ORIGINAL PAPER

ON CHARACTERIZATION OF INVARIANT AND EXACT SOLUTIONS OF HUNTER-SAXON EQUATION

ZELIHA KORPINAR[1], MUSTAFA INC[2]

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Abstract.*In this work, Lie symmetry analysis is applied on Lax's fifth-order KdV and Hunter-Saxton equations. By using point symmetry, all of the geometric vector fields of these equations are presented. Then, on the basis of the optimal system, similarity reductions and exact solutions are obtained.*

Keywords: *Lax's fifth-order KdV equation, Hunter-Saxton equation, Symmetry analysis, Exact solutions, Soliton.*

1. INTRODUCTION

Lie defined theory of Lie symmetry groups of differential equations [1]. Thus, Lie groups are reversible point transformations of the some differential equations. On the other hand, Lie symmetry method have been extensively performed to some nonlinear partial differential equations arising in mathematics, applied physics and in many other scientific fields. Therefore, Lie symmetry groups may deal with symmetry reductions, similarity solutions of nonlinear differential partial equations. Since the method of Lie symmetry analysis is the most important approach for characterizing analytical solutions of some nonlinear partial differential equations. Many practices of Lie groups and algebras in the theory of differential equations were constructed [2-4].

Recently, there has been considerable interest in Lie symmetry analysis. Lie symmetry analysis method have been applied for the Kudryashov Sinelshchikov equation in [5]. They obtained ones and two dimensional optimal systems of Lie subalgebras by group-invariant solutions. By symbolic computation, Lie symmetry analysis, Painlevé test method, some conservation laws and similarity solutions for the Nizhnik Novikov Veselov equation have been performed in [6]. They derived the group classifications and their symmetry reductions by Lie group method.

Jin-qian et al. obtained the symmetry properties by using modified CK's direct method, and some new exact solutions of (2+1) dimensional Boiti Leon Pempinelli equation [7]. Bruzon et al. applied the classical Lie method of infinitesimals for BBM equation [8]. Badali et al. have studied Lie symmety analiysis for Kawahara-KdV equations [9]. Wang et al. have studied the generalized fifth order KdV equation using group methods and conservation laws [10]. The readers can look in [11-14] for Lie symmetry and group invariant solutions.

Intention of this work is to use Lie group analysis method to obtain some exact solutions of the Lax's fifth order KdV and the Hunter Saxton equations.

The generalized fifth-order KdV equation [10] is given by

[1] Mus Alparslan University, Faculty of Economic and Administrative Sciences, Department of Administration, Mus, Turkey. E-mail: zelihakorpinar@gmail.com.
[2] Firat University,Science Faculty, Department of Mathematics, Elazig, Turkey.E-mail: minc@firat.edu.tr.

$$u_t + \alpha u^n u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0. \tag{1.1}$$

In Eq. (1.1), we can acquire, for $n = 2$,

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0, \tag{1.2}$$

where $\alpha,\ \beta$ and γ are nonzero constants. If we choose $\alpha = 30,\ \beta = 20$ and $\gamma = 10,$ then we obtain the Lax's fifth-order KdV equation

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10u u_{xxx} + u_{xxxxx} = 0. \tag{1.3}$$

Hunter Saxton equation [15] is given by

$$(u_t + u u_x)_x = \frac{1}{2} u_x^2. \tag{1.4}$$

Eq. (1.4) defines the spread of weakly nonlinear waves in a massive nematic liquid crystal director field. This equation was solved in [15] using the method of characteristics. A generalization of Eq. (1.4) was studied by Pavlov [16] and it was also solved. Eq. (1.4) was investigated by Hunter and Zheng [17], and it was proven that equation is a entirely integrable, bi variational, bi Hamiltonian system. In [18], Penskoi studied Lagrangian time-discretizations of the Hunter-Saxton equation using the Moser-Veselov approach.

In the second section of this work, method of Lie symmetries are described. The vector fields of the Lax's fifth-order KdV and Hunter-Saxton equations are symbolized by using Lie symmetry analysis method and exact solutions to these equations are given. In Section 3, exact solutions to the Hunter-Saxton equation is investigated and the exact solutions to the Lax's fifth-order KdV equation is obtained by using Kudryashov method and some exact solutions to Hunter-Saxton equation are obtained by using power series method. Finally, the main conclusions are introduced in Section 4.

2. MATERIALS AND METHODS

2.1 LIE SYMMETRY ANALYSIS FOR THE LAX'S FIFTH-ORDER KDV EQUATION

We now recall some basic concepts of Lie group method (symmetry analysis). Thus, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other new solutions. By using this method, one may directly use the defining property of such a group and construct new solutions to the system from known ones.

We imagine a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} &x \to x + \lambda\zeta(x,t,u), \\ &t \to t + \lambda\eta(x,t,u), \\ &u \to u + \lambda\varphi(x,t,u), \end{aligned} \tag{2.1}$$

where $\lambda = 1$. Therefore, vector field associated with the above transformations can be shown as

$$V = \zeta(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}. \tag{2.2}$$

From Lie symmetry analysis method, we construct that the coefficient functions ζ, η and φ must satisfy the symmetry condition. Its fifth prolongation is given [2]

$$Pr^{(5)}V = V + \varphi^t \partial_{u_t} + \varphi^x \partial_{u_x} + \varphi^{xx} \partial_{u_{xx}} + \varphi^{xt} \partial_{u_{xt}} + \varphi^{tt} \partial_{u_{tt}} + \ldots + \varphi^{xt^4} \partial_{u_{xt^4}} + \varphi^{t^5} \partial_{u_{t^5}}, \tag{2.3}$$

where

$$\varphi^x = D_x(\varphi - \zeta u_x - \eta u_t) + \zeta u_{xx} + \eta u_{xt},$$
$$\varphi^t = D_t(\varphi - \zeta u_x - \eta u_t) + \zeta u_{xt} + \eta u_{tt},$$

$$\vdots \tag{2.4}$$
$$\varphi^{t^5} = D_{t^5}(\varphi - \zeta u_x - \eta u_t) + \zeta u_{x^5 t} + \eta u_{t^5},$$

where D_x and D_t are the total differentation with respect to x and t, respectively [2].

V vector field constructs a one-parameter symmetry group of (1.3) if

$$Pr^{(5)}V[u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{xxxxx}] = 0 \tag{2.5}$$

whenever

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0.$$

Substituting (2.5) into (2.3), we have following forms of the coefficient functions:

$$\zeta = c_1, \eta = c_2, \varphi = 0,$$

where c_1, c_2 are arbitrary constants.

Lie algebra of infinitesimal symmetry of (1.3) is spanned by the following vectors:

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}. \tag{2.6}$$

By bracket, relations between these vector fields are

$$[V_1, V_2] = 0.$$

Therefore, Lie algebra is solvable. By the adjoint representing of this vector fields, we have optimal systems of three Lax's fifth-order KdV equation as follows:

For Eq. (1.3), we have

$$\{V_1, V_2, V_2 + \alpha V_1\} \tag{2.7}$$

where α is an arbitrary constant.

On the basis of the Classical symmetry analysis for the Lax's fifth-order KdV equation, we can get

$$\zeta = c_1 + c_3 x, \eta = c_2 + 5tc_3, \varphi = -2uc_3, \tag{2.8}$$

where c_1, c_2 and c_3 are arbitrary constants. Hence, Lie algebra of infinitesimal symmetry of (1.3) is generated by the three vector fields:

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}, V_3 = 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x}. \tag{2.9}$$

If we take the vector field of Eq. (1.3), we have Table 1.

Table 1. Commutator for the Lie algebra of the Lax's fifth-order KdV equation.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	V_1
V_2	0	0	$5V_2$
V_3	$-V_1$	$-5V_2$	0

Table 2. Adjoint representation for Lie algebra of Lax's fifth-order KdV equation.

$Ad(ex(\varepsilon V_i))V_j$	V_1	V_2	V_3
V_1	0	V_2	$V_3 - \varepsilon V_1$
V_2	V_1	0	$V_3 - 5\varepsilon V_2$
V_3	$V_1 + \varepsilon V_1$	$V_2(1+5\varepsilon+25/2\varepsilon^2+\ldots)$	0

By the adjoint representing of this vector fields, we find optimal systems of Lax's fifth-order KdV equation as follows:

$$\{V_1, V_2, V_3, V_2 + \alpha V_1, V_3 + \alpha V_1, V_3 + \alpha V_2\} \tag{2.10}$$

where α is an arbitrary constant.

2.2 LIE SYMMETRY METHOD FOR THE HUNTER-SAXTON EQUATION

We find the vector field of the Hunter Saxton equation (1.4) as follows:

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial t},\ V_3 = 2t\frac{\partial}{\partial t} + t^2\frac{\partial}{\partial x},\ V_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{2.11}$$

For the vector field of Eq. (1.4) under the Lie bracket., we have Table 3.

Table 3. Commutator for the Lie algebra of the Hunter Saxton equation.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	$2V_4$	V_1
V_3	0	$-2V_4$	0	0
V_4	0	$-V_1$	0	0

For the Hunter Saxton equation (1.4), for any $\varepsilon \in R$, we have Table 4.

Table 4. Adjoint representation for Lie algebra of Hunter Saxton equation.

$Ad(ex(\varepsilon V_i))V_j$	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	V_4
V_2	V_1	V_2	$V_3-2\varepsilon V_4+\varepsilon^2 V_1$	$V_4-\varepsilon V_1$
V_3	V_1	$V_2+2\varepsilon V_4$	V_3	V_4
V_4	V_1	$V_2+\varepsilon V_1$	V_3	V_4

By the adjoint representing of this vector fields, we find optimal systems of Hunter Saxton equation as follows:

$$\{V_1, V_2, V_2 + \alpha V_3, V_2 + \alpha V_4\} \tag{2.12}$$

where α is an arbitrary constant.

On the basis of the Classical symmetry analysis for the Hunter Saxton equation, we can get

$$\zeta = (c_2 + c_3)x + 2c_4 xt, \eta = c_1 + tc_3 + t^2 c_4, \varphi = uc_2 + 2c_4 x, \tag{2.13}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Hence, Lie algebra of infinitesimal symmetry of (1.4) is generated by the four vector fields:

$$V_1 = \frac{\partial}{\partial t}, V_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \tag{2.14}$$
$$V_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, V_4 = t^2\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial u} + 2tx\frac{\partial}{\partial x}.$$

3. REDUCED ODES AND EXACT SOLUTIONS

3.1 REDUCED ODES AND EXACT SOLUTIONS FOR THE LAX'S FIFTH-ORDER KDV EQUATION

(i) For V_1, we obtain trivial solution of Eq.(1.3) is

$$u(x,t) = k, \tag{3.1}$$

where k is an arbitrary constant.

(ii) For V_2, we have

$$u(x,t) = f(\xi) \tag{3.2}$$

where $\xi = x$.

Substituting (3.2) into Eq. (1.3) and once integral, we reduce equation to the following ODE

$$10f^3 + 5(f')^2 + 10ff'' + f^{(4)} + C = 0. \tag{3.3}$$

where $f' = \dfrac{df}{d\xi}$.

(iii) For $V_2 + \alpha V_1$, we have

$$u = f(\xi)$$

where $\xi = x - \alpha t$. We obtain to following ODE

$$-\alpha f + 10f^3 + 5(f')^2 + 10ff'' + f^{(4)} + C = 0. \tag{3.4}$$

(iv) For V_3, we have

$$u = t^{-\frac{2}{5}} f(\xi)$$

where $\xi = xt^{-\frac{1}{5}}$. We obtain to following ODE

$$-\frac{2}{5}f - \frac{\xi}{5}f' + 30f^2 f' + 20f'f'' + 10ff^{(3)} + f^{(5)} = 0. \tag{3.5}$$

(v) For $V_3 + \alpha V_1$, this linear combination and generator V_2 have the same ODE for Eq.(1.3).

(vi) For $V_3 + \alpha V_2$, we have

$$-2f - \xi f' + 30f^2 f' + 20f'f'' + 10ff^{(3)} + f^{(5)} = 0. \tag{3.6}$$

Now, we seek a new solution of Eq. (3.4) with the Kudryashov method [19]

$$f(\xi) = c_0 + c_1 Q(\xi) + c_2 Q^2(\xi) \tag{3.7}$$

where c_0, c_1 and c_3 are unknown constants, $Q(\xi)$ is the following function

$$Q(\xi) = \frac{1}{1+e^{\xi}}. \tag{3.8}$$

Substituting (3.7) in Eq. (3.4) and taking (3.8) into account we obtain the polynomial of function $Q(\xi)$. Collecting all terms with the same power of function $Q(\xi)$ and equate this expressions to zero we obtain the system of algebraic equations. Solving this system we find that solution of Eq. (3.4). They are,

$$f_1(\xi) = 2\left(\frac{1}{1+e^{\xi}}\right) - 2\frac{1}{(1+e^{\xi})^2},$$

$$f_2(\xi) = \frac{1}{20}(-5-\sqrt{5}) + 2\left(\frac{1}{1+e^{\xi}}\right) - 2\frac{1}{(1+e^{\xi})^2},$$

$$f_3(\xi) = \frac{1}{20}(-5+\sqrt{5}) + 2\left(\frac{1}{1+e^{\xi}}\right) - 2\frac{1}{(1+e^{\xi})^2}.$$

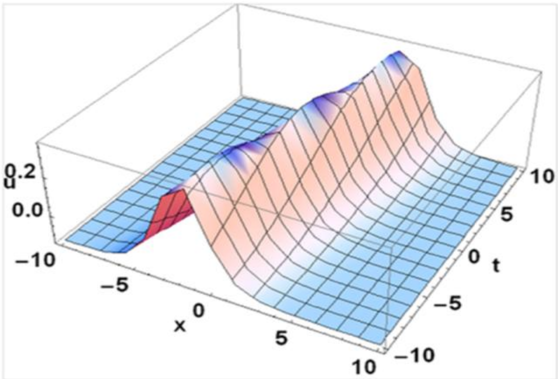

Figure 1. The surface graph of bell-shaped solitary wave solution $f_3(\xi)$ for equation (1.3).

3.2 REDUCED ODES AND EXACT SOLUTIONS FOR THE HUNTER SAXTON EQUATION

In previous section, we have been obtained vector fields and optimal systems of Hunter Saxton equation.

(i) For V_1, we obtain trivial solution of Eq. (1.4) is

$$u(x,t)=k,$$

where k is an arbitrary constant.

(ii) For V_2, we have

$$u(x,t)=f(\xi) \tag{3.9}$$

where $\xi = x$.

Substituting (3.9) into Eq. (1.4), we reduce this equation to the following ODE

$$ff''+\frac{1}{2}(f')^2=0, \tag{3.10}$$

where $f'=\dfrac{df}{d\xi}$.

Eq. (3.10) has the solution $f(\xi)=c_2(3\xi-2c_1)^{\frac{2}{3}}$. Solution of Eq. (1.4) is

$$u(x,t)=c_2(3x-2c_1)^{\frac{2}{3}}, \tag{3.11}$$

where c_1 and c_2 are two arbitrary constant.

(iii) For $V=V_2+\alpha V_3$, we get

$$u(x,t)=\alpha t^2+f(\xi), \tag{3.12}$$

where $\xi = x-\dfrac{\alpha t^3}{3}$.

Substituting (3.12) into Eq. (1.4), we reduce to the following ODE

$$ff''+\frac{1}{2}(f')^2=0, \tag{3.13}$$

where $f'=\dfrac{df}{d\xi}$. The linear combination $V=V_2+\alpha V_3$ and the generator V_2 have the same solution for Eq.(1.4).

(iv) For $V=V_3+\alpha V_4$, we obtain

$$u(x,t)=\frac{2t+\alpha}{t^2+\alpha t}x+f(\xi), \tag{3.14}$$

where $\xi = t$. There isn't solution of Eq. (1.4) for this linear combination.

For (2.14) vector fields which are obtained with classical symmetry analysis of the Hunter-Saxton equation, we can get

$i)$ $V_2 = x\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial u}$; we obtain $u(x,t)=xf(\xi)$, where $\xi = t$. We reduce Eq. (1.4) to following ODE

$$\frac{f^2}{2}+f'=0. \tag{3.15}$$

ii) $V_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$; we obtain $u(x,t) = f(\xi)$, where $\xi = \frac{x}{t}$. We reduce Eq. (1.4) to following ODE

$$-f' + \frac{\left(f'\right)^2}{2} + ff'' - \xi f'' = 0. \tag{3.16}$$

Now, we apply power series method to Eq. (3.16)

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{3.17}$$

Substituting (3.17) into (3.16), we have

$$\begin{aligned} &-c_1 - \sum_{n=1}^{\infty}(n+1)c_{n+1}\xi^n + \frac{1}{2}c_0c_1 + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+1-k)c_{n+1-k}c_k\xi^n \\ &-\sum_{n=1}^{\infty}c_{n+1}n(n+1)\xi^n + 2c_0c_2 + \sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+2-k)(n+1-k) \\ &c_{n+2-k}c_k\xi^n l = 0. \end{aligned} \tag{3.18}$$

where

$$-c_1 + \frac{1}{2}c_0c_1 + 2c_0c_2 = 0,$$

$$c_2 = \frac{1}{4c_0}(2c_1 - c_1c_0)$$

For $n \geq 1$;

$$\begin{aligned} &-\sum_{n=1}^{\infty}c_{n+1}(n+1)\xi^n + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+1-k)c_{n+1-k}c_k\xi^n \\ &-\sum_{n=1}^{\infty}c_{n+1}n(n+1)\xi^n + \sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+2-k)(n+1-k) \\ &c_{n+2-k}c_k\xi^n l = 0. \end{aligned}$$

$$\begin{aligned} c_{n+1} = &\left(\frac{1}{n+1}\right)^2\Big(\frac{1}{2}\sum_{k=0}^{n}c_{n+1-k}c_k(n+1-k) + \\ &\sum_{k=0}^{n}(n+2-k)(n+1-k)c_{n+2-k}c_k\Big) \end{aligned} \tag{3.19}$$

Therefore, the power series solution of Eq. (3.16) can be written as follows:

$$\begin{aligned} f(\xi) = &c_0 + c_1\xi + \left(\frac{1}{n+1}\right)^2\Big(\frac{1}{2}\sum_{k=0}^{n}c_{n+1-k}c_k(n+1-k) + \\ &\sum_{k=0}^{n}c_{n+2-k}c_k(n+2-k)(n+1-k)\Big)\xi^{n+1} \end{aligned} \tag{3.20}$$

Therefore,

$$u(x,t) = c_0 + c_1(\frac{x}{t}) + \frac{1}{(n+1)^2}(\frac{1}{2}\sum_{k=0}^{n} c_{n+1-k} c_k (n+1-k) + \sum_{k=0}^{n} c_{n+2-k} c_k (n+2-k)(n+1-k))(\frac{x}{t})^{n+1}, \quad (3.21)$$

where c_{n+1} $(n = 1,2,...)$ are given by (3.19).

4. RESULTS AND DISCUSSION

In this work, we have obtained similarity reductions and symmetries of Lax's fifth-order KdV equation and Hunter-Saxton equation by using Lie symmetry analysis method. We obtain the exact solutions of these equations corresponding to reduced ODEs, which have been verified by placing them back into the essential equation and then the exact solutions are studied. By using this symmetries, we have imagined that this equation may be converted into a ODE. Finally, this symmetry analysis based on the Lie group method is a very powerful method and is importance of studying further.

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