*ORIGINAL PAPER*

# FUZZY EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER CONTAINED IN THE BOUNDARY CONDITION

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***Abstract.*** *In this article two point fuzzy boundary value problems is defined under the approach generalized Hukuhara differentiability (gH-differentiability). The solution method of the fuzzy boundary value problem is examined with the aid of some initial value problems. This type of problem is associated with an operator in fuzzy Hilbert space. So, several theorems are given for the problem eigenvalues.*

***Keywords:*** *gH-derivative, Eigenvalue, Fuzzy eigenfunction.*

## 1. INTRODUCTION

In this paper we consider the two point fuzzy boundary value problem

$$L=-\frac{d^2}{dt^2}$$

$$L\hat{u}=\lambda\hat{u},\ \ t\in[a,b] \tag{1}$$

which satisfy the conditions

$$a_1\hat{u}(a)=a_2\hat{u}'(a) \tag{2}$$

$$b_1\hat{u}(b)=\lambda b_2\hat{u}'(b) \tag{3}$$

where $a_1,a_2,b_1,b_2\geq 0$, $\lambda>0$, $a_1^2+a_2^2\neq 0$ and $b_1^2+b_2^2\neq 0$, $\hat{u}(x)$ fuzzy functions.

The term "fuzzy differential equation" was first introduced in 1978 by Kandel and Byatt [1] and later an extended version of this equation was published many papers in [5, 12, 14]. There are many suggestions to define a fuzzy derivative and to study fuzzy differential equation [3, 6, 10, 11]. One of the most well-known definitions of difference and derivative for fuzzy set value functions was given by Hukuhara in [10]. By using the H-derivative, Kaleva in [15] started to develop a theory for fuzzy differential equations. Many works have been done by several authers in theoretical and applied fields for fuzzy differential equations with the Hukuhara derivative [13, 15]. But in some cases this approach suffers certain disadvantages since the diameter of the solutions is unbounded as time t increases [11, 14]. So here we use gH-difference and gH-derivative to solve FDE under much less restrictive conditions [11].

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The paper is organized as follows; section 2 introduces the basic concept of fuzzy function spaces, fuzzy Hilbert spaces and fuzzy gH- derivative. In section 3, we define operator formulation of the problem in the adequate fuzzy Hilbert spaces. In section 4, we show how to solve the eigenvalue problem with a numerical example.

## 2. PRELIMINARIES AND NOTATION

In this section, we give some concepts and results besides the essential notations which will be used throughout the paper.

Let $\hat{u}$ be a fuzzy subset on $\mathbb{R}$, i.e. a mapping $\hat{u}:\mathbb{R}\to[0,1]$ associating with each real number $t$ its grade of membership $\hat{u}(t)$.

In this paper, the concept of fuzzy real numbers (fuzzy intervals) is considered in the sense of Xiaoand Zhu which is defined below:

**Definition 1.** A fuzzy subset $\hat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy intervals), whose $\alpha$- cut set is denoted by $[\hat{u}]^{\alpha}$, i.e., $[\hat{u}]^{\alpha}=\{t:\hat{u}(t)\geq 0\}$, if it satisfies two axioms:

i. There exists $r\in\mathbb{R}$ such that $\hat{u}(r)=1$,

ii. For all $0<\alpha\leq 1$, there exist real numbers $-\infty<u_{\alpha}^{-}\leq u_{\alpha}^{+}<+\infty$ such that $[\hat{u}]^{\alpha}$ is equal to the closed interval $\left[u_{\alpha}^{-},u_{\alpha}^{+}\right]$ [2].

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{F}(\mathbb{R})$. $\mathbb{F}_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha$- cuts are nonempty compact convex subsets of $\mathbb{R}$. If $\hat{u}\in\mathbb{F}(\mathbb{R})$ and $\hat{u}(t)=0$ whenever $t<0$, then $\hat{u}$ is called a non-negative fuzzy real number and $\mathbb{F}^{+}(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. For all $\hat{u}\in F^{+}(\mathbb{R})$ and each $\alpha\in(0,1]$, real number $u_{\alpha}^{-}$ is positive.

The fuzzy real number $\hat{r}\in\mathbb{F}(\mathbb{R})$ defined by

$$\hat{r}(t)=\begin{cases}1, & \text{t=r}\\ 0, & \text{t}\neq r\end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $\mathbb{F}(\mathbb{R})$, that is if $\hat{r}\in(-\infty,+\infty)$, then $\hat{r}\in\mathbb{F}(\mathbb{R})$ satisfies $\hat{r}(t)=\hat{0}(t-r)$ and $\alpha$- cut of $\hat{r}$ is given by $[\hat{r}]^{\alpha}=[r,r]$, $\alpha\in(0,1]$.

**Definition 2.** An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $\left(u_{\alpha}^{-},u_{\alpha}^{+}\right)$, $0\leq\alpha\leq 1$, which satisfy the following requirements

i. $u_{\alpha}^{-}$ is bounded non-decreasing left continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

ii. $u_{\alpha}^{+}$ is bounded non- increasing left continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

iii. $u_{\alpha}^{-}\leq u_{\alpha}^{+}$, $0\leq\alpha\leq 1$ [3].

**Definition 3.** Let $\left[a^{\alpha}, b^{\alpha}\right]$, $0 < \alpha \leq 1$, be a given family of non-empty intervals. Assume that

(a) $\left[a^{\alpha_1}, b^{\alpha_1}\right] \supset \left[a^{\alpha_2}, b^{\alpha_2}\right]$ for all $0 < \alpha_1 \leq \alpha_2$,

(b) $\left[\lim_{k\to\infty} a^{\alpha_k}, \lim_{k\to\infty} b^{\alpha_k}\right] = \left[a^{\alpha}, b^{\alpha}\right]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to $\alpha$,

(c) $-\infty < a^{\alpha} \leq b^{\alpha}$, for all $\alpha \in (0,1]$.

Then the family $\left[a^{\alpha}, b^{\alpha}\right]$ represents the $\alpha$- cut sets of a fuzzy number $\hat{u} \in \mathbb{F}(\mathbb{R})$, then the conditions (a), (b) and (c) are satisfied [4].

**Definition 4.** For $\hat{u}, \hat{v} \in \mathbb{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the sum $\hat{u} = \hat{v} \oplus \hat{w}$ and the product $\lambda \odot \hat{u}$ are defined by for all $\alpha \in [0,1]$,

$$[\hat{u} \oplus \hat{v}]^{\alpha} = [\hat{u}]^{\alpha} + [\hat{v}]^{\alpha} = \left\{x + y : x \in [\hat{u}]^{\alpha}, y \in [\hat{v}]^{\alpha}\right\},$$
$$[\lambda \odot \hat{u}]^{\alpha} = \lambda \odot [\hat{u}]^{\alpha} = \left\{\lambda x : x \in [\hat{u}]^{\alpha}\right\},$$

Define $D : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{R}^{+} \cup \{0\}$ by the equation

$$D(\hat{u}, \hat{v}) = \sup_{0<\alpha\leq 1} \left\{\max\left[\left|u_{\alpha}^{-} - v_{\alpha}^{-}\right|, \left|u_{\alpha}^{+} - v_{\alpha}^{+}\right|\right]\right\}$$

where $[\hat{u}]^{\alpha} = \left[u_{\alpha}^{-}, u_{\alpha}^{-}\right]$, $[\hat{v}]^{\alpha} = \left[v_{\alpha}^{-}, v_{\alpha}^{-}\right]$. Then it is easy to show that D is a metric in $\mathbb{F}(\mathbb{R})$ [3].

Let $T \subset \mathbb{R}$ be an interval. We denote by $\mathbb{F}\left(C\left(T;\mathbb{R}^{n}\right)\right)$ the space of all continuous fuzzy functions on $T$. For $\hat{u},\ \hat{v} \in \mathbb{F}\left(C\left(T;\mathbb{R}^{n}\right)\right)$, define a metric

$$D^{*}(\hat{u}, \hat{v}) = \sup_{t\in T} D\left(\hat{u}(t), \hat{v}(t)\right).$$

From [3], $\left(\mathbb{F}\left(C\left(T;\mathbb{R}^{n}\right)\right), D^{*}\right)$ is a complete metric space.

A fuzzy function $\hat{F} : T \to \mathbb{F}(\mathbb{R})$ is measurable if for all $\alpha \in [0,1]$, the set valued mapping $F_{\alpha} : T \to \mathbb{F}_{K}(\mathbb{R})$ defined by $F_{\alpha}(t) = \left[\hat{F}(t)\right]^{\alpha}$ is measurable.

Let $L^{1}(T;\mathbb{R})$ denote the space of Lebesque integrable functions. We denote by $S_{F}^{1}$ the set of all Lebesque integrable selections of $F_{\alpha} : T \to \mathbb{F}_{K}(\mathbb{R})$, that is

$$S_{F}^{1} = \left\{f \in L^{1}(T;\mathbb{R}) : f(t) \in F_{\alpha}(t) \ \ a.e.\right\}. \tag{4}$$

A fuzzy function $\hat{F}:T\to\mathbb{F}(\mathbb{R})$ is integrably bounded if there exists an integrable function $h$ such that $\|x\|\le h(t)$ for all $x\in F_0(t)$. A measurable and integrably bounded fuzzy functions $F_\alpha:T\to\mathbb{F}_K(\mathbb{R})$ is said to be integrable over $T$ if there exists $\hat{F}\in\mathbb{F}(\mathbb{R})$ such that

$$\int_T F_\alpha(t)\,dt=\left\{\int_T f(t)\,dt : f\in S_F^1\right\}$$

for all $\alpha\in[0,1]$ [7].

For measurable functions $f:T\to\mathbb{R}$ define the norm

$$\|f\|_{L_T^p}=\begin{cases}\left(\int_T |f(t)|^p\,dt\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \inf_{\mu(T_0)=0}\sup_{t\in T-T_0}|f(t)|, & p=\infty,\end{cases}$$

where $\mu(T)$ is the Lebesque measure on T and $T_0\subset T$. Let $L^p(T;\mathbb{R})$ be the Banach space of all measurable functions $f:T\to\mathbb{R}$ with $\|f\|_{L_T^p}<\infty$. By $\mathbb{F}\left(L^p(T;\mathbb{R})\right)$, $1\le p<\infty$, it is denoted the space of all functions $\hat{u}:T\to\mathbb{F}(\mathbb{R})$ such that the function $t\to D\left(\hat{u}(t),\hat{0}\right)$ belong to $L^p(T;\mathbb{R}_+)$. Then

$$D_p(\hat{u},\hat{v})=\left(\int_0^1\left(D_H\left([\hat{u}]^\alpha,[\hat{v}]^\alpha\right)\right)^p d\alpha\right)^{\frac{1}{p}} \tag{5}$$

is a metric on $\mathbb{F}\left(L^p(T;\mathbb{R})\right)$, for $1\le p<\infty$ and $D_\infty(\hat{u},\hat{v})=D(\hat{u},\hat{v})=\sup_{0<\alpha\le 1}D_H\left([\hat{u}]^\alpha,[\hat{v}]^\alpha\right)$ is a metric on $\mathbb{F}\left(L^\infty(T;\mathbb{R})\right)$ [7].

So for $p=2$, let $L^2(T;\mathbb{R})$ be the Banach space of all measurable functions $f:T\to\mathbb{R}$ with $\|f\|_{L_T^2}<\infty$. By $\mathbb{F}\left(L^2(T;\mathbb{R})\right)$, we denote the space of all square-integrable fuzzy functions $\hat{u}:T\to\mathbb{F}(\mathbb{R})$ such that the function $t\to D\left(\hat{u}(t),\hat{0}\right)$ belong to $L^2(T;\mathbb{R}_+)$. Then from (5)

$$D_2(\hat{u},\hat{v})=\left(\int_0^1\left(D_H\left([\hat{u}]^\alpha,[\hat{v}]^\alpha\right)\right)^2 d\alpha\right)^{\frac{1}{2}}$$

is a metric on $\mathbb{F}\left(L^2(T;\mathbb{R})\right)$, for $p=2$.

**Definition 5.** Let $X$ be a vector space over $\mathbb{R}$. A Felbin-fuzzy inner product on $X$ is a mapping $\langle .\,,\,.\rangle : X\times X\to\mathbb{F}^+(\mathbb{R})$ such that for all vectors $x,y,z\in X$ and all $r\in\mathbb{R}$, have (see [8])

$(FIP1)$ $\langle x+y,z\rangle=\langle x,z\rangle\oplus\langle y,z\rangle$

$(FIP2)$ $\langle rx,y\rangle=|\tilde{r}|\langle x,y\rangle$

$(FIP3)$ $\langle x,y\rangle=\langle y,x\rangle$

$(FIP4)$ $\mathrm{x}\neq 0\Rightarrow\langle x,x\rangle(t)=0,\quad \text{for all } t<0$

$(FIP5)$ $\langle x,x\rangle=0$ if and only if x=0.

**Remark 6.** Condition (FIP4) in the above definition is equivalent to the condition for all $(0\neq)x\in X$ için $\langle x,x\rangle_\alpha^->0$, for each $\alpha\in(0,1]$ where $[\langle x,x\rangle]^\alpha=\left[\langle x,x\rangle_\alpha^-,\langle x,x\rangle_\alpha^+\right]$ [9].

The vector space $X$ equipped with a Felbin-fuzzy inner product is called a Felbin-fuzzy inner space. A Felbin-fuzzy inner product on $X$ defines a fuzzy number

$$\|x\|=\sqrt{\langle x,x\rangle},\ \text{ for all } x\in X\ . \tag{6}$$

A fuzzy Hilbert space is a complete Felbin-fuzzy inner product space with the fuzzy norm defined by (6).

**Example 7.** Consider the lineer space $L^2([0,1];\mathbb{R})$ of all square-integrable functions. Define

$$\langle f,g\rangle(t)=\sup\left\{\alpha\in(0,1]\,\middle|\, t\in\left[\int_0^1 f_\alpha^-(x)g_\alpha^-(x)dx,\int_0^1 f_\alpha^+(x)g_\alpha^+(x)dx\right]\right\} \tag{7}$$

for $f,g\in L^2([0,1];\mathbb{R})$ such that $\hat{F},\hat{G}\in\mathbb{F}\left(L^2([0,1];\mathbb{R})\right)$. It can be showed that $\langle f,g\rangle$ is a Felbin-fuzzy inner product space on $L^2([0,1];\mathbb{R})$ and $\alpha-$cut set of $\langle f,g\rangle$ is given by

$$[\langle f,g\rangle]^\alpha=\left[\langle f,g\rangle_\alpha^-,\langle f,g\rangle_\alpha^+\right]=\left[\int_0^1 f_\alpha^-(x)g_\alpha^-(x),\int_0^1 f_\alpha^+(x)g_\alpha^+(x)\right] \tag{8}$$

For all $\alpha\in[0,1]$. From Theorem 6 of [8], it is clear that $\langle f,g\rangle$ is a fuzzy real number. So $L^2([0,1];\mathbb{R})$ is a fuzzy Hilbert space.

**Definition 8.** Let $\hat{u},\hat{v}\in\mathbb{F}(\mathbb{R})$. If there exist $\hat{w}\in\mathbb{F}(\mathbb{R})$ such that $\hat{u}=\hat{v}\oplus\hat{w}$, then $\hat{w}$ is called the Hukuhara difference of $\hat{u}$ and $\hat{v}$ and it is denoted by $\hat{u}\ \Theta_H\hat{v}$. If $\hat{u}\ \Theta_H\hat{v}$ exists, its $\alpha$- cuts are

$$[\hat{u}\ \Theta_H\hat{v}]^\alpha=\left[u_\alpha^--v_\alpha^-,u_\alpha^+-v_\alpha^+\right]$$

for all $\alpha\in[0,1]$ [10].

**Definition 9.** The generalized Hukuhara difference of two fuzzy numbers $\hat{u},\hat{v}\in\mathbb{F}(\mathbb{R})$ is defined as follows

$$\left[\hat{u}\ \Theta_{gH}\hat{v}\right]=\hat{w}\Leftrightarrow\begin{cases}(i)\ \hat{u}=\hat{v}\oplus\hat{w}\\ or(ii)\ \hat{v}=\hat{u}\oplus(-1)\hat{w}.\end{cases}$$

In terms of α-cuts we have

$$\left[\hat{u}\ \Theta_{gH}\hat{v}\right]^{\alpha}=\left[\min\left\{u_{\alpha}^{-}-v_{\alpha}^{-},u_{\alpha}^{+}-v_{\alpha}^{+}\right\},\max\left\{u_{\alpha}^{-}-v_{\alpha}^{-},u_{\alpha}^{+}-v_{\alpha}^{+}\right\}\right]$$

and if the H -difference exists, then $\hat{u}\ \Theta_{H}\hat{v}=\hat{u}\ \Theta_{gH}\hat{v}$; the conditions for the existence of $\hat{w}=\hat{u}\ \Theta_{gH}\hat{v}\in\mathbb{F}(\mathbb{R})$ are

$$case(i)\begin{cases}w_{\alpha}^{-}=u_{\alpha}^{-}-v_{\alpha}^{-}\text{ and }w_{\alpha}^{+}=u_{\alpha}^{+}-v_{\alpha}^{+},\\ with\ w_{\alpha}^{-}\text{ increasing, }w_{\alpha}^{+}\text{ decreasing , }w_{\alpha}^{-}\leq w_{\alpha}^{+}\end{cases}$$

$$case(ii)\begin{cases}w_{\alpha}^{-}=u_{\alpha}^{+}-v_{\alpha}^{+}\text{ and }w_{\alpha}^{+}=u_{\alpha}^{-}-v_{\alpha}^{-},\\ with\ w_{\alpha}^{-}\text{ increasing, }w_{\alpha}^{+}\text{ decreasing , }w_{\alpha}^{-}\leq w_{\alpha}^{+}.\end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if $\hat{w}$ is a crisp number [11].

**Remark 10.** Throughout the rest of this paper, we assume that $\hat{u}\ \Theta_{gH}\hat{v}\in\mathbb{F}(\mathbb{R})$ and $\alpha$-cut representation of fuzzy-valued function $\hat{f}:(a,b)\to\mathbb{F}(\mathbb{R})$ is expressed by $\left[\hat{f}(t)\right]^{\alpha}=\left[\left(f_{\alpha}^{-}\right)(t),\left(f_{\alpha}^{+}\right)(t)\right]$, $t\in[a,b]$ for each $\alpha\in[0,1]$.

**Definition 11.** Let $t_0\in(a,b)$ and $h$ be such that $t_0+h\in(a,b)$, then the gH-derivative of a function $\hat{f}:(a,b)\to\mathbb{F}(\mathbb{R})$ at $t_0$ is defined as

$$\hat{f}'_{gH}(t_0)=\lim_{h\to 0}\frac{\hat{f}(t_0+h)\Theta_{gH}\hat{f}(t_0)}{h}. \tag{9}$$

If $\hat{f}'_{gH}(t_0)\in\mathbb{F}(\mathbb{R})$ satisfying (9) exist, we say that $\hat{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at $t_0$ [11].

**Definition 12.** Let $\hat{f}:(a,b)\to\mathbb{F}(\mathbb{R})$ and $t_0\in(a,b)$ eith $f_{\alpha}^{-}(t)$ and $f_{\alpha}^{+}(t)$ both differentiable at $t_0$. We say that in [11]

- $\hat{f}$ is $[(i)-gH]$-differentiable at $t_0$ if
$$\left[\hat{f}'_{gH}(t)\right]^{\alpha}=\left[\left\{\left(f_{\alpha}^{-}\right)'(t),\left(f_{\alpha}^{+}\right)'(t)\right\}\right]\quad\forall\ \alpha\in[0,1] \tag{10}$$
- $\hat{f}$ is $[(ii)-gH]$-differentiable at $t_0$ if
$$\left[\hat{f}'_{gH}(t)\right]^{\alpha}=\left[\left\{\left(f_{\alpha}^{+}\right)'(t),\left(f_{\alpha}^{-}\right)'(t)\right\}\right]\quad\forall\ \alpha\in[0,1]. \tag{11}$$

**Definition 13.** The second generalized Hukuhara derivative of a fuzzy function $\hat{f}:(a,b)\to\mathbb{F}(\mathbb{R})$ at $t_0$ is defined as

$$\hat{f}''_{gH}(t_0)=\lim_{h\to 0}\frac{\hat{f}'(t_0+h)\Theta_{gH}\hat{f}'(t_0)}{h},$$

if $\hat{f}''_{gH}(t_0)\in\mathbb{F}(\mathbb{R})$, we say that $\hat{f}'_{gH}(t)$ is generalized Hukuhara derivative at $t_0$.

Also we say that $\hat{f}'_{gH}(t)$ is $\left[(i)-gH\right]$-differentiable at $t_0$ if

$$\hat{f}''_{i.gH}(t_0,\alpha)=\begin{cases}\left[\left(f_\alpha^-\right)''(t_0),\left(f_\alpha^-\right)''(t_0)\right], & \textit{if } f \textit{ be } \left[(i)-gH\right]\textit{-differentiable} \text{ on } (a,b)\\ \left[\left(f_\alpha^+\right)''(t_0),\left(f_\alpha^-\right)''(t_0)\right], & \textit{if } f \textit{ be } \left[(ii)-gH\right]\textit{-differentiable} \text{ on } (a,b)\end{cases}$$

for all $\alpha\in[0,1]$, and that $\hat{f}'_{gH}(t)$ is $\left[(ii)-gH\right]$-differentiable at $t_0$ if

$$\hat{f}''_{ii.gH}(t_0,\alpha)=\begin{cases}\left[\left(f_\alpha^+\right)''(t_0),\left(f_\alpha^-\right)''(t_0)\right], & \textit{if } f \textit{ be } \left[(i)-gH\right]\textit{-differentiable} \text{ on } (a,b)\\ \left[\left(f_\alpha^-\right)''(t_0),\left(f_\alpha^+\right)''(t_0)\right], & \textit{if } f \textit{ be } \left[(ii)-gH\right]\textit{-differentiable} \text{ on } (a,b)\end{cases}$$

for all $\alpha\in[0,1]$ [12].

## 3. OPERATOR-THEORETIC FORMULATION OF THE PROBLEM

Consider the (1)-(3) eigenvalue problem.

This is not the usual type of eigenvalue problem because the eigenvalue appears in the boundary conditions; so, we cannot put $L=-\frac{d^2}{dt^2}$ and consider problem as a special case of $L\hat{u}=\lambda\hat{u}$ because $D$, the domain of $L$, depends upon $\lambda$. So we enlarge our defination of $L$. To do this we formulate a theoretic approach to the problem.

From Theorem 6 of [8], we define a fuzzy Hilbert space $\hat{H}=L^2([0,1];\mathbb{R})\oplus\mathbb{R}$ with a Felbin-fuzzy inner product

$$\langle F,G\rangle=\left[\int_0^1 f_\alpha^-(t)g_\alpha^-(t)dt+(f_1)_\alpha^-(g_1)_\alpha^-,\int_0^1 f_\alpha^+(t)g_\alpha^+(t)dt+(f_1)_\alpha^+(g_1)_\alpha^+\right]$$

where $F\in\left[\hat{F}\right]^\alpha=\begin{pmatrix}\left[f_\alpha^-(t),f_\alpha^+(t)\right]\\ \left[(f_1)_\alpha^-,(f_1)_\alpha^+\right]\end{pmatrix}$, $G\in\left[\hat{G}\right]^\alpha=\begin{pmatrix}\left[g_\alpha^-(t),g_\alpha^+(t)\right]\\ \left[(g_1)_\alpha^-,(g_1)_\alpha^+\right]\end{pmatrix}\subset H$ and $\hat{F},\hat{G}\in\mathbb{F}^+\left(L^2([0,1];\mathbb{R})\right)$ such that $f_\alpha^-(t)$, $g_\alpha^-(t)$, $f_\alpha^+(t)$, $g_\alpha^+(t)\in L^2([0,1];\mathbb{R})$ and $(f_1)_\alpha^-$, $(g_1)_\alpha^-$, $(f_1)_\alpha^+$, $(g_1)_\alpha^+\in\mathbb{R}$.

Consider the space of two- component fuzzy vectors $\hat{F} \in \hat{H}$ , whose first component is a twice gH-differentiable fuzzy function $\hat{f}(t) \in \mathbb{F}^{+}\left(L^{2}\left([0,1];\mathbb{R}\right)\right)$ and whose second component is a fuzzy real number $\hat{f}_1 \in \mathbb{F}^{+}(\mathbb{R})$. Here $\hat{F}$ , $\hat{f}(t)$ and $\hat{f}_1$ are pozitive fuzzy numbers and their $\alpha$- cut sets are respectively $\left[F_\alpha^-, F_\alpha^+\right]$, $\left[f_\alpha^-(t), f_\alpha^+(t)\right]$ and $\left[(f_1)_\alpha^-, (f_1)_\alpha^+\right]$.

In this fuzzy Hilbert space we construct the lineer operator $A: \hat{H} \to \hat{H}$ with domain

$$D(A) = \left\{ \begin{pmatrix} \hat{f}(t) \\ \hat{f}_1 \end{pmatrix} \middle| \begin{array}{c} \hat{f}(t),\ \hat{f}'(t) \text{ fuzzy absolutely continuous (see [6]) in } [0,1]; \\ a_1 \hat{f}(a) = a_2 \hat{f}'(a),\ \hat{f}_1 = b_2 \hat{f}'(b) \end{array} \right\}$$

which acts by the rule $A\hat{F} := A\begin{pmatrix} \hat{f}(t) \\ \hat{f}_1 \end{pmatrix} = A\begin{pmatrix} \hat{f}(t) \\ b_2 \hat{f}'(b) \end{pmatrix} = \begin{pmatrix} -\hat{f}''(t) \\ b_1 \hat{f}(b) \end{pmatrix}$ with $\hat{F} = \begin{pmatrix} \hat{f} \\ \hat{f}_1 \end{pmatrix} \in D(A)$.

Thus, the problem (1) -(3) can be written in the form

$$A\hat{F} = \lambda \hat{F}$$

where $\hat{F} = \begin{pmatrix} \hat{f} \\ \hat{f}_1 \end{pmatrix} \in D(A)$. Then the eigenvalues and the eigenfunctions of the problem (1)-(3) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator $A$, respectively.

## 4. SOLUTION METHOD OF THE FUZZY PROBLEM

In this section we concern with fuzzy eigenvalues and eigenfunctions of .two-Point fuzzy boundary value problems with eigenvalue parameter contained in the boundary conditions. To do this, at first we need to use fuzzy derivatives. So here we use gH-difference and gH-derivative to solve fuzzy problem [11].

**Definition 14.** Let $\hat{u}: [a,b] \subset \mathbb{R} \to \mathbb{F}(\mathbb{R})$ be a fuzzy function and $\left[\hat{u}(t,\lambda)\right]^\alpha = \left[u_\alpha^-(t,\lambda), u_\alpha^+(t,\lambda)\right]$ be the $\alpha$- cut representation of the fuzzy function $\hat{u}(t)$ for all $t \in [0,1]$ and $\alpha \in [0,1]$. If the fuzzy differential equation (1) has the nontrivial solutions such that $u_\alpha^-(t,\lambda) \neq 0$, $u_\alpha^+(t,\lambda) \neq 0$ then $\lambda$ is eigenvalue of (1).

Consider the eigenvalues of the fuzzy boundary value problem (1)-(3). Using the $\alpha$-cut sets and $\left[(i)-gH\right]$- differentiable of $\hat{u}(t)$, $\left[(ii)-gH\right]$- differentiable of $\hat{u}'(t)$, we get from (1)-(3):

$$\left[-\left(u_\alpha^-\right)''(t), -\left(u_\alpha^+\right)''(t)\right] = \lambda\left[u_\alpha^-(t), u_\alpha^+(t)\right] \tag{12}$$

$$a_1\left[u_\alpha^-(a), u_\alpha^+(a)\right] = a_2\left[\left(u_\alpha^-\right)'(a), \left(u_\alpha^+\right)'(a)\right] \tag{13}$$

$$b_1\left[u_\alpha^-(b), u_\alpha^+(b)\right] = \lambda b_2\left[\left(u_\alpha^-\right)'(b), \left(u_\alpha^+\right)'(b)\right] \tag{14}$$

Suppose that the two linerly independent solutions of $u''+\lambda u=0$ classic differential equation are $u_1(t,\lambda)$ and $u_2(t,\lambda)$. The general solution of the fuzzy differential equation (12) is

$$\left[\hat{u}(t,\lambda)\right]^{\alpha}=\left[u_{\alpha}^{-}(t,\lambda),u_{\alpha}^{+}(t,\lambda)\right]$$

where

$$u_{\alpha}^{-}(t,\lambda)=C_1(\alpha,\lambda)u_1(t,\lambda)+C_2(\alpha,\lambda)u_2(t,\lambda)$$
$$u_{\alpha}^{+}(t,\lambda)=C_3(\alpha,\lambda)u_1(t,\lambda)+C_4(\alpha,\lambda)u_2(t,\lambda).$$

Let $\left[\hat{\phi}(t,\lambda)\right]^{\alpha}$ be a solution which is satisfying

$$\begin{aligned}\left[u_{\alpha}^{-}(a),u_{\alpha}^{+}(a)\right]&=a_2\\ \left[(u_{\alpha}^{-})'(a),(u_{\alpha}^{+})'(a)\right]&=a_1\end{aligned}\tag{15}$$

initial conditions of fuzzy differential equations (12). This solution can be expressed as

$$\begin{aligned}\phi_{\alpha}^{-}(t,\lambda)&=C_{11}(\alpha,\lambda)\cos(kt)+C_{12}(\alpha,\lambda)\sin(kt)\\ \phi_{\alpha}^{+}(t,\lambda)&=C_{13}(\alpha,\lambda)\cos(kt)+C_{14}(\alpha,\lambda)\sin(kt)\end{aligned}\tag{16}$$

and $\left[\hat{\chi}(t,\lambda)\right]^{\alpha}$ be a solution which is satisfying

$$\begin{aligned}\left[u_{\alpha}^{-}(b),u_{\alpha}^{+}(b)\right]&=\lambda b_2\\ \left[(u_{\alpha}^{-})'(b),(u_{\alpha}^{+})'(b)\right]&=b_1\end{aligned}\tag{17}$$

initial conditions of fuzzy differential equations (12). This solution can be expressed as

$$\begin{aligned}\chi_{\alpha}^{-}(t,\lambda)&=C_{21}(\alpha,\lambda)\cos(kt)+C_{22}(\alpha,\lambda)\sin(kt)\\ \chi_{\alpha}^{+}(t,\lambda)&=C_{23}(\alpha,\lambda)\cos(kt)+C_{24}(\alpha,\lambda)\sin(kt).\end{aligned}\tag{18}$$

Then, we have

$$\begin{aligned}W\left(\phi_{\alpha}^{-},\chi_{\alpha}^{-}\right)(t,\lambda)&=\phi_{\alpha}^{-}(t,\lambda)\chi'^{-}_{\alpha}(t,\lambda)-\chi_{\alpha}^{-}(t,\lambda)\phi'^{-}_{\alpha}(t,\lambda)\\ &=\left(C_{11}(\alpha,\lambda)C_{22}(\alpha,\lambda)-C_{12}(\alpha,\lambda)C_{21}(\alpha,\lambda)\right)\left(u_1(t,\lambda)u_2'(t,\lambda)-u_2(t,\lambda)u_1'(t,\lambda)\right)\\ &=\left(C_{11}(\alpha,\lambda)C_{22}(\alpha,\lambda)-C_{12}(\alpha,\lambda)C_{21}(\alpha,\lambda)\right)W(u_1,u_2)(t,\lambda)\end{aligned}$$

and similarly,

$$W\left(\phi_\alpha^+,\chi_\alpha^+\right)(t,\lambda)=\phi_\alpha^+(t,\lambda)\chi'^+_\alpha(t,\lambda)-\chi_\alpha^+(t,\lambda)\phi'^+_\alpha(t,\lambda)$$
$$=\left(C_{13}(\alpha,\lambda)C_{24}(\alpha,\lambda)-C_{14}(\alpha,\lambda)C_{23}(\alpha,\lambda)\right)W(u_1,u_2)(t,\lambda)$$

Now let $\phi(t,\lambda)$ be the solution of the classical differential equation $Lu=\lambda u$ satisfying the conditions $u(a)=a_2$, $u'(a)=a_1$. Then, the solution of the equation is

$$u(t,\lambda)=C_1(\lambda)u_1(t,\lambda)+C_2(\lambda)u_2(t,\lambda).$$
(19)

Using conditions, the following linear system of equation

$$u(a,\lambda)=C_1(\lambda)u_1(a,\lambda)+C_2(\lambda)u_2(a,\lambda)=a_2$$
$$u'(a,\lambda)=C_1(\lambda)u_1(a,\lambda)+C_2(\lambda)u_1(a,\lambda)=a_1.$$

is obtained. From the determinant of the coefficients matrix of the above linear system we get $C_1$ and $C_2$ such that

$$C_1(\lambda)=\frac{\begin{vmatrix} a_2 & u_2(a,\lambda)\\ a_1 & u_2'(a,\lambda)\end{vmatrix}}{W(u_1,u_2)(a,\lambda)}=\frac{a_2u_2'(a,\lambda)-a_1u_2(a,\lambda)}{W(u_1,u_2)(a,\lambda)} \tag{20}$$

$$C_2(\lambda)=\frac{\begin{vmatrix} u_1(a,\lambda) & a_2\\ u_1'(a,\lambda) & a_1\end{vmatrix}}{W(u_1,u_2)(a,\lambda)}=\frac{a_1u_1(a,\lambda)-a_2u_1'(a,\lambda)}{W(u_1,u_2)(a,\lambda)}. \tag{21}$$

Substituting this $C_1(\lambda)$ and $C_2(\lambda)$ coefficients, the above equations in (19), the general solution is obtained as

$$\phi(t,\lambda)=\frac{1}{W(u_1,u_2)(a,\lambda)}\left\{\left(a_2u_2'(a,\lambda)-a_1u_2(a,\lambda)\right)u_1(t,\lambda)+\left(a_1u_1(a,\lambda)-a_2u_1'(a,\lambda)\right)u_2(t,\lambda)\right\}.$$

Let $\chi(t,\lambda)$ be the solution of the classical differential equation $Lu=\lambda u$ satisfying the conditions $u(b)=\lambda b_2$, $u'(b)=b_1$. Similarly

$$\chi(t,\lambda)=\frac{1}{W(u_1,u_2)(b,\lambda)}\left\{\left(\lambda b_2u_2'(b,\lambda)-b_1u_2(b,\lambda)\right)u_1(t,\lambda)\right.$$
$$\left.+\left(b_1u_1(b,\lambda)-\lambda b_2u_1'(b,\lambda)\right)u_2(t,\lambda)\right\}.$$

is obtained. So

$$\left[\hat{\phi}(t,\lambda)\right]^{\alpha}=\left[\phi_{\alpha}^{-}(t,\lambda),\phi_{\alpha}^{+}(t,\lambda)\right]=\left[c_{1}(\alpha),c_{2}(\alpha)\right]\phi(t,\lambda)$$

is the solution of fuzzy differential equations (12) satisfying the conditions (15) and

$$\left[\hat{\chi}(t,\lambda)\right]^{\alpha}=\left[\chi_{\alpha}^{-}(t,\lambda),\chi_{\alpha}^{+}(t,\lambda)\right]=\left[c_{1}(\alpha),c_{2}(\alpha)\right]\chi(t,\lambda)$$

is the solution satisfying the conditions (17), where $c_1{}'(\alpha)\geq 0$, $c_2{}'(\alpha)\leq 0$, and for $\alpha=1$ $c_1(\alpha)=c_2(\alpha)=1$, $[1]^{\alpha}=\left[c_1(\alpha),c_2(\alpha)\right]$. Specially, we take $\left[c_1(\alpha),c_2(\alpha)\right]=\left[\alpha,2-\alpha\right]$.

Then Wronskian functions $W_{\alpha}^{-}(t,\lambda)$ and $W_{\alpha}^{+}(t,\lambda)$ are obtained as

$$W_{\alpha}^{-}(t,\lambda)=\alpha^{2}\left(\phi(t,\lambda).\chi'(t,\lambda)-\chi(t,\lambda)\phi'(t,\lambda)\right) \tag{22}$$

$$W_{\alpha}^{+}(t,\lambda)=(2-\alpha)^{2}\left(\phi(t,\lambda).\chi'(t,\lambda)-\chi(t,\lambda)\phi'(t,\lambda)\right). \tag{23}$$

Consequently, the equation

$$W\left(\phi_{\alpha}^{-},\chi_{\alpha}^{-}\right)(t,\lambda)=\frac{\alpha^{2}}{(2-\alpha)^{2}}W\left(\phi_{\alpha}^{+},\chi_{\alpha}^{+}\right)(t,\lambda)$$

is obtained [13].

**Teorem 15.** The Wronskian functions $W\left(\phi_{\alpha}^{-},\chi_{\alpha}^{-}\right)(t,\lambda)$ and $W\left(\phi_{\alpha}^{+},\chi_{\alpha}^{+}\right)(t,\lambda)$ are independent of variable $x$ for $x\in[a,b]$, where functions $\phi_{\alpha}^{-},\chi_{\alpha}^{-},\phi_{\alpha}^{+},\chi_{\alpha}^{+}$ are the solution of the fuzzy boundary value problem (1)-(3) and it is show that

$$\left[\hat{W}\right]^{\alpha}=\left[W_{\alpha}^{-}(\lambda),W_{\alpha}^{+}(\lambda)\right]$$

for each $\alpha\in[0,1]$ [13].

**Teorem 16.** The eigenvalues of the fuzzy boundary value problem (1)-(3) if and only if consist of the zeros of functions $W_{\alpha}^{-}(\lambda)$ and $W_{\alpha}^{+}(\lambda)$ [13].

**Example 17.** Consider the two point fuzzy boundary problem

$$-\hat{u}''=\lambda\hat{u} \tag{24}$$
$$\hat{u}(0)=0 \tag{25}$$
$$2\hat{u}(1)=\lambda 3\hat{u}'(1) \tag{26}$$

where $\lambda=k^{2},k>0$ and $\hat{u}(t)$ is $\left[(i)-gH\right]$- differentiable and $\hat{u}'(t)$ is $\left[(ii)-gH\right]$-differentiable fuzzy functions.

Let

$$\phi(t,\lambda)=\sin(kt)$$

be the solution of the classical differential equation $u''+\lambda u=0$ satisfying the condition $u(0)=0$ and

$$\chi(t,\lambda)=\left(\left(k^2 3\cos(k)-\frac{2}{k}\sin(k)\right)\cos(kt)+\left(3k^2\sin(k)+\frac{2}{k}\cos(k)\right)\sin(kt)\right)$$

be the solution satisfying the conditions $u(1)=3\lambda$, $u'(1)=2$. Then

$$\left[\hat{\phi}(t,\lambda)\right]^{\alpha}=\left[\phi_{\alpha}^{-}(t,\lambda),\phi_{\alpha}^{+}(t,\lambda)\right]=[\alpha,2-\alpha]\sin(kt) \qquad (27)$$

and

$$\begin{aligned}\left[\hat{\chi}(t,\lambda)\right]^{\alpha}&=\left[\chi_{\alpha}^{-}(t,\lambda),\chi_{\alpha}^{+}(t,\lambda)\right]\\&=[\alpha,2-\alpha]\left(\left(k^2 3\cos(k)-\frac{2}{k}\sin(k)\right)\cos(kt)\right.\\&\quad\left.+\left(3k^2\sin(k)+\frac{2}{k}\cos(k)\right)\sin(kt)\right).\end{aligned} \qquad (28)$$

From theorem 19, the eigenvalues of the problem (24)-(26) are zeros of the functions $W_{\alpha}^{-}(\lambda)$ and $W_{\alpha}^{+}(\lambda)$. So we get

$$W_{\alpha}^{-}(\lambda)=\alpha^2\left(-3k^2\cos(k)+\frac{2}{k}\sin(k)\right)=0 \qquad (29)$$

$$W_{\alpha}^{+}(\lambda)=(2-\alpha)^2\left(-3k^2\cos(k)+\frac{2}{k}\sin(k)\right)=0. \qquad (30)$$

Then, if the $k$ values satisfying

$$-3k^2\cos(k)+\frac{2}{k}\sin(k)=0 \qquad (31)$$

equation are computed with Matlab Program, then an infinite number of values are obtained such that

$$k_1=4.7060,\ k_2=7.8526,\ k_3=10.9951,\ k_4=14.1369,\ k_5=17.2786,\ \ldots$$

We show this values with $k_n$, $n=1,2,\ldots$. Substituing this values in (27)-(28), we obtain

$$\left[\hat{\phi}_n(t,\lambda)\right]^{\alpha}=\left[(\phi_n)_{\alpha}^{-}(t,\lambda),(\phi_n)_{\alpha}^{+}(t,\lambda)\right]=[\alpha,2-\alpha]\sin(k_n t)$$

$$\begin{aligned}\left[\hat{\chi}_n(t,\lambda)\right]^{\alpha}&=\left[(\chi_n)_{\alpha}^{-}(t,\lambda),(\chi_n)_{\alpha}^{+}(t,\lambda)\right]\\&=[\alpha,2-\alpha]\left(3k_n^{\,2}\sin(k_n)+\frac{2}{k_n}\cos(k_n)\right)\sin(k_n t).\end{aligned}$$

If $\sin(k_n t) \geq 0$ and $\left(3k_n^2 \sin(k_n) + \frac{2}{k_n}\cos(k_n)\right)\sin(k_n t) \geq 0$, then $\left[\hat{\phi}_n(t,\lambda)\right]^\alpha$ and $\left[\hat{\chi}_n(t,\lambda)\right]^\alpha$ are a valid $\alpha$- cut set.

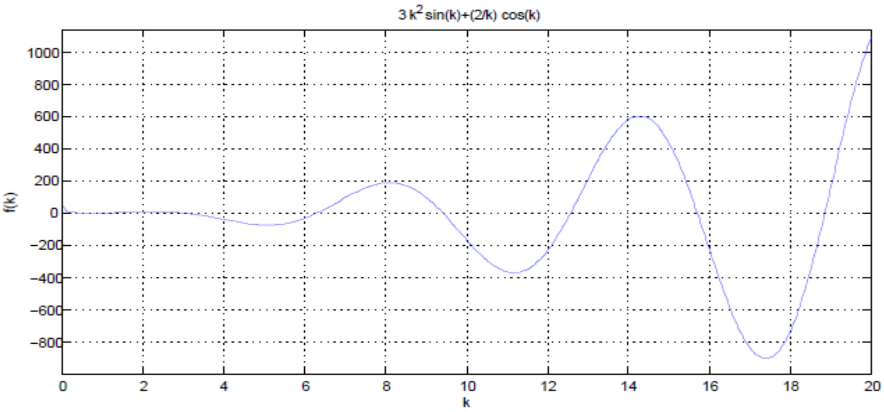


**Figure 1.** $f(k) = 3k^2 \sin(k) + \frac{2}{k}\cos(k)$

Let be $k_n t \in [(n-1)\pi, n\pi]$, $n = 1, 2, \ldots$

i. If n is odd, $\sin(k_n t) \geq 0$. So $\left[\hat{\phi}_n(t,\lambda)\right]^\alpha$ is a valid $\alpha$- cut set.

ii. If $n$ is even, $\sin(k_n t) \leq 0$, Also for $x \in [0,1]$, $k_n \in [(n-1)\pi, n\pi]$ and from Figure 1, $\left(3k_n^2 \sin(k_n) + \frac{2}{k_n}\cos(k_n)\right) \leq 0$. So $\left[\hat{\chi}_n(t,\lambda)\right]^\alpha$ is a valid $\alpha$- cut set.

Consequently, for $k_n t \in [(n-1)\pi, n\pi]$, $n = 1, 2, \ldots$

i. If $n$ is odd, the eigenfunctions corresponding to the eigenvalues $\lambda_n = k_n^2$ are

$$\left[\hat{u}_{1n}(t)\right]^\alpha = [\alpha, 2-\alpha]\sin(k_n t),$$

ii. If $n$ is even, the eigenfunctions corresponding to the eigenvalues $\lambda_n = k_n^2$ are

$$\left[\hat{u}_{2n}(t)\right]^\alpha = [\alpha, 2-\alpha]\left(3k_n^2 \sin(k_n) + \frac{2}{k_n}\cos(k_n)\right)\sin(k_n t),$$

iii. If $\alpha = 1$, eigenfunctions corresponding to the eigenvalues $\lambda_n = k_n^2$ are

$$\left[\hat{u}_n(t)\right]^\alpha = \sin(k_n t).$$

## 5. CONCLUSION

Eigenvalues and eigenfunctions of the fuzzy boundary problem are introduced and examined by using generalized Hukuhara differentiability concept. In this problem, the eigenvalue parameter is contained in the boundary condition at $b$. So we define lineer operator in fuzzy Hilbert space. To solve this problem, we use some initial value problems. Then we give numerical example.

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