_2^2+x_3^2$, where $X=(x_1,x_2,x_3)\in IR^3$. A vector $X=(x_1,x_2,x_3)\in IR^3$ is said to be timelike if $g(X,X)<0$, spacelike if $g(X,X)>0$ and lightlike (or null) if $g(X,X)=0$. Similarly, an arbitrary curve $\eta=\eta(s)$ in IR_1^3 where s is a pseudo-arclenght parameter, can locally be timelike spacelike or null (lightlike), if all of its velocity vectors $\eta'(s)$ are respectively timelike, spacelike or null, for every $s\in I\subset IR$. The norm of a vector X is defined by

$$\|X\|_{IL}=\sqrt{|g(X,X)|}$$

The vectors $X=(x_1,x_2,x_3)$, $Y=(y_1,y_2,y_3)\in IR_1^3$ are orthogonal if and only if

$$g(X,Y)=0, [6].$$

Lorentzian cross product of X and Y is given by

$$X\times Y=(x_3y_2-x_2y_3, x_1y_3-x_3y_1, x_1y_2-x_2y_1), [7]. \tag{1}$$

We denote by $\{T(s),N(s),B(s)\}$ the moving Frenet frame along the curve η. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve η, respectively. Depending on the causal character of the curve η, we have the following Frenet formulae and instantaneous rotation vectors:

i) Let η be a unit speed timelike space curve with curvature κ and torsion τ. Let Frenet frames of η be $\{T,N,B\}$.

In this trihedron, T is timelike vector, N and B are spacelike vectors. For this vectors, we can write

$$T\times N=-B\ ,\ N\times B=T\ ,\ B\times T=-N,$$

where $\times$ is the Lorentzian cross product, [7] in space IR_1^3. In this situation, the Frenet formulas are given by

$$T'=\kappa N,\qquad N'=\kappa T-\tau B,\qquad B'=\tau N, [8].$$

The Darboux vector for the timelike curve is given by

$$W=\tau T-\kappa B, [9]. \tag{2}$$

ii) Let η be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B spacelike vectors and N timelike vector. In this situation,

$$T\times N=-B\ ,\ N\times B=-T\ ,\ B\times T=N.$$

The Frenet formulas are given by

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N\text{, [8].}$$

The Darboux vector for the spacelike curve is given by

$$W = -\tau T + \kappa B\text{, [9].} \tag{3}$$

iii) Let η be a unit speed spacelike space curve. In this trihedron, we assume that T and N spacelike vektors and B timelike vector. For this trihedron we write

$$T \times N = B\ ,\ N \times B = -T\ ,\ B \times T = -N,$$

The Frenet formulas are given by

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = \tau N\text{, [8].}$$

The Darboux vector for the η curve is given by

$$W = \tau T - \kappa B\text{, [9].} \tag{4}$$

Lemma 1. Let (η^*, η) be the involute-evolute curve couple which are given by (I, η) and (I, η^*) coordinate neighbourhoods, respectively. The distance between the curves η^* and η are given by

$$d_{IL}(\eta(s), \eta^*(s)) = |c - s|, \quad \text{c=constant, } \forall s \in I\text{, [5].}$$

Lemma 2. Let (η^*, η) be the involute-evolute curve couple which are given by (I, η) and (I, η^*) coordinate neighbourhoods, respectively. For the curvature and torsion of the curve η^*, we have

$$\kappa^{*2} = \frac{\kappa^2 + \tau^2}{(c-s)^2 \kappa^2}, \quad \tau^* = \frac{\kappa'\tau - \kappa\tau'}{|c-s|\kappa(\tau^2 + \kappa^2)}\text{, [5].}$$

Lemma 3. Let the curve η^* be involute of the curve η, then

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \left(\sqrt{\kappa^2 + \tau^2}\right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -\tau \\ -\tau & & -\kappa \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}\text{, [5].}$$

Lemma 4. Let X and Y be nonzero Lorentz orthogonal vektors in IR_1^3. If X is timelike, then Y is spacelike, [10].

Lemma 5. i) The Timelike Angle between Timelike Vectors

Let X and Y be positive (negative) timelike vectors in IR_1^3. Then there is a unique nonnegative real number $\varphi(X, Y)$ such that

$$g(X,Y) = \|X\|\|Y\| \cosh \varphi(X,Y)$$

The **Lorentzian timelike angle** between X and Y is defined to be $\varphi(X,Y)$.

ii) The Spacelike Angle between Spacelike Vectors

Let X and Y be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then we have $|g(X,Y)| \leq \|X\|\|Y\|$. Hece, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\|Y\| \cos \varphi(X,Y)$$

The **Lorentzian spacelike angle** between X and Y is defined to be $\varphi(X,Y)$.

iii) The Timelike Angle between Spacelike Vectors

Let X and Y be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then we have $|g(X,Y)| > \|X\|\|Y\|$. Hence, there is a unique positive real number $\varphi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\|Y\| \cosh \varphi(X,Y)$$

The **Lorentzian timelike angle** between X and Y is defined to be $\varphi(X,Y)$.

iv) The Angle between Spacelike and Timelike Vectors

Let X be a spacelike vector and Y a positive timelike vector in IR_1^3. Then there is a unique nonnegative real number $\varphi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\|Y\| \sinh \varphi(X,Y).$$

The **Lorentzian timelike angle** between X and Y is defined to be $\varphi(X,Y)$, [10].

Definition 1. Let $M_1, M_2 \subset IR_1^3$ be two curves which are given by (I,η) and (I,η^*) coordinate neighbourhoods, resp. Let Frenet frames of M_1 and M_2 be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, resp. M_2 is called the involüte of M_1 (M_1 is called the evolute of M_2) if

$$g(T,T^*) = 0, \text{[11]}. \tag{5}$$

We should note here that according to reference [5], if η is a unit speed spacelike curve with spacelike binormal then η^* is a timelike curve with spacelike binormal. In this case, the causal characteristics of the Frenet frames of the curves η and η^* must be of the form

$$\{T \text{ spacelike}, N \text{ timelike}, B \text{ spacelike}\}$$

and

$$\{T^* \text{ timelike}, N^* \text{ spacelike}, B^* \text{ spacelike}\}.$$

Definition 2. (Unit Vectors C of Direction W for Nonnull Curves):
i) For the curve η with a timelike tangent, θ being a Lorentzian timelike angle between the spacelike binormal unit vector -B and the Darboux vector W,
a) If $|\kappa| > |\tau|$, then W is a spacelike vector. In this situation, from the Lemma 5. iii) we can write

$$\begin{cases} \kappa = \|W\| \cosh\theta \\ \tau = \|W\| \sinh\theta \end{cases} , \quad \|W\|^2 = g(W, W) = \kappa^2 - \tau^2 \tag{6}$$

and

$$C = \frac{W}{\|W\|} = \sinh\theta T - \cosh\theta B ,$$

(7)

where C is unit vector of direction W.

b) If $|\kappa| < |\tau|$, then W is a timelike vector. In this situation, from the Lemma 5.iv) we can write

$$\begin{cases} \kappa = \|W\| \sinh\theta \\ \tau = \|W\| \cosh\theta \end{cases} , \quad \|W\|^2 = -g(W, W) = -(\kappa^2 - \tau^2) \tag{8}$$

and

$$C = \cosh\theta T - \sinh\theta B . \tag{9}$$

ii) For the curve η with a timelike principal normal, θ being a Lorentzian spacelike angle between the B and the W, if B and W spacelike vectors that span a spacelike vector subspace then by the Lemma 5. ii) we can write

$$\begin{cases} \kappa = \|W\| \cos\theta \\ \tau = \|W\| \sin\theta \end{cases} , \quad \|W\|^2 = g(W, W) = \kappa^2 + \tau^2 . \tag{10}$$

and

$$C = -\sin\theta T + \cos\theta B \tag{11}$$

iii) For the curve η with a timelike binormal, θ being a Lorentzian timelike angle between the -B and the W,
a) If $|\tau| > |\kappa|$, then W is a spacelike vector. From the Lemma 5.iv), we can write

$$\begin{cases} \kappa = \|W\| \sinh\theta \\ \tau = \|W\| \cosh\theta \end{cases} , \quad g(W, W) = \|W\|^2 = (\tau^2 - \kappa^2) \tag{12}$$

and

$$C = \cosh\theta T - \sinh\theta B \tag{13}$$

b) If $|\tau| < |\kappa|$, then W is a timelike vector. In this situation, from the Lemma 5.i) we have

$$\begin{cases}\kappa = \|W\|\cosh\theta \\ \tau = \|W\|\sinh\theta\end{cases}, \quad g(W,W) = -\|W\|^2 = -\left(\tau^2 - \kappa^2\right) \tag{14}$$

and

$$C = \sinh\theta T - \cosh\theta B,\ [11]. \tag{15}$$

3. A NEW PERSPECTIVE ON THE INVOLUTES OF THE SPACELIKE CURVE WITH A SPACELIKE BINORMAL

Theorem 1. Let $\left(\eta^*, \eta\right)$ be the involute-evolute curve couple. The Frenet vectors of the curve couple as follow:

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{16}$$

where θ $\left(0 < \theta < \pi\right)$ is a Lorentzian spacelike angle between the binormal unit spacelike vector B and spacelike Darboux vector W.

Proof: If η^* is the involute of η, we have

$$\eta^*(s) = \eta(s) + \lambda T(s), \ \ \lambda = c - s, \ \ \eta' = T \ \text{(c=constant)}.$$

Let us derivative both side with respect to s:

$$\eta^{*\prime} = \frac{d\eta^*}{ds} = \frac{d\eta^*}{ds^*}\frac{ds^*}{ds} = T^*\frac{ds^*}{ds} = \lambda\kappa N, \tag{17}$$

where s and s^* are arc parameters of η and η^*, respectively. We can find

$$\frac{ds^*}{ds} = \lambda\kappa$$

thus we have

$$T^* = N. \tag{18}$$

On the other hand, we have

$$\eta^{*\prime\prime} = \lambda\kappa^2 T + (\lambda\kappa' - \kappa)N + \lambda\kappa\tau B.$$

If we calculate vector $\beta' \times \beta''$, then we get

$$\eta^{*\prime} \times \eta^{*\prime\prime} = -\lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B$$

and

$$g\left(\eta^{*'} \times \eta^{*''}, \eta^{*'} \times \eta^{*''}\right) = \lambda^4 \kappa^4 \left(\kappa^2 + \tau^2\right).$$

Furthermore, we have

$$B^* = \frac{1}{\left\| \eta^{*'} \times \eta^{*''} \right\|_{IL}} \left(\eta^{*'} \times \eta^{*''}\right) = -\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B$$

substituting Eq. (10) into the last equation, we obtain

$$B^* = -\sin\theta T + \cos\theta B. \tag{19}$$

Since $N^* = -\left(B^* \times T^*\right)$, then we have

$$N^* = -\cos\theta T - \sin\theta B. \tag{20}$$

If the Eqs. (18), (19), (20) are written by matrix form, then, the theorem is proved.

Theorem 2. Let $\left(\eta^*, \eta\right)$ be the involute-evolute curve couple. If W and W^* are the Darboux vectors of η and η^* respectively, then

$$W^* = \frac{1}{|c-s|\kappa}\left(-\theta' N + W\right).$$

Proof: For the Darboux vector of the curve η^*, from the Eq. (2) we can write

$$W^* = \tau^* T^* - \kappa^* B^* \tag{21}$$

Using the Lemma 2. and Theorem 1. in the Eq. (21), we have

$$W^* = \frac{\kappa'\tau - \kappa\tau'}{|c-s|\kappa\left(\tau^2 + \kappa^2\right)} N + \sqrt{\frac{\kappa^2 + \tau^2}{\kappa^2 (c-s)^2}} (-\sin\theta T + \cos\theta B)$$

$$W^* = \frac{\left(\frac{\kappa}{\tau}\right)' \tau^2}{|c-s|\kappa\left(\tau^2 + \kappa^2\right)} N + \frac{\sqrt{\kappa^2 + \tau^2}}{|c-s|\kappa} \left(-\sin\theta T + \cos\theta B\right).$$

Using the equation (10) in the last equation, then we obtain

$$W^* = \frac{1}{|c-s|\kappa} \left\{-\theta' N + \left(-\tau T + \kappa B\right)\right\}.$$

and from Eq. (3), we get

$$W^* = \frac{1}{|c-s|\kappa}\left(-\theta' N + W\right).$$

Corollary 1. Let (η^*, η) be the involute-evolute curve couple. If η evolute curve is helix, then the relations between Darboux vectors of the curve couple as follows

$$W^* = \frac{1}{|c-s|\kappa} W.$$

Proof: **i)** If η evolute curve is helix, then we have

$$\frac{\tau}{\kappa} = \tanh\theta = \text{constant}$$

and then we have

$$\theta' = 0. \tag{22}$$

So, combining the equation (22) and Theorem 2., we get

$$W^* = \frac{1}{|c-s|\kappa} W$$

Theorem 3. Let (η^*, η) be the involute-evolute curve couple. If C and C^* are unit Darboux vectors, respectively, then we have

(i) $$C^* = -\frac{\theta'}{\sqrt{\left|-\theta'^2+\kappa^2+\tau^2\right|}} N + \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{\left|-\theta'^2+\kappa^2+\tau^2\right|}} C, \quad (\text{for } |\kappa| > |\tau|)$$

(ii) $$C^* = \frac{\theta'}{\sqrt{\left|-\theta'^2+\kappa^2+\tau^2\right|}} N - \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{\left|-\theta'^2+\kappa^2+\tau^2\right|}} C, \quad (\text{for } |\kappa| < |\tau|).$$

Proof: **(i)** Let θ^* be the Lorentzian timelike angle between $-B^*$ and W^*. From Eq. (7) we can write

$$C^* = \sinh\theta^* T^* - \cosh\theta^* B^*. \tag{23}$$

From Eqs. (11), (19) and (20) we can write

$$C^* = \sinh\theta^* N - \cosh\theta^* C \tag{24}$$

On the other hand, for the curvatures and torsions of the curve η^*, we have

$$\begin{cases} \kappa^* = \left\|W^*\right\| \cosh\theta^* \\ \tau^* = \left\|W^*\right\| \sinh\theta^* \end{cases} \tag{25}$$

Using the Lemma2. and Theorem 2. into the Eq. (25) , we get

$$\sinh\theta^* = -\frac{\theta'}{\sqrt{\left|-\theta'^2 + \kappa^2 + \tau^2\right|}} , \tag{26}$$

$$\cosh\theta^* = \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\left|-\theta'^2 + \kappa^2 + \tau^2\right|}} . \tag{27}$$

Substituting by the Eqs. (26) , (27) into the equation (24), the theorem is proved.

(ii) The proof is analogous to the proof of the statement (i).

Corollary 2. Let (η^*, η) be the involute-evolute curve couple. If η evolute curve is helix, then

i) The vectors W^* and B^* of the involute curve η^* are linearly dependent.

ii) $C = \mp C^*$

Proof: **i)** If α evolute curve is helix, then from the Eq. (22), we can write

$$\theta' = 0 .$$

Substituting by the last result into the Eqs. (26), (27)

$$\sinh\theta^* = 0$$
$$\cosh\theta^* = 1$$

are found. Thus we have

$$\theta^* = 0 .$$

ii) Substituting by the Eq. (22) into the Theorem 3. (i) and **(**ii), we have

$$C = \mp C^*$$

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