ORIGINAL PAPER

CERTAIN INTEGRAL ASSOCIATED WITH THE BESSEL FUNCTION

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Manuscript received: 28.03.2018; Accepted paper: 15.05.2018;
Published online: 30.09.2018.

Abstract. *The objective of this paper is to set up some finite integral formulas involving the generalized Bessel first kind function negative order of m, modified Bessel function and spherical Bessel function. The result is given in terms of generalized Wright's Hypergeometric functions pΨq. These results are procured with the help of finite integral due to Lavoie and Trottier. Some interesting particular cases in terms of corollary are established here.*

Keywords: *Lavoie and Trottier integral formula, Gamma function, General Wright's Hypergeometric Functions. Bessel first kind negative order Function, Modified Bessel function, Spherical Bessel function.*

2010 Mathematics Subject Classification: *26A33, 33B15, 33C10, 33C20.*

1. INTRODUCTION

In Applied sciences, many important functions are defined via improper integrals or series (or finite products). These important functions are generally known as special functions. In recent years, one of the most important functions (Bessel function) is widely used in Engineering and Physics; therefore, they are of interest to Engineers and Physicists as well as Mathematicians. A variety of special functions involved in a large number of integral formulas and their particular cases have been developed by many authors: Brychkov [3], Choi and Agarwal [4], Choi et al. [5], Agarwal et al. [1], Manaria et al. [10], Khan and Kashmin [8], and Nisar et al. [11, 12]. Our aim here to presenting some generalized integral formulas involving the Bessel first kind negative [7] order, modified Bessel function and spherical Bessel function.

Let us consider the second –order differential equation [16]

$$z^2w''(z) + zw'(z) + (z^2 - p^2)w(z) = 0 \tag{1.1}$$

which is called Bessel's equation, where p is an unrestricted real (or complex) number. The function J_p, which is called the Bessel function of the first kind of order p, is defined as a particular solution of (1.1). This function has the form [7]

$$J_{-p}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-p+n)} \left(\frac{z}{2}\right)^{2n-p} \qquad for\ all\ z\epsilon\mathbb{C}. \tag{1.2}$$

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Two interesting and familiar special cases of (1.2) are worth mentioning. To do this, we recall the following known formula [2]

$$J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} sinx \quad (1.3)$$

To do this, we recall the following known formula [2]

$$J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} cosx \quad (1.4)$$

The differential equation [15]

$$z^2 w''(z) + zw'(z) - (z^2 + p^2)w\,(z) = 0 \quad (1.5)$$

which differs from Bessel's equation only in the coefficient of w. is of frequent occurrence in problems of mathematical physics. The particular solutions of (1.3) is called the modified Bessel function of the first kind of order p, and is defined by the formula [15]

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(1+p+n)} \left(\frac{z}{2}\right)^{2n+p} \qquad for\ all\ z\epsilon\mathbb{C}. \quad (1.6)$$

It is worth mentioning that in particular we have

$$\ell_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{\frac{1}{2}}\ I_{-\frac{1}{2}}(x) \quad 1.7)$$

$$\ell_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}}\ I_{\frac{1}{2}}(x) \quad (1.8)$$

The Generalized Wright Hypergeometric function ${}_p\Psi_q(z)$ [14]. For $z \in \mathbb{C}$, $a_i , b_j \in \mathbb{C}$ and $\alpha_i , \beta_j \in \mathbb{R}$ where $(\alpha_i , \beta_j \neq 0; i = 1,2,3, . . , p ; j = 1,2,3, \dots q)$ is defined as bellow:

$${}_p\Psi_q\left[\begin{matrix}(a_i, \alpha_i)_{1,p};\\ \left(b_j, \beta_j\right)_{1,q};\end{matrix} Z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q} \Gamma\left(b_j+\beta_j k\right)} \frac{z^k}{k!} \quad (1.9)$$

Introduced by Wright [13], the above generalized Wright function used to provide several theorems on the asymptotic expansion of pΨq (z) for all values of the argument z, under the condition:

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_j > -1 \quad (1.10)$$

where $(\lambda)_n$ is the pochhammer symbol defined (for$\lambda\epsilon\mathbb{C}$) by [13]

$$(\lambda)_n \coloneqq 1 \qquad (n = 0) \quad (1.11)$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\lambda\epsilon\mathbb{C}\backslash\mathbb{Z}_0^-) \quad (1.12)$$

and $\mathbb{Z}_0^-$ denotes the set of non positive integers. For our present paper, we also need to use the following Lavoie-Trottier integral formula [9]:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.13}$$

Provided $\Re(\alpha)>0\,,\Re(\beta)>0$

2. INTEGRAL FORMULA INVOLVING FIRST KIND BESSEL'S FUNCTION

In this section, we established some generalized integral formulas; the results are expressed in terms of generalized (Wright) Hypergeometric functions (1.9), by using the generalized Bessel-first kind function of the order of m in view of (1.2) with the suitable argument into the integrand of (1.13).

Theorem 2.1. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with $\Re(\alpha)>0\,,\Re(\beta)>0$ $\Re(m)>0$ and x >0, we have,

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}J_{-m}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$=\left(\frac{2}{y}\right)^{m}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\;{}_1\Psi_2\left[{}^{(\beta-m,2)}_{(1-m,1)(\alpha-m+\beta,2)};-\left(\frac{y}{2}\right)^2\right] \tag{2.1}$$

Now using (1.2) to the integrand of (2.1) and then on changing the order of integration and summation, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}J_{-m}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n-m+1)}\left(\frac{y}{2}\right)^{2n-m}\int_0^1 x^{\alpha-1}(1-x)^{2(2n-m+\beta)-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{(2n-m+\beta)-1}dx$$

By considering the condition given in theorem (2.1), since $\Re(\alpha)>0\,,\Re(\beta)>0$ and applying (1.13),

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-m+1)}\left(\frac{y}{2}\right)^{2n-m}\left(\frac{2}{3}\right)^{2\alpha}\frac{(\Gamma\alpha)\Gamma(2n-m+\beta)}{\Gamma(\alpha+2n-m+\beta)}$$

$$=\left(\frac{2}{y}\right)^{m}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-m+1)}\frac{\Gamma(2n-m+\beta)}{\Gamma(\alpha+2n-m+\beta)}\left(\frac{y}{2}\right)^{2n}$$

Which upon using the definition (1.9), we get the desired result (2.1).

Theorem 2.2. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with $\Re(\alpha)>0,\Re(\beta)>0$, $\Re(m)>0$ and x >0 , we have

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{-m}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$
$$=\left(\frac{2}{y}\right)^m\left(\frac{2}{3}\right)^{2(\alpha-m)}\Gamma(\beta)\ {}_1\Psi_2\left[{}^{(\alpha-m,2)}_{(1-m,1)(\alpha-m+\beta,2)};-\left(\frac{2y}{9}\right)^2\right] \quad (2.2)$$

Now applying (1.2) to the integrand of (2.2) and then interchanging the order of integration and summation, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} J_{-m}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n-m+1)}\left(\frac{y}{2}\right)^{2n-m}\int_0^1 x^{(\alpha+2n-m)-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+2n-m)-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx$$

By considering the condition given in theorem (2.2), since $\Re(\alpha)>0\,,\Re(\beta)>0$ and applying (1.11),

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-m+1)}\left(\frac{y}{2}\right)^{2n-m}\left(\frac{2}{3}\right)^{2(\alpha+2n-m)}\frac{(\Gamma\alpha+2n-m)\Gamma(\beta)}{\Gamma(\alpha+2n-m+\beta)}$$
$$=\left(\frac{2}{y}\right)^m\left(\frac{2}{3}\right)^{2(\alpha-m)}\Gamma(\beta)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-m+1)}\frac{\Gamma(2n-m+\alpha)}{\Gamma(\alpha+2n-m+\beta)}\left(\frac{y}{3}\right)^{2n}$$

which upon using the definition (1.9), we get the desired result (2.2).

Theorem 2.3. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with with $\Re(\alpha)>0,\Re(\beta)>0$, $\Re(m)>0$ and x >0, we have,

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ J_{-m}\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$=\left(\frac{2}{y}\right)^m\left(\frac{2}{3}\right)^{2(\alpha-m)}\ {}_2\Psi_2\left[{}^{(\alpha-m,2)\ ,(\beta-m\,,2)}_{(1-m,1)\,,(\alpha+\beta-2m\,,\ 4)}\ ;-\left(\frac{2y}{9}\right)^2\right]$$
$$(2.3)$$

Now applying (1.2) to the integrand of (2.3) and then interchanging the order of integration and summation, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ J_{-m}\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$
$$=\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n-m+1)}\left[\frac{yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)}{2}\right]^{2n-m}dx$$

By considering the condition given in theorem (2.3), since $\Re(\alpha)>0\,,\Re(\beta)>0$ and applying (1.11),

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left(\frac{y}{2}\right)^{2n-m} \left(\frac{2}{3}\right)^{2(\alpha+2n-m)} \frac{(\Gamma\alpha+2n-m)\Gamma(\beta+2n-m)}{\Gamma(\alpha+\beta-2m+4n)}$$

$$= \left(\frac{2}{y}\right)^{m} \left(\frac{2}{3}\right)^{2(\alpha-m)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \frac{(\Gamma\alpha+2n-m)\Gamma(\beta+2n-m)}{\Gamma(\alpha+\beta-2m+4n)} \left(\frac{2y}{9}\right)^{2n}$$

which upon using the definition (1.9), we get the desired result (2.3).

3. SPECIAL CASE

In this section, we shall mention some of the very interesting special cases in the form of the many corollaries. Under the conditions easily obtained from the respective integrals (2.1) to (2.3), the following integrals hold true.

Corollary 3.1. In (2.1) if we put $\alpha = \rho + j$ and $\beta = \rho$, then after some algebra, we get the following results

$$\int_0^1 x^{\rho+j-1} (1-x)^{2\rho-1} \left(1-\frac{x}{3}\right)^{2(\rho+j)-1} \left(1-\frac{x}{4}\right)^{\rho-1} J_{-m}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right] dx$$

$$= \left(\frac{2}{y}\right)^{m} \left(\frac{2}{3}\right)^{2(\rho+j)} \Gamma(\rho+j)\ {}_1\Psi_2\left[{}_{(1-m,1)\,,(2\rho+j-m,2)}^{(\rho-m,2)}; -\left(\frac{y}{2}\right)^2\right] \tag{3.1}$$

Corollary 3.2. In (2.2) if we put $\alpha = \rho$ and $\beta = \rho + j$, then after some algebra, we get the following results

$$\int_0^1 x^{\rho-1} (1-x)^{2(\rho+j)-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{(\rho+j)-1} J_{-m}\left[yx\left(1-\frac{x}{3}\right)^2\right] dx$$

$$= \left(\frac{2}{y}\right)^{m} \left(\frac{2}{3}\right)^{2(\rho-m)} \Gamma(\rho+j)\ {}_1\Psi_2\left[{}_{(1-m,1)(2\rho+j-m,2)}^{(\rho-m,2)}; -\left(\frac{2y}{9}\right)^2\right] \tag{3.2}$$

Corollary 3.3. In (2.1) if we put $m = \frac{1}{2}$, and making the use of the result (1.4), we get the following results:

$$\int_0^1 x^{\alpha-1} (1-x)^{2\beta-1} \left(1-\frac{x}{3}\right)^{2\alpha-1} \left(1-\frac{x}{4}\right)^{\beta-1} \frac{\cos\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]}{\sqrt{y\left(1-\frac{x}{4}\right)(1-x)^2}} dx$$

$$= \sqrt{\pi}\ \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\alpha)\ {}_1\Psi_2\left[{}_{\left(\frac{1}{2},1\right),\left(\alpha+\beta-\frac{1}{2},2\right)}^{\left(\beta-\frac{1}{2},2\right)}; -\left(\frac{y}{2}\right)^2\right] \tag{3.3}$$

Corollary 3.4. In (2.2) if we put $m = \frac{1}{2}$, and making the use of the result (1.4), we get the following results:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\frac{\cos\left[yx\left(1-\frac{x}{3}\right)^2\right]}{\sqrt{yx\left(1-\frac{x}{3}\right)^2}}dx$$

$$=\left(\frac{2}{3}\right)^{2\left(\alpha-\frac{1}{2}\right)}\Gamma(\beta)\ {}_1\Psi_2\left[\begin{matrix}\left(\alpha-\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right)\left(\alpha+\beta-\frac{1}{2}\ ,2\right)\end{matrix};-\left(\frac{2y}{9}\right)^2\right] \tag{3.4}$$

Corollary 3.5. In (2.3) if we put $m=\frac{1}{2}$, and making the use of the result (1.4), we get the following results:

$$\int_0^1 \mathrm{x}^{\alpha-1}(1-\mathrm{x})^{2\beta-1}\left(1-\frac{\mathrm{x}}{3}\right)^{2\alpha-1}\left(1-\frac{\mathrm{x}}{4}\right)^{\beta-1}\frac{\cos\left[\mathrm{yx}(1-\mathrm{x})^2\left(1-\frac{\mathrm{x}}{3}\right)^2\left(1-\frac{\mathrm{x}}{4}\right)\right]}{\sqrt{\mathrm{yx}(1-\mathrm{x})^2\left(1-\frac{\mathrm{x}}{3}\right)^2\left(1-\frac{\mathrm{x}}{4}\right)}}\mathrm{dx}$$

$$=\left(\frac{2}{3}\right)^{2\alpha-\frac{1}{2}}\ {}_2\Psi_2\left[\begin{matrix}\left(\alpha-\frac{1}{2},2\right)\ ,\left(\beta-\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right),(\alpha+\beta-1\ ,\ 4)\end{matrix}\ ;-\left(\frac{2\mathrm{y}}{9}\right)^2\right] \tag{3.5}$$

4. INTEGRAL FORMULA INVOLVING MODIFIED BESSEL'S FUNCTION OF THE FIRST KIND

Theorem 4.1. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with with $\Re(\alpha)>0,\Re(\beta)>0$, $\Re(m)>0$ and x >0, we have:

$$\int_0^1 x^{\alpha-1}\ (1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\quad I_p\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$=\left(\frac{y}{2}\right)^p\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\ \ {}_1\Psi_2\left[\begin{matrix}(p+\beta,2)\\ (p+1,1)\ (\alpha+p+\beta,2)\end{matrix};\left(\frac{y}{2}\right)^2\right] \tag{4.1}$$

Now applying (1.6) to the integrand of (4.1) and then interchanging the order of integration and summation, we get

$$=\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(p+m+1)}\left[\frac{y\left(1-\frac{x}{4}\right)(1-x)^2}{2}\right]^{2m+p}dx$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(p+m+1)}\left(\frac{y}{2}\right)^{2m+p}\int_0^1 x^{\alpha-1}(1-x)^{2(2m+p+\beta)-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{(2m+p+\beta)-1}dx$$

By considering the condition given in theorem (4.1), since $\Re(\alpha)>0\ ,\Re(\beta)>0$ and applying (1.11)

$$=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(p+m+1)}\left(\frac{y}{2}\right)^{2m+p}\left(\frac{2}{3}\right)^{2\alpha}\ \frac{\Gamma(\alpha)\,\Gamma(2m+p+\beta)}{\Gamma(\alpha+2m+p+\beta)}$$

$$=\left(\frac{y}{2}\right)^p\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(p+m+1)}\ \frac{\Gamma(2m+p+\beta)}{\Gamma(\alpha+2m+p+\beta)}\left(\frac{y}{2}\right)^{2m}$$

which upon using the definition (1.15), we get the desired result (4.1).

Theorem 4.2. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with with $\Re(\alpha)>0,\Re(\beta)>0$, $\Re(m)>0$ and x >0 , we have:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \quad I_p\left[yx\left(1-\frac{x}{3}\right)^2\right]dx$$

$$=\left(\frac{y}{2}\right)^p\left(\frac{2}{3}\right)^{2(\alpha+p)}\Gamma(\beta)\ {}_1\Psi_2\left[\begin{matrix}(p+\alpha,2)\\(p+1,1)\ (\alpha+p+\beta,2)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{4.2}$$

Now applying (1.6) to the integrand of (4.2) and then interchanging the order of integration and summation, we get

$$=\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+p+1)}\left[\frac{yx\left(1-\frac{x}{3}\right)^2}{2}\right]^{2m+p}dx$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+p+1)}\left(\frac{y}{2}\right)^{2m+v}\int_0^1 x^{(\alpha+2m+p)-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+2m+p)-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ dx$$

By considering the condition given in theorem (4.2), since $\Re(\alpha)>0\,,\Re(\beta)>0$ and applying (1.15)

$$=\sum_{m=0}^{\infty}\frac{1}{n!\,\Gamma(m+p+1)}\left(\frac{y}{2}\right)^{2m+p}\left(\frac{2}{3}\right)^{2(\alpha+2m+p)}\ \frac{(\Gamma\alpha+2m+p\,)\Gamma(\beta)}{\Gamma(\alpha+2m+p+\beta)}$$

$$=\left(\frac{y}{2}\right)^p\left(\frac{2}{3}\right)^{2(\alpha+p)}\Gamma(\beta)\sum_{n=0}^{\infty}\frac{1}{m!\,\Gamma(m+p+1)}\ \frac{\Gamma(2m+p+\alpha)}{\Gamma(\alpha+2m+p+\beta)}\left(\frac{y}{3}\right)^{2m}$$

which upon using the definition (1.11), we get the desired result (4.2).

Theorem 4.3. The following integral holds true for $\alpha,\beta\epsilon\mathbb{C}$ with with $\Re(\alpha)>0,\Re(\beta)>0$, $\Re(m)>0$ and x >0, we have:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\quad I_p\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\left(\frac{y}{2}\right)^p\left(\frac{2}{3}\right)^{2(\alpha+p)}\ {}_2\Psi_2\left[\begin{matrix}(\alpha+p\,,2)\ \ ,(\beta+p\,,2)\\(1+p\,,1)\,,(\alpha+\beta+2p\,,\ 4)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{4.3}$$

Now applying (1.6) to the integrand of (4.3) and then interchanging the order of integration and summation, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\quad I_p\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]dx$$

$$=\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+p+1)}\left[\frac{yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)}{2}\right]^{2n+p}dx$$

By considering the condition given in theorem (4.3), since $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and applying (1.15)

$$=\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+p+1)}\left(\frac{y}{2}\right)^{2n+p}\left(\frac{2}{3}\right)^{2(\alpha+2n+p)}\frac{(\Gamma\alpha+2n+p\,)\Gamma(\beta+2n+p)}{\Gamma(\alpha+\beta+2p+4n)}$$

$$=\left(\frac{y}{2}\right)^{p}\left(\frac{2}{3}\right)^{2(\alpha+p)}\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+p+1)}\frac{(\Gamma\alpha+2n+p\,)\Gamma(\beta+2n+p)}{\Gamma(\alpha+\beta+2p+4n)}\left(\frac{2y}{9}\right)^{2n}$$

which upon using the definition (1.11), we get the desired result (4.3).

5. SPECIAL CASE

In this section, we shall mention some of the very interesting special cases in the form of the many corollaries. Under the conditions easily obtained from the respective integrals (4.1) to (4.3), the following integrals hold true.

Corollary 5.1. In (4.1) if we put $p=-\frac{1}{2}$, and then making the use of the result in (1.7), we get the following results.

$$\int_0^1 x^{\alpha-1}\ (1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\frac{\ell_{-\frac{1}{2}}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]}{\sqrt{y\left(1-\frac{x}{4}\right)(1-x)^2}}dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\sqrt{\frac{\pi}{y}}\ {}_1\Psi_2\left[\begin{matrix}\left(\beta-\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right)\left(\alpha+\beta-\frac{1}{2},2\right)\end{matrix};\left(\frac{y}{2}\right)^2\right] \tag{5.1}$$

Corollary 5.2. In (4.2) if we put $p=-\frac{1}{2}$, and then making the use of the result in (1.7), we get the following results.

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\frac{\ell_{\frac{-1}{2}}\left[yx\left(1-\frac{x}{3}\right)^2\right]dx}{\sqrt{yx\left(1-\frac{x}{3}\right)^2}}$$

$$=\left(\frac{2}{3}\right)^{2\alpha-1}\sqrt{\frac{\pi}{y}}\ \Gamma(\beta)\ {}_1\Psi_2\left[\begin{matrix}\left(\alpha-\frac{1}{2},\ 2\right)\\ \left(\frac{1}{2},1\right)\left(\alpha+\beta-\frac{1}{2},2\right)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{5.2}$$

Corollary 5.3. In (4.1) if we put $p=\frac{1}{2}$, and then making the use of the result in (1.8), we get the following results.

$$\int_0^1 x^{\alpha-1}\ (1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \frac{\ell_{\frac{1}{2}}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]}{\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^{-\frac{1}{2}}}dx$$

$$=\frac{(\pi y)^{\frac{1}{2}}}{2}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\ {}_1\Psi_2\left[\begin{matrix}\left(\beta+\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right)\left(\alpha+\frac{1}{2}+\beta,2\right)\end{matrix};\left(\frac{y}{2}\right)^2\right] \tag{5.3}$$

Corollary 5.4. In (4.2) if we put $p=\frac{1}{2}$, and then making the use of the result in (1.8), we get the following results.

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \frac{\ell_{\frac{1}{2},}\left[yx\left(1-\frac{x}{3}\right)^2\right]}{\left(yx\left(1-\frac{x}{3}\right)^2\right)^{-\frac{1}{2}}}dx$$

$$=\frac{(\pi y)^{\frac{1}{2}}}{2}\left(\frac{2}{3}\right)^{(2\alpha+1)}\Gamma(\beta)\ {}_1\Psi_2\left[\begin{matrix}\left(\alpha+\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right)\left(\alpha+\frac{1}{2}+\beta,2\right)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{5.4}$$

Corollary 5.5. In (4.3) if we put $p=-\frac{1}{2}$, and then making the use of the result (1.7), we get the following results:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \frac{\ell_{\frac{-1}{2}}\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]}{\sqrt{yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)}}dx$$

$$=\sqrt{\frac{\pi}{y}}\left(\frac{2}{3}\right)^{(2\alpha-1)}\ {}_2\Psi_2\left[\begin{matrix}\left(\alpha-\frac{1}{2},2\right),\left(\beta-\frac{1}{2},2\right)\\ \left(\frac{1}{2},1\right),(\alpha+\beta-1,\ 4)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{5.5}$$

Corollary 5.6. In (4.3) if we put $p=\frac{1}{2}$, and then making the use of the result in (1.8), we get the following results.

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}\ \frac{\ell_{\frac{1}{2}}\left[yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right]}{\left(yx(1-x)^2\left(1-\frac{x}{3}\right)^2\left(1-\frac{x}{4}\right)\right)^{-\frac{1}{2}}}dx$$

$$=\frac{(\pi y)^{\frac{1}{2}}}{2}\left(\frac{2}{3}\right)^{(2\alpha+1)}\ {}_2\Psi_2\left[\begin{matrix}\left(\alpha+\frac{1}{2},2\right),\left(\beta+\frac{1}{2},2\right)\\ \left(\frac{3}{2},1\right),(\alpha+\beta+1,\ 4)\end{matrix};\left(\frac{2y}{9}\right)^2\right] \tag{5.6}$$

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