ORIGINAL PAPER

ON GENERATING FUNCTIONS OF QUADRUPLE HYPERGEOMETRIC FUNCTION $X_7^{(4)}$

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Abstract. *In this work, by using Laplace integral representation of quadruple function $X_7^{(4)}$ defined in [3], we introduced new generating functions involving some quadruple hypergeometric functions. Some particular cases and consequences of our main results are also considered.*

Keywords: *Laplace transforms, quadruple hypergeometric series, generating functions.*

2010 Mathematics subject classification: *33C50, 33C56.*

1. INTRODUCTION

Recently, Bin-Saad et al. [3] introduced five new quadruple hypergeometric functions whose names are $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$ to investigate their five Laplace integral representations which include the confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$, a Humbert functions Φ_2, Φ_3 and Ψ_2 in their kernels. Very recently Bin-Saad and Younis [2] established new integral representations of Euler type for some hypergeometric functions of four variables, whose kernels include the quadruple hypergeometric functions $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$, of which $X_7^{(4)}$ is defined as follows

$$X_7^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol given by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)...(a+n-1)\ (n\in N:=\{1,2,3,...\})\text{ and }(a)_0=1.$$

The following is the Laplace integral representation of the function $X_7^{(4)}$ [3]:

$$X_7^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s}s^{a_1-1}$$
$$\times {}_0F_1\left(-;c_1;s^2x\right)\,{}_1F_1\left(a_2;c_2;sy\right)\,{}_1F_1\left(a_3;c_3;sz\right)\,{}_0F_1\left(-;c_4;s^2u\right)ds, \tag{1.2}$$
$$\left(\operatorname{Re}(a_1)>0\right).$$

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In diverse areas in applied mathematics and mathematical physics, generating functions play an important role in the investigation of various useful properties of the sequences which they generate. These are used to find certain properties and formulas for numbers and polynomials in a wide range of research subjects such as modern combinatorics. One can refer the extensive work of Srivastava and Manocha [12] for a systematic introduction to, and several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of hypergeometric functions and polynomials in one, two or more variables, among much abundant literature. In fact, a remarkable large number of generating functions involving a variety of hypergeometric functions have been developed by many authors (for example [1, 4, 9, 10] and the related references therein).Here, we use the integral representation of the hypergeometric function of four variables $X_7^{(4)}$ to obtain new generating functions involving generalized Horn's function ${}^{(p)}H_4^{(4)}$ of four variables, the Lauricella functions of four variables $F_A^{(4)}, F_C^{(4)}$ and the quadruple function $X_7^{(4)}$ itself. Some special cases of the main results here are also considered.

2. GENERATING FUNCTIONS

For our purpose, we begin by recalling generalized Horn's function of four variables ${}^{(p)}H_4^{(4)}$ defined by (see [11])

$$ {}^{(p)}H_4^{(4)}(a, a_{p+1},..., a_4; c_1, c_2, c_3, c_4; x_1,..., x_4) = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{2(m_1+...+m_p)+m_{p+1}+...+m_4} (a_{p+1})_{m_{p+1}} \cdots (a_4)_{m_4}}{(c_1)_{m_1} \cdots (c_4)_{m_4}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_4^{m_4}}{n!}. \tag{2.1}$$

Lauricella hypergeometric functions of four variables $F_A^{(4)}, F_C^{(4)}$ are as below [7]:

$$F_A^{(4)}(a, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (a_1)_m (a_2)_n (a_3)_p (a_4)_q}{(c_1)_m (c_2)_n (c_3)_p (c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{2.2}$$

$$F_C^{(4)}(a_1, a_2; c_1, c_2, c_3, c_4; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q} (a_2)_{m+n+p+q}}{(c_1)_m (c_2)_n (c_3)_p (c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{2.3}$$

Now, we begin the following theorem:

Theorem 2.1. Each of the following generating functions for $X_7^{(4)}$ holds true.

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{w^k}{k!} X_7^{(4)}(a_1+k, a_1+k, a_1+k, a_1+k, a_1+k, a_2, a_3, a_1+k; c_1, c_2, c_3, c_4; x, y, z, u^2) \\ &= (1+2u)^{-a_1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{w}{1+2u}\right)^k {}^{(1)}H_4^{(4)}\left(a_1+k, a_2, a_3, c_4-\tfrac{1}{2}; c_1, c_2, c_3, 2c_4-1; \frac{x}{(1+2u)^2}, \frac{y}{(1+2u)}, \frac{z}{1+2u}, \frac{4u}{1+2u}\right); \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{w^k}{k!} X_7^{(4)}(a_1+k, a_1+k, a_1+k, a_1+k, a_1+k, a_2, a_3, a_1+k; c_1, c_2, c_3, c_4; x, y, z, u^2) = (1+2u-y)^{-a_1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{w}{1+2u-y}\right)^k \\ &\times {}^{(1)}H_4^{(4)}\left(a_1+k, c_2-a_2, a_3, c_4-\tfrac{1}{2}; c_1, c_2, c_3, 2c_4-1; \frac{x}{(1+2u-y)^2}, \frac{y}{y-2u-1}, \frac{z}{1+2u-y}, \frac{4u}{1+2u-y}\right); \end{aligned} \tag{2.5}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,c_2,a_3,a_1+k;c_1,2c_2,c_3,c_4;x,2y,z,u)=(1-y)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-y}\right)^k \tag{2.6}$$
$$\times\ {}^{(3)}H_4^{(4)}\left(a_1+k,\,a_3;c_1,\,c_2+\tfrac{1}{2},c_4,c_3;\frac{x}{(1-y)^2},\frac{y^2}{4(1-y)^2},\frac{u}{(1-y)^2},\frac{z}{1-y}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,c_2,a_3,a_1+k;c_1,2c_2,c_3,c_4;x,2y,z,u)=(1-y-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-y-z}\right)^k \tag{2.7}$$
$$\times\ {}^{(3)}H_4^{(4)}\left(a_1+k,\,c_3-a_3;c_1,\,c_2+\tfrac{1}{2},c_4,c_3;\frac{x}{(1-y-z)^2},\frac{y^2}{4(1-y-z)^2},\frac{u}{(1-y-z)^2},\frac{z}{y+z-1}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k\,;c_1,c_2,c_3,c_4;x^2,y,z,u^2)=(1+2x+2u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x+2u}\right)^k \tag{2.8}$$
$$\times F_A^{(4)}\left(a_1+k,\,c_1-\tfrac{1}{2},a_2,a_3,c_4-\tfrac{1}{2}\,;\,2c_1-1,\,c_2,c_3,2c_4-1;4\lambda x,\lambda y,\lambda z,4\lambda u\right),$$
$$\left(\lambda_1=\frac{1}{1+2x+2u},\,\lambda_2=\frac{1}{1+2x+2u}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k\,;c_1,c_2,c_3,c_4;x^2,y,z,u^2)=(1+2x+2u-y)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x+2u-y}\right)^k \tag{2.9}$$
$$\times F_A^{(4)}\left(a_1+k,\,c_1-\tfrac{1}{2},c_2-a_2,a_3,c_4-\tfrac{1}{2}\,;\,2c_1-1,\,c_2,c_3,2c_4-1;4\lambda x,-\lambda y,\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u-y}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k\,;c_1,c_2,c_3,c_4;x^2,y,z,u^2)=(1+2x+2u-y-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x+2u-y-z}\right)^k \tag{2.10}$$
$$\times F_A^{(4)}\left(a_1+k,\,c_1-\tfrac{1}{2},c_2-a_2,c_3-a_3,c_4-\tfrac{1}{2}\,;\,2c_1-1,\,c_2,c_3,2c_4-1;4\lambda x,-\lambda y,-\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u-y-z}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,c_2,c_3,a_1+k;c_1,2c_2,2c_3,c_4;x,2y,2z,u)=(1-y-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-y-z}\right)^k \tag{2.11}$$
$$\times F_C^{(4)}\left(\frac{a_1+k}{2},\frac{a_1+k+1}{2};c_1,\,c_2+\tfrac{1}{2},c_3+\tfrac{1}{2},c_4;\frac{4x}{(1-y-z)^2},\frac{y^2}{(1-y-z)^2},\frac{z^2}{(1-y-z)^2},\frac{4u}{(1-y-z)^2}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;\,c_1,c_2,c_3,c_4;x,y,z,u)=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k \tag{2.12}$$
$$\times X_7^{(4)}\left(a_1+k,\,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_3-a_3,a_1;\,c_1,c_2,\,c_3,\,c_4;\,\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1},\frac{u}{(1-z)^2}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_7^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;\,c_1,c_2,c_3,c_4;x,y,z,u)=(1-y-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-y-z}\right)^k \tag{2.13}$$
$$\times X_7^{(4)}\left(a_1+k,\,a_1+k,a_1+k,a_1+k,a_1+k,c_2-a_2,c_3-a_3,a_1;\,c_1,c_2,\,c_3,\,c_4;\,\frac{x}{(1-y-z)^2},\frac{y}{y+z-1},\frac{z}{y+z-1},\frac{u}{(1-y-z)^2}\right).$$

Proof: To prove the above relations, we need the following formulae [5, 8, 11, 12]:

$$\Gamma(z)=s^z\int_0^{\infty}e^{-st}\,t^{z-1}\,dt\,,\ \ \mathrm{Re}\,(z)>0; \tag{2. 14}$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\qquad a\neq 0,-1,-2,\ldots; \tag{2.15}$$

$$(a)_{2m}=2^{2m}\left(\frac{a}{2}\right)_m\left(\frac{a+1}{2}\right)_m,\quad m=0,1,2\ldots; \tag{2.16}$$

$${}_0F_1\left(-;a;x^2\right)=e^{-2x}\,{}_1F_1\left(a-\tfrac{1}{2};\,2a-1;\,4x\right); \tag{2.17}$$

$${}_0F_1\left(-;a+\tfrac{1}{2};\frac{x^2}{4}\right)=e^{-x}\,{}_1F_1\left(a;\,2a;\,2x\right); \tag{2.18}$$

$${}_1F_1\left(a;b;x\right)=e^{x}\,{}_1F_1\left(b-a;\,b;\,-x\right). \tag{2.19}$$

For the convenience, we denote the left hand side of (2.4) with δ, using (1.2)

$$\delta=\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(a_1+k)}\int_0^{\infty}e^{-s}\,s^{a_1+k-1}\;{}_0F_1\left(-;c_1;s^2x\right)\,{}_1F_1(a_2;c_2;sy)\,{}_1F_1(a_3;c_3;sz)\,{}_0F_1\left(-;c_4;s^2u\right)ds\ ,$$

by using (2.17), we have

$$\delta=\sum_{k,m,n,p=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,w^k\,x^m\,y^n\,z^p}{(c_1)_m\,(c_2)_n\,(c_3)_p\,k!\,m!\,n!\,p!\,\Gamma(a_1+k)}\int_0^{\infty}e^{-s(1+2u)}\,s^{a_1+k+2m+n+p-1}\;{}_1F_1\left(c_4-\tfrac{1}{2};2c_4-1;4u\right)ds\ .$$

The function ${}_1F_1$ which appears in above equation can be replaced by its series form and then interchanging the order of the summation and integral sign which is permissible here, we get

$$\delta=\sum_{k,m,n,q=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,\left(c_4-\tfrac{1}{2}\right)_q\,w^k\,x^m\,y^n\,z^p\,(4u)^q}{(c_1)_m\,(c_2)_n\,(c_3)_p\,(2c_4-1)k!\;m!\,n!\,p!\,q!\,\Gamma(a_1+k)}\int_0^{\infty}e^{-s(1+2u)}\,s^{a_1+k+2m+n+p+q-1}\ ds\ .$$

Now, use of (2.14) and (2.15), in above equation and then simplified with series manipulation completes the proof of relation (2.4). From the relations (2.14) to (2.19), one can easily obtain the other generating functions.

3. SPECIAL CASES

It is easy to observe that the main results (2.4) to (2.13) gave a number generating functions for the hypergeometric series of four variables $X_7^{(4)}$. In the present section, we will mention only some special cases.

Letting $k=0$ in (2.4) to (2.13), we obtain the following formulae:

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u^2\right)$$
$$=(1+2u)^{-a_1}\,{}^{(1)}H_4^{(4)}\left(a_1,a_2,a_3,c_4-\tfrac{1}{2};c_1,c_2,c_3,2c_4-1;\frac{x}{(1+2u)^2},\frac{y}{(1+2u)},\frac{z}{1+2u},\frac{4u}{1+2u}\right),\qquad(3.1)$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u^2\right)$$
$$=(1+2u-y)^{-a_1}\,{}^{(1)}H_4^{(4)}\left(a_1,c_2-a_2,a_3,c_4-\tfrac{1}{2};c_1,c_2,c_3,2c_4-1;\frac{x}{(1+2u-y)^2},\frac{y}{y-2u-1},\frac{z}{1+2u-y},\frac{4u}{1+2u-y}\right),\qquad(3.2)$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,c_2,a_3,a_1;c_1,2c_2,c_3,c_4;x,2y,z,u\right)$$
$$=(1-y)^{-a_1}\,{}^{(3)}H_4^{(4)}\left(a_1,a_3;c_1,c_2+\tfrac{1}{2},c_4,c_3;\frac{x}{(1-y)^2},\frac{y^2}{4(1-y)^2},\frac{u}{(1-y)^2},\frac{z}{1-y}\right),\qquad(3.3)$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,c_2,a_3,a_1;c_1,2c_2,c_3,c_4;x,2y,z,u\right)$$
$$=(1-y-z)^{-a_1}\,{}^{(3)}H_4^{(4)}\left(a_1,c_3-a_3;c_1,c_2+\tfrac{1}{2},c_4,c_3;\frac{x}{(1-y-z)^2},\frac{y^2}{4(1-y-z)^2},\frac{u}{(1-y-z)^2},\frac{z}{y+z-1}\right),\qquad(3.4)$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x^2,y,z,u^2\right)\qquad(3.5)$$
$$=(1+2x+2u)^{-a_1}F_A^{(4)}\left(a_1,c_1-\tfrac{1}{2},a_2,a_3,c_4-\tfrac{1}{2};2c_1-1,c_2,c_3,2c_4-1;4\lambda x,\lambda y,\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u}\right),$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x^2,y,z,u^2\right)\qquad(3.6)$$
$$=(1+2x+2u-y)^{-a_1}F_A^{(4)}\left(a_1,c_1-\tfrac{1}{2},c_2-a_2,a_3,c_4-\tfrac{1}{2};2c_1-1,c_2,c_3,2c_4-1;4\lambda x,-\lambda y,\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u-y}\right),$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x^2,y,z,u^2\right) \tag{3.7}$$
$$=\left(1+2x+2u-y-z\right)^{-a_1}F_A^{(4)}\left(a_1,c_1-\tfrac{1}{2},c_2-a_2,c_3-a_3,c_4-\tfrac{1}{2};2c_1-1,c_2,c_3,2c_4-1;4\lambda x,-\lambda y,-\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u-y-z}\right),$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,c_2,c_3,a_1;c_1,2c_2,2c_3,c_4;x,2y,2z,u\right) \tag{3.8}$$
$$=\left(1-y-z\right)^{-a_1}F_C^{(4)}\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1,c_2+\tfrac{1}{2},c_3+\tfrac{1}{2},c_4;4\lambda^2x,(\lambda y)^2,(\lambda z)^2,4\lambda^2u\right),$$
$$\left(\lambda=\frac{1}{1-y-z}\right),$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u\right) \tag{3.9}$$
$$=\left(1-z\right)^{-a_1}X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,c_3-a_3,a_1;c_1,c_2,c_3,c_4;\lambda^2x,\lambda y,-\lambda z,\lambda^2u\right),$$
$$\left(\lambda=\frac{1}{1-z}\right),$$

$$X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u\right) \tag{3.10}$$
$$=\left(1-y-z\right)^{-a_1}X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,c_2-a_2,c_3-a_3,a_1;c_1,c_2,c_3,c_4;\lambda^2x,-\lambda y,-\lambda z,\lambda^2u\right),$$
$$\left(\lambda=\frac{1}{1-y-z}\right).$$

Equations (3.1), (3.5) and (3.9) with $y=0$, yield Exton's results [6]. Equations (3.2), (3.5), (3.8)-(3.10) with $u=0$, yield the known results [6].

If we set $y=0$ in (2.10) and (2.11), we shall obtain the following generating functions:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_2(a_1+k,a_3;c_1,c_4,c_3;x^2,u^2,z)=\left(1+2x+2u-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x+2u-z}\right)^k \tag{3.11}$$
$$\times F_A^{(3)}\left(a_1+k,c_1-\tfrac{1}{2},c_3-a_3,c_4-\tfrac{1}{2};2c_1-1,c_3,2c_4-1;4\lambda x,-\lambda z,4\lambda u\right),$$
$$\left(\lambda=\frac{1}{1+2x+2u-z}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_2(a_1+k,c_3;c_1,c_4,2c_3;x,u,2z)=\left(1-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k \tag{3.12}$$
$$\times F_C^{(3)}\left(\frac{a_1+k}{2},\frac{a_1+k+1}{2};c_1,c_3+\tfrac{1}{2},c_4;\frac{4x}{(1-z)^2},\frac{z^2}{(1-z)^2},\frac{4u}{(1-z)^2}\right).$$

A special case of (3.12) when $x=u=0$ yields the elegant relation

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}{}_2F_1(a_1+k,c_3;2c_3;2z)=\left(1-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k{}_2F_1\left(\frac{a_1+k}{2},\frac{a_1+k+1}{2};c_3+\tfrac{1}{2};\frac{z^2}{(1-z)^2}\right), \tag{3.13}$$

by assigning the value zero to k ,we have the following transformation:

$${}_2F_1\left(a_1,c_3;2c_3;2z\right)=\left(1-z\right)^{-a_1}{}_2F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_3+\tfrac{1}{2};\frac{z^2}{(1-z)^2}\right), \tag{3.14}$$

Now, if in (2.9), we take $x=0$, we get

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8(a_1+k,a_2,a_3;c_1,c_2,c_3;x^2,y,z)=\left(1+2x-y\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x-y}\right)^k \tag{3.15}$$
$$\times F_A^{(3)}\left(a_1+k,c_1-\tfrac{1}{2},c_2-a_2,a_3;2c_1-1,c_2,c_3;4\lambda x,-\lambda y,\lambda z\right),$$
$$\left(\lambda=\frac{1}{1+2x-y}\right).$$

where X_2, X_8 are Exton's functions of three variables [6], $F_A^{(3)}, F_C^{(3)}$ are Lauricella's functions of three variables [7] and Gauss hypergeometric function ${}_2F_1$ [11].

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