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MODULES THAT HAVE A WEAK *RAD*-SUPPLEMENT IN EVERY EXTENSION

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Abstract. *As a proper generalization of the modules with the properties* (E) *and* (EE) *that were introduced by Zöschinger in terms of supplements, we say that a module* M *has the property* (WRE) *(respectively,*$(WREE)$*) if* M *has a weak Rad-supplement (respectively, ample weak Rad-supplements) in every extension. In this paper, we prove that if every submodule of a module* M *has the property* (WRE)*, then* M *has the property* $(WREE)$*. We show that a ring* R *is semilocal if and only if every left* R*-module has the property* (WRE)*. Also we prove that over a commutative Von Neumann regular ring a module* M *has the property* (WRE) *if and only if* M *is injective.*

Keywords: *weak Rad-supplement; extension; semilocal ring; Von Neumann regular ring.*

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1. INTRODUCTION

Throughout this paper, R is an associative ring with identity and all modules are unital left R-modules. Let M be an R-module. By $U \leq M$, we mean that U is a submodule of M. A submodule U of M is said to be *small* in M, denoted as $U \ll M$, if $M \neq U + L$ for every proper submodule L of M. By $Rad(M)$, we denote the intersection of all maximal submodules of M or, equivalently the sum of all small submodules of M. A module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$.

As a proper generalization of direct summands of a module, the notion of supplement submodules is defined. For U, V submodules of a module M, V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$. Then, it is natural to introduce a generalization of supplement submodules by [11, 19.3.(2)]. A submodule V of M is called a *weak supplement* of U in M if $M = U + V$ and $U \cap V \ll M$. A submodule U of M has *ample (weak) supplements* in M if, whenever $M = U + L$, L contains a (weak) supplement of U in M. A module M is called *weakly supplemented* if every submodule of M has a weak supplement in M (see [5]). By [4, 17.9(6)], if a module M is weakly supplemented, every submodule of M has ample weak supplements in M. A submodule V of M is called *radical supplement* or briefly *Rad-supplement* (according to [10], *generalized supplement*) of U in M if $M = U + V$ and $U \cap V \leq Rad(V)$ (see [4, 10.14]). A submodule U of M has *ample Rad-supplements* in M if every submodule L of M with $M = U + L$ contains a *Rad*-supplement of U in M. A submodule V of M is called *weak Rad-supplement* of a

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submodule U in M if $M = U + V$ and $U \cap V \leq Rad(M)$. A submodule U of M has *ample weak Rad-supplements* in M if every submodule L of M with $M = U + L$ contains a weak *Rad*-supplement of U in M . A module M is called *weakly Rad-supplemented* (according to [10], *weakly generalized supplemented*) if every submodule of M has a weak *Rad*-supplement in M [6]. Note that every submodule of a weakly *Rad*-supplemented module M has ample weak *Rad*-supplements in M. In the view of given definitions, we clearly have the following implication on submodules:

$$direct\ summand \Rightarrow supplement \Rightarrow Rad - supplement \Rightarrow weak\ Rad - supplement$$

Let R be a ring and M be an R-module. An R-module N is called an *extension* of M provided $M \subseteq N$.

It is well known that a module M is *injective* if and only if it is a direct summand in every extension. H. Zöschinger initiated the study of the modules that have a supplement (resp. ample supplements) in every extension, i.e. modules with *the property* (E) (resp. (EE)) in [12], as a generalization of injective modules. The author determined in the same paper the structure of modules with these properties.

In [9], S. Özdemir defined the modules that have (ample) a *Rad*-supplement in every extension, namely *(ample) Rad-supplementing modules*. He gave various properties of these modules. He provided that every left R-module is (ample) *Rad*-supplementing if and only if $R/_{P(R)}$ is left perfect, where $P(R)$ is the sum of all left ideals I of R such that $Rad(I) = I$.

In [8], E. Önal, H. Çalışıcı and E. Türkmen called a module M has *the property* (WE) (resp. (WEE)) if M has a weak supplement (resp. ample weak supplements) in every extension. Also they gave new characterizations of left perfect rings via the modules that have the property (WE).

Adapting the modules whose definitions were given above, we define and study the modules that have the property (WRE) (resp. $(WREE)$) as a proper generalization of the modules with the property (E) (resp. (EE)). A module M is called to have *the property* (WRE) (resp. $(WREE)$) if it has a weak *Rad*-supplement (resp. ample weak *Rad*-supplements) in every extension.

In this paper, we provide some properties of the modules with the properties (WRE) and $(WREE)$. We prove that if every submodule of a module M has the property (WRE), then M has the property $(WREE)$. We show that every direct summand of a module with the property (WRE) has the property (WRE). A ring R is semilocal if and only if every left R-module has the property (WRE).

2. MAIN RESULTS

It is shown in [12, Lemma 1.3(a)] that direct summands of modules with the property (E) have the property (E). Now we give an analogue of this fact for the modules with the property (WRE).

Proposition 1: Every direct summand of a module with the property (WRE) has the property (WRE).

Proof: Let M_1 be a direct summand of M. Then there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N be an extension of M_1. Let N' be the external direct sum $N \oplus M_2$ and

$\vartheta: M \longrightarrow N'$ be the canonical embedding. Then $M \cong \vartheta(M)$ has the property (WRE). Hence, there exists a submodule V of N' such that $N' = \vartheta(M) + V$ and $\vartheta(M) \cap V \leq Rad(N')$. By the projection $\pi: N' \to N$, we have that $M_1 + \pi(V) = N$. Since $Ker(\pi) \subseteq \vartheta(M)$, $\pi(\vartheta(M) \cap V) = \pi(\vartheta(M)) \cap \pi(V) = M_1 \cap \pi(V) \leq \pi(Rad(N')) = Rad(\pi(N')) = Rad(N)$ by [11, 21.6]. Hence $\pi(V)$ is a weak *Rad* -supplement of M_1 in N.

Proposition 2: Let M be a module. If every submodule of M has the property (WRE), then M has the property $(WREE)$.

Proof: Suppose that every submodule of M has the property (WRE). For any extension N of M, let $N = M + K$ for some submodule K of N. Since $M \cap K$ has the property (WRE), there exists a submodule L of K such that $(M \cap K) + L = K$ and $(M \cap K) \cap L = M \cap L \leq Rad(K)$. Note that $N = M + K = M + ((M \cap K) + L) = M + L$ and $M \cap L \leq Rad(N)$ by [11, 19.3]. It follows that L is a weak *Rad*-supplement of M in N.

Proposition 3: Let M be a module and U be a radical submodule of M. If the factor module M/U has the property (WRE), then M has the property (WRE).

Proof: Let N be any extension of M. Since M/U has the property (WRE), there exists a submodule V/U of N/U such that $M/U + V/U = N/U$ and $(M \cap V)/U \leq Rad(N/U)$. Since U is a radical submodule of M, it follows that $U \leq P$ for every maximal submodule P of M and so $Rad(N/U) = Rad(N)/U$. Note that $N = M + V$ and $M \cap V \leq Rad(N)$. Hence V is a weak *Rad*-supplement of M in N .

Proposition 4: Every radical module has the property (WRE).

Proof: Let M be a radical module and N be any extension of M. Since $Rad(M) = M$, then $N = M + N$ and $M \cap N = M = Rad(M) \leq Rad(N)$. Then N is a weak *Rad*-supplement of M in N.

Corollary 5: Every finite direct sum of radical modules has the property (WRE).

Proof: By [11, 21.6(5)], every finite direct sum of radical modules is a radical module and so it has the property (WRE) by Proposition 4.

Let $P(M) = \sum\{U \leq M | Rad(U) = U\}$. A module M is *reduced* if $P(M) = 0$. Since $P(M)$ is a radical submodule of M, we obtain the next result which is a direct consequence of Proposition 4.

Corollary 6: For a module M, $P(M)$ has the property (WRE).

Proposition 7: Let $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ be a short exact sequence and K be a radical module. If L has the property (WRE), M also has the property (WRE). If the sequence splits, the converse holds.

Proof: Without restriction of generality we will assume that $K \leq M$. Since $M/_K \cong L$ has the property (WRE) and K is radical, then we have M has the property (WRE) by Proposition 3. On the other hand, suppose that the sequence splits. Then $M \cong K \oplus L$. If M has the property (WRE), then L also has the property (WRE) by Proposition 1.

A ring R is called a *left V-ring* if every simple left R-module is injective. It is known that R is a left V-ring if and only if for any left R-module M, $Rad(M) = 0$.

Proposition 8: Let R be a left V-ring and M be an R-module. The following statements are equivalent:

(1) M has the property (WRE),
(2) M is injective,
(3) M has the property (WE).

Proof: $(1) \Rightarrow (2)$ Let N be any extension of M . Then there exists a weak *Rad*-supplement V of M in N, that is, $M + V = N$, $M \cap V \leq Rad(N)$. Since R is left V-ring, $Rad(N) = 0$. Hence we have that $N = M \oplus V$.
$(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are clear.

A ring R is called *Von Neumann regular* if every element a of R can be written in the form axa for some $x \in R$. A commutative ring is a left V-ring if and only if it is Von Neumann regular [11, 23.5]. The next result is a direct consequence of this fact and Proposition 8.

Corollary 9: Let R be a commutative Von Neumann regular ring. Then an R-module M is injective if and only if it has the property (WRE).

In general, a module with the property (WRE) does not need to have the property (WE). To see this, we shall consider the following example:

Example 10: (see [2, Example 6.2]) Let k be a field. In the polynomial ring $k[x_1, x_2, \dots]$ with countably many indeterminates x_n, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \dots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for every $n \in \mathbb{Z}^+$. Then the quotient ring $R = {k[x_1, x_2, \dots]}/_I$ is a local ring and the ideal $J = {(x_1, x_2, \dots)}/_I$ of R generated by all $x_n + I$, $n \in \mathbb{Z}^+$ is the unique maximal ideal of R. Then $M = J^{(\mathbb{N})}$ is a radical R-module and so M has the property (WRE) by Proposition 4. Since R is not a left perfect ring, $M = J^{(\mathbb{N})} = Rad\left(R^{(\mathbb{N})}\right)$ does not have a weak supplement in $R^{(\mathbb{N})}$ by [1, Theorem 1]. Thus M does not have the property (WE).

We recall from [4] that a ring R is a *left Bass ring* if and only if for every nonzero left R-module M, $Rad(M) \ll M$.

Lemma 11: Let R be a left Bass ring and M be an R-module. Then every weak *Rad*-supplement in M is a weak supplement in M.

Proof: Let V be a weak Rad-supplement submodule in M. Then there exists a submodule U of M such that $M = U + V$ and $U \cap V \leq Rad(M)$. Since R is a left Bass ring, $Rad(M) \ll M$. Hence V is a weak supplement of U in M.

Theorem 12: Let R be a left Bass ring. Then the followings are equivalent:

(1) Every left R-module has the property (WRE),

(2) Every left R-module has the property (WE).

Proof: $(1) \Rightarrow (2)$ Let M be an R-module with the property (WRE), and N be any extension of M. By hypothesis, there exists a weak *Rad*-supplement V of M in N. V is a weak supplement of M in N by Lemma 11. Thus M has the property (WE).

$(2) \Rightarrow (1)$ is clear.

Recall from [3] that a module M is called *strongly radical supplemented* if every submodule of M containing $Rad(M)$ has a supplement in M. It is shown in [3, Corollary 2.1] that finite sums of strongly radical supplemented modules are strongly radical supplemented. Moreover every radical module is strongly radical supplemented by [3, Lemma 2.2].

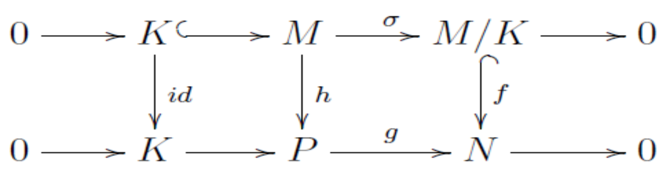
In general, modules with the property (WRE) need not to have the property (E) as the following example shows.

Example 13: (see [7, Example 1]) For a non-complete local dedekind domain R, let M be the direct sum of left R-modules R^*, $K^{(I)}$ and R, where R^* is the completion of R, K is the quotient field of R and I is an index set, respectively. Since injective modules over a dedekind domain are strongly radical supplemented, it follows from [12, Lemma 3.3] that M has the property (WRE). On the other hand, M doesn't have the property (E) by [12, Theorem 3.5].

Next we prove that a factor module of a module with the property (WRE) has the property (WRE), under a special condition.

Proposition 14: Let $K \subseteq M \subseteq L$ be modules with L/K injective. If M has the property (WRE), then M/K also has the property (WRE).

Proof: Let N be any extension of M/K. Since L/K is injective, by [9, Lemma 2.16] we have the following commutative diagram with exact rows:



Since h is monomorphism and M has the property (WRE), $M \cong h(M)$ has a weak *Rad*-supplement V in P, that is, $h(M) + V = P$ and $h(M) \cap V \leq Rad(P)$. Note that $N = g(P) = g(h(M)) + g(V) = (f\sigma)(M) + g(V) = M/K + g(V)$ and by [11, 21.6], $M/K \cap g(V) = f(\sigma(M)) \cap g(V) = g[h(M) \cap V] \leq g(Rad(P)) = Rad(N)$. Hence $g(V)$ is a weak *Rad*-supplement of M/K in N, that is, M/K has the property (WRE).

A ring R is called *left hereditary* if every left ideal of R is projective in ${}_RR$. It is well known that a ring R is left hereditary if and only if every factor module of an injective R-module is injective [11, 39.16].

Corollary 15: If R is a left hereditary ring and M is an R-module with the property (WRE), then every factor module of M has the property (WRE).

We also obtain the following result by Proposition 3 and Proposition 14.

Corollary 16: Let R be a left hereditary ring and M be an R-module. Then $M/_{P(M)}$ has the property (WRE) if and only if M has the property (WRE).

Theorem 17: For a ring R the following statements are equivalent:

(1) R is semilocal,

(2) Every left R-module is weakly *Rad*-supplemented,

(3) Every left R-module has the property (WRE),

(4) Every left R-module has the property $(WREE)$.

Proof: (1) $\Rightarrow$ (2) Let M be an R-module and U be a submodule of M. Since R is semilocal ring, M is a semilocal module by [5, Theorem 3.5]. Then there exists a submodule V of M such that $M = U + V$ and $U \cap V \leq Rad(M)$ by [5, Proposition 2.1]. Hence M is weakly *Rad*-supplemented.

(2) $\Rightarrow$ (3) Let M be a left R-module and N be any extension of M. By hypothesis, N is weakly *Rad*-supplemented, and so M has a weak *Rad*-supplement in N.

(3) $\Rightarrow$ (4) Let M be a left R-module. By hypothesis, every submodule of M has the property (WRE). Then M has the property $(WREE)$ by Proposition 2.

(4) $\Rightarrow$ (1) Let M be a left R-module with small radical and U be a submodule of M. Since U has the property $(WREE)$, there exists a submodule V of M such that $M = U + V$ and $U \cap V \leq Rad(M)$. Then V is a weak supplement of U in M, because $Rad(M) \ll M$. Thus M is weakly supplemented. Hence R is semilocal by [5, Theorem 3.5].

REFERENCES

[1] Büyükaşık, E., and Lomp, C., *Mathematica Scandinavica*, **106,** 25, 2009.

[2] Büyükaşık, E., Mermut, E., and Özdemir, S., *Rendiconti Del Seminario Matematico Della Universita di Padova*, **124 ,** 157, 2010.

[3] Büyükaşık, E., and Türkmen, E., *Ukrainian Mathematical Journal*, **63**(8), 1306, 2012.

[4] Clark, J., Lomp, C., Vanaja, N., and Wisbauer, R., *Lifting modules. Supplements and Projectivity in Module Theory*, Frontiers in Mathematics, Basel. Birkhauser, 2006.

[5] Lomp, C., *Communications in Algebra*, **27**(4), 1921, 1999.

[6] Nişancı, B., Türkmen, E., and Pancar, A., *International Journal of Computational Cognition*, **7**(2), 48, 2009.

[7] Nişancı Türkmen, B., *Miskolc Mathematical Notes*, **16**(1), 543, 2015.

[8] Önal, E., Çalışıcı, H., and Türkmen, E., *Miskolc Mathematical Notes*, **17**(1), 471, 2016.

[9] Özdemir, S., *Journal of the Korean Mathematical Society*, **53**(2), 403, 2016.

[10] Wang, Y., and Ding, N., *Taiwanese Journal of Mathematics*, **10**(6), 1589, 2006.

[11] Wisbauer, R., *Foundations of Modules and Rings*, Gordon and Breach, Philadelphia, 1991.

[12] Zöschinger, H., *Mathematica Scandinavica*, **35**, 267, 1974.