ORIGINAL PAPER

NEGATION OF BETA FUNCTION

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Abstract. *Beta functions are used in many fields such as computational and applied mathematics, statistics, physics and chemistry. In this paper we generalized Beta function to negative integers. We present some new relationships for the negation of Beta function for negative integers. We also give an algorithm for the computation of Beta function in terms of negative numbers.*
Keywords: *Beta function, Gamma function, Negative factorial*

1. INTRODUCTION

Factorial of a number n is the product of positive integers from l to n The domain of this factorial function is natural numbers. Euler generalized this factorial function from natural numbers to the gamma function as [1,2,9]

$$\Gamma(x) = \int_0^x t^{x-1} e^{-t} dt, \mathrm{Re}(x) > 0 \tag{1}$$

over the right half of the complex plane. From this definition it is possible to obtain factorial of negative and fractional numbers [3].

Beta function was first studied by Euler and Legendre. It is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, p > 0, q > 0\,. \tag{2}$$

It is a definite integral and also can be written in trigonometric form as

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta, p > 0, q > 0 \tag{3}$$

Other forms of beta function can be found for example in [3,5]. The most known form of Beta function is

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{4}$$

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where $\Gamma(p)=(p-1)!$ and again $p>0, q>0$. It is one of the special functions which emerges in mathematics, especially applied mathematics and physics. Beside that Beta function is related to binomial coefficients which occurs in atomic physics [6] and chemistry [7]. Beta function also appears in the calculations of Quantum Field Theory [8] and QCD Sum rules as well as in classical mechanics [2]. For example period of a simple pendulum with a mass m suspended by a string of length l is

$$T=4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\theta}{\cos\theta} \tag{5}$$

One can handle this integrand in usual manner but from Eqn. (3) it can be seen that this integral is a Beta function.

In the definition of the Beta function p and q are positive integers. However, it is possible to calculate Beta function for negative integers since negative factorials are defined. In the following section we derive Beta function formulas for negative integers and give an algorithm to compare our results.

2. NEGATION OF BETA FUNCTION

In order to obtain negative Beta function, we used Weierstrass infinite-product definition of gamma function. This definition gives an important identity [5]

$$z!(-z)!=\frac{\pi z}{\sin \pi z} \tag{6}$$

With this definition we obtained two theorems regarding of Beta function in negative integers.

Theorem 2.1. For $p<0, q>0$ and $B(-p,q)=B(p,-q)$, $B(p,q)$ can be written as

$$B(-p,q)=(-1)^q\Gamma(q)\frac{(p-q)!}{p!} \tag{7}$$

Proof 2.1.

$$\begin{aligned}B(-p,q)&=\frac{\Gamma(-p)\Gamma(q)}{\Gamma(-p+q)}\\&=\Gamma(q)\frac{(-p-1)!}{(-p+q-1)!}\\&=\Gamma(q)\frac{(p-q)!}{p!}\frac{\sin\pi(p-q+1)}{\sin\pi(q+1)}\\&=(-1)^q\Gamma(q)\frac{(p-q)!}{p!}\end{aligned}$$

where from second step to third, we used Eqn. 6 and the identity $\frac{\sin(\pi z_1)}{\sin(\pi z_2)}=(-1)^{z_2-z_1}$.

In other cases the following theorem occurs:

Theorem 2.2. For $p>0, q<0$ and $|q|>p$, $B(p,q)$ can be written as

$$B(p,-q)=(-1)^{p}\Gamma(p)\frac{(q-p)!}{q!} \tag{8}$$

Proof 2.2.

$$B(p,-q)=\frac{\Gamma(p)\Gamma(-q)}{\Gamma(p-q)}$$
$$=\Gamma(p)\frac{(-q-1)!}{(-q+p-1)!}$$
$$=\Gamma(p)\frac{(q-p)!}{q!}\frac{\sin\pi(q+p+1)}{\sin\pi(-q+1)}$$
$$=(-1)^{p}\Gamma(p)\frac{(q-p)!}{q!}$$

At this point it will be useful to give some examples regarding our theorems. We will compare our results with the Mathematica programming language.

3. RESULTS AND DISCUSSION

Before giving examples, we first write our algorithm in order to compute Beta function in negative numbers by using Mathematica programming language.

Algorithm:

```
p=Input["Enter p"];
q=Input["Enter q"];
If[p<0,If[And[q>0,Abs[p]>q],betaour=((-1)^q)*(q-
1)!*(Abs[p]-q)!/(Abs[p])!]];
If[q<0,If[And[p>0,Abs[q]>p],betaour=((-1)^p)*(p-
1)!*(Abs[q]-p)!/(Abs[q])!]];
mathbet=Beta[p,q];
If[And[p<0,q<0],betaour="ComplexInfinity"];
If[And[p>0,q>0],betaour=Beta[p,q]];
mathbet=Beta[p,q];
Print["betaour = ",betaour,"    Mathematica= ",mathbet]
```

When you enter p and q, our algorithm computes Beta function as `betaour` and compares in with the result of Mathematica as `mathbet`

Example 1. For $p=-5$ and $q=3$ the algorithms gives

```
p=Input["Enter p"];
q=Input["Enter q"];
If[p<0,If[And[q>0,Abs[p]>q],betaour=((-1)^q)*(q-1)!*(Abs[p]-
q)!/(Abs[p])!]];
If[q<0,If[And[p>0,Abs[q]>p],betaour=((-1)^p)*(p-1)!*(Abs[q]-
p)!/(Abs[q])!]];
mathbet=Beta[p,q];
```

```
If[And[p<0,q<0],betaour="ComplexInfinity"];
If[And[p>0,q>0],betaour=Beta[p,q]];
mathbet=Beta[p,q];
Print["betaour = ",betaour,"    Mathematica= ",mathbet]
betaour =  -(1/30)     Mathematica=  -(1/30)
```

For $p=-5$ and $q=3$ the algorithms gives

```
p=Input["Enter p"];
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If[p<0,If[And[q>0,Abs[p]>q],betaour=((-1)^q)*(q-1)!*(Abs[p]-
q)!/(Abs[p])!]];
If[q<0,If[And[p>0,Abs[q]>p],betaour=((-1)^p)*(p-1)!*(Abs[q]-
p)!/(Abs[q])!]];
mathbet=Beta[p,q];
If[And[p<0,q<0],betaour="ComplexInfinity"];
If[And[p>0,q>0],betaour=Beta[p,q]];
mathbet=Beta[p,q];
Print["betaour = ",betaour,"    Mathematica= ",mathbet]
     betaour =  -(1/30)     Mathematica=  -(1/30)
```

From these two examples and Eqn. (7) and Eqn. (8) one can see that there is a symmetry relation as

$$B(-p,q)=B(p,-q) \tag{9}$$

4. CONCLUSION

In this paper we obtained the negation of Beta function. It is related to gamma function which is also called *Euler's integral of second kind*. Our algorithm can also be applied to other programming languages.

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