ORIGINAL PAPER

BOUNDS FOR SIGNLESS LAPLACIAN SPECTRAL RADIUS

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Abstract. *We consider weighted graphs, such as graphs where the edge weights are positive definite matrices of the same order in this paper. The signless Laplacian eigenvalues of a graph are the eigenvalues of the signless Laplacian matrix of a graph G. Then, we have give some bound and found different upper bounds for the signless Laplacian radius of weighted graphs. We have obtained some results by characterizing these upper bounds.*
Keywords: *Weighted graph, signless Laplacian eigenvalue, upper bound*

1. INTRODUCTION

A weighted graph is a graph each edge of which has been assigned to a square matrix called the weight of the edge. All the weight matrices are assumed to be of the same order and to be positive definite.

Let G be a weighted graph on n vertices. Denote by w_{ij} the positive definite weight matrix of order p of the edge ij, and assume that $w_{ij} = w_{ji}$. We write $i \sim j$ if vertices i and j are adjacent.

The signless Laplacian matrix of a graph G is defined as $Q(G) = \left(q_{ij}\right)$, where

$$q_{ij} = \begin{cases} w_i & ; \quad i = j \\ w_{ij} & ; \quad i \sim j \\ 0 & ; \quad otherwise. \end{cases}$$

In the definition above, the zero denotes the $p \times p$ zero matrix. Hence $Q(G)$ is square and symmetric matrix of order np. The eigenvalues of G are the eigenvalues of $Q(G)$. Let $q_1, q_2, \dots, q_{np}$ be the eigenvalues of G, and let q_1 denote the largest eigenvalue of $Q(G)$.

In this study firstly we give an upper bound for the largest eigenvalue of a weighted graph G.

2. PRELIMINARIES

In this section, we shall give some previously known results that will be needed in the next sections.

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Theorem 1. (Rayleigh − Ritz [2]) Let $A \in M_n$ be Hermitian and let the eigenvalues of A be ordered such that $\lambda_n \leq \cdots \leq \lambda_1$. Then for all $x \in \mathbb{C}^n$

$$\lambda_n x^T x \leq x^T A x \leq \lambda_1 x^T x$$
$$\lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x^T x = 1} x^T A x$$
$$\lambda_n = \min_{x \neq 0} \frac{x^T A x}{x^T x} = \min_{x^T x = 1} x^T A x$$

Theorem 2. (Horn − Johnson [2]) Let $A \in M_n$ be Hermitian matrix with eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$. Then for all $x, y \in \mathbb{R}^n$

$$|x^T A y| = \lambda_1 \sqrt{x^T x}\,\sqrt{y^T y}. \qquad (2.1)$$

3. BOUNDS FOR SIGNLESS LAPLACIAN SPECTRAL RADIUS

In this section we give some bounds for the signless Laplacian radius for weighted graphs. We adopt the proof of the following theorem from Theorem 2.4 in [1].

Theorem 3.1 Let G be a simple connected weighted graph. Then

$$q_1 \leq \max_{i \sim j} \sqrt[4]{\begin{array}{c}\left(q_1(w_i^2) + \left(\sum_{k \in N_i} q_1(w_{ik}^2)\right) + q_1\left(w_i w_{ij} + w_{ji} w_j\right)\right) \\ \left(q_1\left(w_j^2\right) + \left(\sum_{s \in N_j} q_1\left(w_{js}^2\right)\right) + q_1\left(w_j w_{js} + w_{sj} w_s\right) + \sum_{s \in N_j \cap N_k} q_1\left(w_{jk} w_{sk}\right)\right)\end{array}}$$

Proof: Let $\bar{X} = (\bar{x}_1^T, \bar{x}_2^T, \ldots, \bar{x}_n^T)^T$ be an eigenvector corresponding to the largest eigenvalue q_1 of $Q(G)$. Then q_1^2 is the largest eigenvalue of $Q^2(G)$. We assume that $\bar{x}_i$ is the vector component of $\bar{X}$ such that $\bar{x}_i^T \bar{x}_i = \max_{k \in V}\{\bar{x}_k^T \bar{x}_k\}$. Since $\bar{X}$ is nonzero, so is $\bar{x}_i$. Let $\bar{x}_j^T \bar{x}_j = \sum_{k \in N_i}\{\bar{x}_k^T \bar{x}_k\}$, so that $\bar{x}_j^T \bar{x}_j \geq \bar{x}_k^T \bar{x}_k$ for all $k \in N_i$.

The (i,j) th block of $Q(G)$ is

$$q_{ij} = \begin{cases} w_i & ; \quad i = j \\ w_{ij} & ; \quad i \sim j \\ 0 & ; \quad otherwise. \end{cases}$$

Now, we consider the matrix $Q^2(G)$. The (i,j) th element of $Q^2(G)$ is

$$\begin{cases} w_i^2 + \sum_{k \in N_i} w_{ik}^2 & ; \quad i = j \\ w_i w_{ij} + w_{ji} w_j + \sum_{k \in N_i \cap N_j} w_{ik} w_{kj} & ; \quad otherwise. \end{cases}$$

We have

$$Q^2(G)\bar{X} = q_1^2 \bar{X}.$$

From the $i - th$ equation in the last equality we have

$$q_1^2 \bar{x}_i = \left(w_i^2 + \sum_{k \in N_i} w_{ik}^2 \right) \bar{x}_i + \left(w_i w_{ij} + w_{ji} w_j + \sum_{k \in N_i \cap N_j} w_{ik} w_{kj} \right) \bar{x}_j$$

i.e.

$$\bar{x}_i^T q_1^2 \bar{x}_i = \bar{x}_i^T \left(w_i^2 + \sum_{k \in N_i} w_{ik}^2 \right) \bar{x}_i + \bar{x}_i^T \left(w_i w_{ij} + w_{ji} w_j + \sum_{k \in N_i \cap N_j} w_{ik} w_{kj} \right) \bar{x}_j$$

Taking the modulus on both sides in the last equality we get

$$|q_1^2| \bar{x}_i^T \bar{x}_i \leq \left| \bar{x}_i^T \left(w_i^2 + \sum_{k \in N_i} w_{ik}^2 \right) \bar{x}_i \right| + \left| \bar{x}_i^T \left(w_i w_{ij} + w_{ji} w_j + \sum_{k \in N_i \cap N_j} w_{ik} w_{kj} \right) \bar{x}_j \right|$$

Since w_{ij} is the positive definite matrix for all i, j, w_{ij}^2 matrices are also positive definite matrix. So we have and using Cauchy-Schwarz inequality we get

$$\begin{aligned} |q_1^2| \bar{x}_i^T \bar{x}_i \leq & \left| q_1(w_i^2) \bar{x}_i^T \bar{x}_i + \sum_{k \in N_i} q_1(w_{ik}^2) \bar{x}_i^T \bar{x}_i \right| \\ & + \left| q_1\left(w_i w_{ij} + w_{ji} w_j\right) \bar{x}_i^T \bar{x}_j + \sum_{k \in N_i \cap N_j} q_1\left(w_{ik} w_{kj}\right) \bar{x}_i^T \bar{x}_j \right| \end{aligned}$$

from (2.1),

$$\begin{aligned} |q_1^2| \bar{x}_i^T \bar{x}_i \leq & \; q_1(w_i^2) \bar{x}_i^T \bar{x}_i + \sum_{k \in N_i} q_1(w_{ik}^2) \bar{x}_i^T \bar{x}_i + q_1\left(w_i w_{ij} + w_{ji} w_j\right) \sqrt{\bar{x}_i^T \bar{x}_i} \sqrt{\bar{x}_j^T \bar{x}_j} \\ & + \sum_{k \in N_i \cap N_j} q_1\left(w_{ik} w_{kj}\right) \sqrt{\bar{x}_i^T \bar{x}_i} \sqrt{\bar{x}_j^T \bar{x}_j} \end{aligned}$$

thus we obtain

$$q_1^2 \leq q_1(w_i^2) + \left(\textstyle\sum_{k \in N_i} q_1(w_{ik}^2) \right) + q_1\left(w_i w_{ij} + w_{ji} w_j\right) + \textstyle\sum_{k \in N_i \cap N_j} q_1\left(w_{ik} w_{kj}\right) \qquad (1)$$

From the $j-th$ equation in the last equality we have

$$q_1^2\bar{x}_j = \left(w_j^2 + \sum_{s\in N_j} w_{js}^2\right)\bar{x}_j + \left(w_j w_{js} + w_{sj} w_s + \sum_{s\in N_j\cap N_k} w_{jk} w_{sk}\right)\bar{x}_k$$

i.e.

$$\bar{x}_j^T q_1^2 \bar{x}_j = \bar{x}_j^T \left(w_j^2 + \sum_{s\in N_j} w_{js}^2\right)\bar{x}_j + \bar{x}_j^T\left(w_j w_{js} + w_{sj} w_s + \sum_{s\in N_j\cap N_k} w_{jk} w_{sk}\right)\bar{x}_k$$

Taking the modulus on both sides in the last equality we get

$$|q_1^2|\bar{x}_j^T\bar{x}_j \leq \left|\bar{x}_j^T \left(w_j^2 + \sum_{s\in N_j} w_{js}^2\right)\bar{x}_j\right| + \left|\bar{x}_j^T\left(w_j w_{js} + w_{sj} w_s + \sum_{s\in N_j\cap N_k} w_{jk} w_{sk}\right)\bar{x}_k\right|$$

Since w_{ij} is the positive definition matrix for all i, j, w_{ij}^2 matrices are also positive definite matrix. So we have and using Cauchy-Schwarz inequality we get

$$\begin{aligned}|q_1^2|\bar{x}_j^T\bar{x}_j \leq &\left|q_1(w_j^2)\bar{x}_j^T\bar{x}_j + \sum_{s\in N_j} q_1\,(w_{js}^2)\bar{x}_j^T\bar{x}_j\right| \\ &+\left|q_1(w_j w_{js} + w_{sj} w_s)\bar{x}_j^T\bar{x}_k + \sum_{s\in N_j\cap N_k} q_1(w_{jk} w_{sk})\bar{x}_j^T\bar{x}_k\right|\end{aligned}$$

from (2.1),

$$\begin{aligned}|q_1^2|\bar{x}_j^T\bar{x}_j \leq\ & q_1(w_j^2)\bar{x}_j^T\bar{x}_j + \sum_{s\in N_j} q_1\,(w_{js}^2)\bar{x}_j^T\bar{x}_j + q_1(w_j w_{js} + w_{sj} w_s)\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_k^T\bar{x}_k} \\ &+ \sum_{s\in N_j\cap N_k} q_1\, w_{jk} w_{sk}\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_k^T\bar{x}_k}\end{aligned}$$

thus we obtain

$$q_1^2 \leq q_1(w_j^2) + \left(\sum_{s\in N_j} q_1\,(w_{js}^2)\right) + q_1(w_i w_{ij} + w_{ji} w_j) + \sum_{s\in N_j\cap N_k} q_1\,(w_{jk} w_{sk}) \quad (2)$$

From (1) and (2) we get

$$q_1 \leq \max_{i\sim j} \sqrt[4]{\begin{gathered}\left(q_1(w_i^2) + \left(\sum_{k\in N_i} q_1(w_{ik}^2)\right) + q_1(w_i w_{ij} + w_{ji} w_j)\right) \\ \left(q_1(w_j^2) + \left(\sum_{s\in N_j} q_1(w_{js}^2)\right) + q_1(w_j w_{js} + w_{sj} w_s) + \sum_{s\in N_j\cap N_k} q_1(w_{jk} w_{sk})\right)\end{gathered}}$$

Theorem 3.2 Let G be a simple connected weighted graph. Then,

$$
q_1 \leq \max_{i\sim j} \sqrt[6]{
\begin{pmatrix}
q_1(w_i^2 w_i) + \sum\limits_{k\in N_i} q_1(w_{ik}^2 w_i) + \sum\limits_{k\in N_i} q_1\left[\left(\sum\limits_{k,t\in N_i} w_i\, w_{ik} + w_{ki}w_k + w_{it}w_{tk}\right) w_{ik}\right] \\
+q_1\left[\left(w_i w_{ij} + w_{ji}w_j\right)w_j\right] + \sum\limits_{k\in N_i\cap N_j} q_1\left(w_{ik}w_{kj}w_j\right) + q_1\left(w_i^2 w_{ij}\right) + \sum\limits_{k\in N_i} q_1\left(w_{ik}^2 w_{ij}\right) \\
+\sum\limits_{s\in N_i\cap N_j} q_1\left[\left(\sum\limits_{k\in N_i\cap N_j} w_i\, w_{ik} + w_{ki}w_k\right) + \left(\sum\limits_{t\in N_i\cap N_j} w_{ij}w_{jt}\right) w_{sj}\right] \\
+q_1\left(\sum\limits_{k\in N_i\cap N_j} \left(w_i w_{ik} + w_{ki}w_k\right) w_{kj}\right) + q_1\left[\left(\sum\limits_{k\in N_i\cap N_j} w_{ik}w_{kj}\right) w_j\right] \\
+q_1\left(\sum\limits_{k\in N_i, t\in N_j} w_{ik}w_{kt}w_{tj}\right) + q_1\left(\sum\limits_{\substack{k\in N_i, t\in N_j,\\ k\neq j, t\neq i}} w_{ik}w_{kt}\, w_{tj}\right)
\end{pmatrix}
\begin{pmatrix}
q_1\left(w_j^2 w_j\right) + \sum\limits_{k\in N_j} q_1\left(w_{jk}^2 w_j\right) + q_1\left[\sum\limits_{k\in N_j}\left(\sum\limits_{k,t\in N_j} w_j\, w_{jk} + w_{kj}w_k + w_{jt}w_{tk}\right) w_{jk}\right] \\
+q_1\left[\left(w_j w_{jk} + w_{kj}w_k + \sum\limits_{t\in N_j\cap N_k} w_{jt}\, w_{tk}\right) w_k\right] + q_1\left[\left(w_j^2 + \sum\limits_{t\in N_j} w_{jt}^2\right) w_{jk}\right] \\
+\sum\limits_{s\in N_j\cap N_k} q_1\left[\left(\sum\limits_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj}w_t\right) + \left(\sum\limits_{t\in N_j\cap N_k} w_{jk}\, w_{kt}\right) w_{sk}\right] \\
+q_1\left(\sum\limits_{t\in N_j\cap N_k} \left(w_j w_{jt} + w_{tj}w_t\right) w_{tk}\right) + q_1\left[\left(\sum\limits_{t\in N_j\cap N_k} w_{jt}w_{tk}\right) w_k\right] \\
+q_1\left(\sum\limits_{t\in N_j, s\in N_k} w_{jt}w_{ts}w_{sk}\right) + q_1\left(\sum\limits_{\substack{t\in N_j, s\in N_k,\\ k\neq t, s\neq j}} w_{jt}w_{ts}\, w_{sk}\right)
\end{pmatrix}
}
$$

Proof: Let $\bar{X} = (\bar{x}_1^T, \bar{x}_2^T, \dots, \bar{x}_n^T)^T$ be an eigenvector corresponding to the largest eigenvalue q_1 of $Q(G)$. Then q_1^3 is the largest eigenvalue of $Q^3(G)$. We assume that $\bar{x}_i$ is the vector component of $\bar{X}$ such that $\bar{x}_i^T \bar{x}_i = \max_{k\in V}\{\bar{x}_k^T \bar{x}_k\}$. Since $\bar{X}$ is nonzero, so is $\bar{x}_i$. Let $\bar{x}_j^T \bar{x}_j = \sum_{k\in N_i}\{\bar{x}_k^T \bar{x}_k\}$, so that $\bar{x}_j^T \bar{x}_j \geq \bar{x}_k^T \bar{x}_k$ for all $k \in N_i$.

The (i,j) th block of $Q(G)$ is

$$q_{ij} = \begin{cases} w_i & ; \quad i = j \\ w_{ij} & ; \quad i \sim j \\ 0 & ; \quad otherwise. \end{cases}$$

Now, we consider the matrix $Q^3(G)$. The (i,j) th element of $Q^3(G)$ is

$$\begin{cases} \left(w_i^2 + \sum_{k\in N_i} w_{ik}^2\right) w_i + \sum_{k\in N_i}\left(\sum_{k,t\in N_i} w_i w_{ik} + w_{ki}\, w_k + w_{it} w_{tk}\right) w_{ik} & ; \quad i = j \\ \left(w_i w_{ij} + w_{ji} w_j + \sum_{k\in N_i\cap N_j} w_{ik} w_{kj}\right) w_j + \left(w_i^2 + \sum_{k\in N_i} w_{ik}^2\right) w_{ij} + & \\ \sum_{s\in N_i\cap N_j}\left[\left(\sum_{k\in N_i\cap N_j} w_i\, w_{ik} + w_{ki} w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij}\, w_{jt}\right) w_{sj} - \sum_{\substack{k\in N_j (N_i\cap N_j) \\ i\neq k}} w_{ij}\, w_{jk}^2\right] & ; \quad i \sim j \\ \sum_{k\in N_i\cap N_j} (w_i w_{ik} + w_{ki} w_k) w_{kj} + \left(\sum_{k\in N_i\cap N_j} w_{ik} w_{kj}\right) w_j + & \\ \sum_{k\in N_i, t\in N_j} w_{ik} w_{kt} w_{tj} + \sum_{\substack{k\in N_i, t\in N_j \\ k\neq j, t\neq i}} w_{ik} w_{kt} w_{tj} & ; \quad otherwise. \end{cases}$$

We have

$$Q^3(G)\bar{X} = q_1^3 \bar{X}.$$

From the $i - th$ equation in the last equality we have

$$\begin{aligned} q_1^3 \bar{x}_i = {} & \left(w_i^2 + \sum_{k\in N_i} w_{ik}^2\right) w_i \bar{x}_i + \sum_{k\in N_i}\left(\sum_{k,t\in N_i} w_i w_{ik} + w_{ki}\, w_k + w_{it} w_{tk}\right) w_{ik} \bar{x}_i \\ & + \left(w_i w_{ij} + w_{ji} w_j + \sum_{k\in N_i\cap N_j} w_{ik} w_{kj}\right) w_j \bar{x}_j + \left(w_i^2 + \sum_{k\in N_i} w_{ik}^2\right) w_{ij} \bar{x}_j \\ & + \sum_{s\in N_i\cap N_j}\left[\left(\sum_{k\in N_i\cap N_j} w_i w_{ik} + w_{ki}\, w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij} w_{jt}\right) w_{sj} - \sum_{\substack{k\in N_j\setminus(N_i\cap N_j) \\ i\neq k}} w_{ij} w_{jk}^2\right] \bar{x}_j \\ & + \left(\sum_{k\in N_i\cap N_j} (w_i w_{ik} + w_{ki} w_k) w_{kj}\right) \bar{x}_k + \left(\sum_{k\in N_i\cap N_j} w_{ik}\, w_{kj}\right) w_j \bar{x}_k \end{aligned}$$

$$+\left(\sum_{k\in N_i,t\in N_j} w_{ik}\,w_{kt}w_{tj}\right)\bar{x}_k+\left(\sum_{\substack{k\in N_i,t\in N_j\\ k\neq j,t\neq i}} w_{ik}w_{kt}w_{tj}\right)\bar{x}_k$$

i.e.

$$\bar{x}_i{}^T q_1^3\bar{x}_i=\bar{x}_i{}^T\left(w_i^2+\sum_{k\in N_i} w_{ik}^2\right)w_i\bar{x}_i+\bar{x}_i{}^T\sum_{k\in N_i}\left(\sum_{k,t\in N_i} w_iw_{ik}+w_{ki}\,w_k+w_{it}w_{tk}\right)w_{ik}\bar{x}_i$$

$$+\bar{x}_i{}^T\left(w_iw_{ij}+w_{ji}w_j+\sum_{k\in N_i\cap N_j} w_{ik}w_{kj}\right)w_j\bar{x}_j+\bar{x}_i{}^T\left(w_i^2+\sum_{k\in N_i} w_{ik}^2\right)w_{ij}\bar{x}_j$$

$$+\bar{x}_i{}^T\sum_{s\in N_i\cap N_j}\left[\left(\sum_{k\in N_i\cap N_j} w_iw_{ik}+w_{ki}\,w_k\right)+\left(\sum_{t\in N_i\cap N_j} w_{ij}w_{jt}\right)w_{sj}-\sum_{\substack{k\in N_j\setminus(N_i\cap N_j)\\ i\neq k}} w_{ij}w_{jk}^2\right]\bar{x}_j$$

$$+\bar{x}_i{}^T\left(\sum_{k\in N_i\cap N_j}(w_iw_{ik}+w_{ki}w_k)w_{kj}\right)\bar{x}_k+\bar{x}_i{}^T\left(\sum_{k\in N_i\cap N_j} w_{ik}\,w_{kj}\right)w_j\bar{x}_k$$

$$+\bar{x}_i{}^T\left(\sum_{k\in N_i,t\in N_j} w_{ik}\,w_{kt}w_{tj}\right)\bar{x}_k+\bar{x}_i{}^T\left(\sum_{\substack{k\in N_i,t\in N_j\\ k\neq j,t\neq i}} w_{ik}w_{kt}w_{tj}\right)\bar{x}_k$$

Taking the modulus on both sides in the last equality we get

$$|q_1^3|\bar{x}_i{}^T\bar{x}_i\leq\left|\bar{x}_i{}^T w_i^2w_i\bar{x}_i+\sum_{k\in N_i}\bar{x}_i{}^T w_{ik}^2w_i\bar{x}_i\right|$$

$$+\left|\sum_{k\in N_i}\bar{x}_i{}^T\left(\sum_{k,t\in N_i} w_iw_{ik}+w_{ki}\,w_k+w_{it}w_{tk}\right)w_{ik}\bar{x}_i\right|$$

$$+\left|\bar{x}_i{}^T(w_iw_{ij}+w_{ji}w_j)w_j\bar{x}_j+\sum_{k\in N_i\cap N_j}\bar{x}_i{}^T w_{ik}w_{kj}\,w_j\bar{x}_j\right|+\left|\bar{x}_i{}^T w_i^2w_{ij}\bar{x}_j+\sum_{k\in N_i}\bar{x}_i{}^T w_{ik}^2\,w_{ij}\bar{x}_j\right|$$

$$+\left|\sum_{s\in N_i\cap N_j}\bar{x}_i{}^T\left[\left(\sum_{k\in N_i\cap N_j} w_iw_{ik}+w_{ki}\,w_k\right)+\left(\sum_{t\in N_i\cap N_j} w_{ij}w_{jt}\right)w_{sj}-\sum_{\substack{k\in N_j\setminus(N_i\cap N_j)\\ i\neq k}} w_{ij}w_{jk}^2\right]\bar{x}_j\right|$$

$$+\left|\bar{x}_i{}^T\left(\sum_{k\in N_i\cap N_j}(w_iw_{ik}+w_{ki}w_k)w_{kj}\right)\bar{x}_k\right|+\left|\bar{x}_i{}^T\left(\sum_{k\in N_i\cap N_j} w_{ik}\,w_{kj}\right)w_j\bar{x}_k\right|$$

$$+\left|\bar{x}_i{}^T\left(\sum_{k\in N_i,t\in N_j} w_{ik}\,w_{kt}w_{tj}\right)\bar{x}_k\right| + \left|\bar{x}_i{}^T\left(\sum_{\substack{k\in N_i,t\in N_j\\ k\neq j,t\neq i}} w_{ik}w_{kt}w_{tj}\right)\bar{x}_k\right|$$

Since w_{ij} is the positive definite matrix for all i,j,w_{ij}^3 matrices are also positive definite matrix. So we have and using Cauchy-Schwarz inequality we get

$$\begin{aligned}|q_1^3|\bar{x}_i{}^T\bar{x}_i \le &\left|q_1(w_i^2w_i)\bar{x}_i{}^T\bar{x}_i + \sum_{k\in N_i} q_1(w_{ik}^2w_i)\bar{x}_i{}^T\bar{x}_i\right| \\ &+\left|\sum_{k\in N_i} q_1\left[\left(\sum_{k,t\in N_i} w_iw_{ik} + w_{ki}\,w_k + w_{it}w_{tk}\right)w_{ik}\right]\bar{x}_i{}^T\bar{x}_i\right| \\ &+\left|q_1\left[\left(w_iw_{ij}+w_{ji}w_j\right)w_j\right]\bar{x}_i{}^T\bar{x}_j + \sum_{k\in N_i\cap N_j} q_1\left(w_{ik}w_{kj}w_j\right)\bar{x}_i{}^T\bar{x}_j\right| \\ &+\left|q_1\left(w_i^2w_{ij}\right)\bar{x}_i{}^T\bar{x}_j + \sum_{k\in N_i} q_1\left(w_{ik}^2w_{ij}\right)\bar{x}_i{}^T\bar{x}_j\right| \\ &+\left|\sum_{s\in N_i\cap N_j} q_1\left[\left(\sum_{k\in N_i\cap N_j} w_iw_{ik} + w_{ki}\,w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij}w_{jt}\right)w_{sj}\right]\bar{x}_i{}^T\bar{x}_j\right| \\ &+\left|q_1\left(\sum_{k\in N_i\cap N_j}(w_iw_{ik}+w_{ki}w_k)w_{kj}\right)\bar{x}_i{}^T\bar{x}_k\right| + \left|q_1\left[\left(\sum_{k\in N_i\cap N_j} w_{ik}\,w_{kj}\right)w_j\right]\bar{x}_i{}^T\bar{x}_k\right| \\ &+\left|q_1\left(\sum_{k\in N_i,t\in N_j} w_{ik}\,w_{kt}w_{tj}\right)\bar{x}_i{}^T\bar{x}_k\right| + \left|q_1\left(\sum_{\substack{k\in N_i,t\in N_j\\ k\neq j,t\neq i}} w_{ik}w_{kt}w_{tj}\right)\bar{x}_i{}^T\bar{x}_k\right|\end{aligned}$$

from (2.1),

$$\begin{aligned}|q_1^3|\bar{x}_i{}^T\bar{x}_i \le\ & q_1(w_i^2w_i)\bar{x}_i{}^T\bar{x}_i + \sum_{k\in N_i} q_1(w_{ik}^2w_i)\bar{x}_i{}^T\bar{x}_i \\ &+\sum_{k\in N_i} q_1\left[\left(\sum_{k,t\in N_i} w_iw_{ik} + w_{ki}\,w_k + w_{it}w_{tk}\right)w_{ik}\right]\bar{x}_i{}^T\bar{x}_i \\ &+q_1\left[\left(w_iw_{ij}+w_{ji}w_j\right)w_j\right]\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_j{}^T\bar{x}_j} + \sum_{k\in N_i\cap N_j} q_1\left(w_{ik}w_{kj}w_j\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_j{}^T\bar{x}_j} \\ &+q_1\left(w_i^2w_{ij}\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_j{}^T\bar{x}_j} + \sum_{k\in N_i} q_1\left(w_{ik}^2w_{ij}\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_j{}^T\bar{x}_j}\end{aligned}$$

$$+\sum_{s\in N_i\cap N_j} q_1\left[\left(\sum_{k\in N_i\cap N_j} w_i w_{ik} + w_{ki}\,w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij} w_{jt}\right) w_{sj}\right]\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_j{}^T\bar{x}_j}$$

$$+q_1\left(\sum_{k\in N_i\cap N_j}(w_i w_{ik} + w_{ki} w_k) w_{kj}\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_k{}^T\bar{x}_k}$$

$$+\,q_1\left[\left(\sum_{k\in N_i\cap N_j} w_{ik}\,w_{kj}\right) w_j\right]\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_k{}^T\bar{x}_k}$$

$$+q_1\left(\sum_{k\in N_i, t\in N_j} w_{ik}\,w_{kt} w_{tj}\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_k{}^T\bar{x}_k} + q_1\left(\sum_{\substack{k\in N_i, t\in N_j\\ k\neq j, t\neq i}} w_{ik} w_{kt} w_{tj}\right)\sqrt{\bar{x}_i{}^T\bar{x}_i}\sqrt{\bar{x}_k{}^T\bar{x}_k}$$

thus we obtain

$$q_1^3 \le q_1(w_i^2 w_i) + \sum_{k\in N_i} q_1(w_{ik}^2 w_i) + \sum_{k\in N_i} q_1\left[\left(\sum_{k,t\in N_i} w_i w_{ik} + w_{ki}\,w_k + w_{it} w_{tk}\right) w_{ik}\right]$$
$$+q_1\left[\left(w_i w_{ij} + w_{ji} w_j\right) w_j\right] + \sum_{k\in N_i\cap N_j} q_1\left(w_{ik} w_{kj} w_j\right) + q_1\left(w_i^2 w_{ij}\right) + \sum_{k\in N_i} q_1\left(w_{ik}^2 w_{ij}\right)$$
$$+\sum_{s\in N_i\cap N_j} q_1\left[\left(\sum_{k\in N_i\cap N_j} w_i w_{ik} + w_{ki}\,w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij} w_{jt}\right) w_{sj}\right]$$
$$+q_1\left(\sum_{k\in N_i\cap N_j}(w_i w_{ik} + w_{ki} w_k) w_{kj}\right) + q_1\left[\left(\sum_{k\in N_i\cap N_j} w_{ik}\,w_{kj}\right) w_j\right]$$
$$+q_1\left(\textstyle\sum_{k\in N_i, t\in N_j} w_{ik}\,w_{kt} w_{tj}\right) + q_1\left(\textstyle\sum_{\substack{k\in N_i, t\in N_j\\ k\neq j, t\neq i}} w_{ik} w_{kt} w_{tj}\right) \tag{3}$$

From the j-th equation in the last equality we have

$$q_1^3\bar{x}_j = \left(w_j^2 + \sum_{k\in N_j} w_{jk}^2\right) w_j\bar{x}_j + \sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk} + w_{kj}\,w_k + w_{jt} w_{tk}\right) w_{jk}\,\bar{x}_j$$
$$+\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k\bar{x}_k + \left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk}\bar{x}_k$$
$$+\sum_{s\in N_j\cap N_k}\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj}\,w_t\right) + \left(\sum_{t\in N_j\cap N_k} w_{jk} w_{kt}\right) w_{sk} - \sum_{\substack{t\in N_k\setminus(N_j\cap N_k)\\ j\neq t}} w_{jk} w_{kt}^2\right]\bar{x}_k$$
$$+\left(\sum_{t\in N_j\cap N_k}(w_j w_{jt} + w_{tj} w_t) w_{tk}\right)\bar{x}_t + \left(\sum_{t\in N_j\cap N_k} w_{jt}\,w_{tk}\right) w_k\bar{x}_t$$

$$+\left(\sum_{t\in N_j, s\in N_k} w_{jt}\, w_{ts} w_{sk}\right)\bar{x}_t + \left(\sum_{\substack{t\in N_j, s\in N_k \\ t\neq k, s\neq j}} w_{jt} w_{ts} w_{sk}\right)\bar{x}_t$$

i.e.

$$\begin{aligned}
\bar{x}_j{}^T q_1^3 \bar{x}_j = {}& \bar{x}_j{}^T\left(w_j^2 + \sum_{k\in N_j} w_{jk}^2\right) w_j \bar{x}_j + \bar{x}_j{}^T \sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk} + w_{kj}\, w_k + w_{jt} w_{tk}\right) w_{jk}\, \bar{x}_j \\
&+ \bar{x}_j{}^T\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k \bar{x}_k + \bar{x}_j{}^T\left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk} \bar{x}_k \\
&+ \bar{x}_j{}^T \sum_{s\in N_j\cap N_k}\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj}\, w_t\right) + \left(\sum_{t\in N_j\cap N_k} w_{jk} w_{kt}\right) w_{sk} \right. \\
&\left. - \sum_{\substack{t\in N_k\setminus(N_j\cap N_k) \\ j\neq t}} w_{jk} w_{kt}^2\right]\bar{x}_k \\
&+ \bar{x}_j{}^T\left(\sum_{t\in N_j\cap N_k} (w_j w_{jt} + w_{tj} w_t) w_{tk}\right)\bar{x}_t + \bar{x}_j{}^T\left(\sum_{t\in N_j\cap N_k} w_{jt}\, w_{tk}\right) w_k \bar{x}_t \\
&+ \bar{x}_j{}^T\left(\sum_{t\in N_j, s\in N_k} w_{jt}\, w_{ts} w_{sk}\right)\bar{x}_t + \bar{x}_j{}^T\left(\sum_{\substack{t\in N_j, s\in N_k \\ t\neq k, s\neq j}} w_{jt} w_{ts} w_{sk}\right)\bar{x}_t
\end{aligned}$$

Taking the modulus on both sides in the last equality we get

$$\begin{aligned}
|q_1^3| \bar{x}_j{}^T \bar{x}_j = {}& \left|\bar{x}_j{}^T\left(w_j^2 + \sum_{k\in N_j} w_{jk}^2\right) w_j \bar{x}_j\right| \\
&+ \left|\bar{x}_j{}^T \sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk} + w_{kj}\, w_k + w_{jt} w_{tk}\right) w_{jk}\, \bar{x}_j\right| \\
&+ \left|\bar{x}_j{}^T\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k \bar{x}_k\right| + \left|\bar{x}_j{}^T\left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk} \bar{x}_k\right|
\end{aligned}$$

$$+\left|\bar{x}_j^{\,T}\sum_{s\in N_j\cap N_k}\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt}+w_{tj}\,w_t\right)+\left(\sum_{t\in N_j\cap N_k} w_{jk}w_{kt}\right)w_{sk}\right.\right.$$

$$\left.\left.-\sum_{\substack{t\in N_k\setminus(N_j\cap N_k)\\ j\neq t}} w_{jk}w_{kt}^2\right]\bar{x}_k\right|$$

$$+\left|\bar{x}_j^{\,T}\left(\sum_{t\in N_j\cap N_k}\left(w_j w_{jt}+w_{tj}w_t\right)w_{tk}\right)\bar{x}_t\right|+\left|\bar{x}_j^{\,T}\left(\sum_{t\in N_j\cap N_k} w_{jt}\,w_{tk}\right)w_k\bar{x}_t\right|$$

$$+\left|\bar{x}_j^{\,T}\left(\sum_{t\in N_j,s\in N_k} w_{jt}\,w_{ts}w_{sk}\right)\bar{x}_t\right|+\left|\bar{x}_j^{\,T}\left(\sum_{\substack{t\in N_j,s\in N_k\\ t\neq k,s\neq j}} w_{jt}w_{ts}w_{sk}\right)\bar{x}_t\right|$$

Since w_{ij} is the positive definite matrix for all i,j,w_{ij}^3 matrices are also positive definite matrix. So we have and using Cauchy-Schwarz inequality we get

$$|q_1^3|\bar{x}_j^{\,T}\bar{x}_j=\left|q_1\left(w_j^2+\sum_{k\in N_j} w_{jk}^2\right)w_j\bar{x}_j^{\,T}\bar{x}_j\right|$$

$$+\left|q_1\left[\sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk}+w_{kj}\,w_k+w_{jt}w_{tk}\right)w_{jk}\right]\bar{x}_j^{\,T}\bar{x}_j\right|$$

$$+\left|q_1\left[\left(w_j w_{jk}+w_{kj}w_k+\sum_{t\in N_j\cap N_k} w_{jt}w_{tk}\right)w_k\right]\bar{x}_j^{\,T}\bar{x}_k\right|+\left|q_1\left[\left(w_j^2+\sum_{t\in N_j} w_{jt}^2\right)w_{jk}\right]\bar{x}_j^{\,T}\bar{x}_k\right|$$

$$+\left|\sum_{s\in N_j\cap N_k} q_1\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt}+w_{tj}\,w_t\right)+\left(\sum_{t\in N_j\cap N_k} w_{jk}w_{kt}\right)w_{sk}\right.\right.$$

$$\left.\left.-\sum_{\substack{t\in N_k\setminus(N_j\cap N_k)\\ j\neq t}} w_{jk}w_{kt}^2\right]\bar{x}_j^{\,T}\bar{x}_k\right|$$

$$+\left|q_1\left(\sum_{t\in N_j\cap N_k}\left(w_j w_{jt}+w_{tj}w_t\right)w_{tk}\right)\bar{x}_j^{\,T}\bar{x}_t\right|+\left|q_1\left[\left(\sum_{t\in N_j\cap N_k} w_{jt}\,w_{tk}\right)w_k\right]\bar{x}_j^{\,T}\bar{x}_t\right|$$

$$+\left|q_1\left(\sum_{t\in N_j,s\in N_k} w_{jt}\,w_{ts}w_{sk}\right)\bar{x}_j^{\,T}\bar{x}_t\right|+\left|q_1\left(\sum_{\substack{t\in N_j,s\in N_k\\ t\neq k,s\neq j}} w_{jt}w_{ts}w_{sk}\right)\bar{x}_j^{\,T}\bar{x}_t\right|$$

from (2.1),

$$
\begin{aligned}
|q_1^3|\bar{x}_j{}^T\bar{x}_j \le\ & q_1\left(w_j^2 w_j\right)\bar{x}_j{}^T\bar{x}_j + \sum_{k\in N_j} q_1\left(w_{jk}^2 w_j\right)\bar{x}_j{}^T\bar{x}_j \\
& + q_1\left[\sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk} + w_{kj}\, w_k + w_{jt} w_{tk}\right) w_{jk}\right]\bar{x}_j{}^T\bar{x}_j \\
& + q_1\left[\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k\right]\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_k{}^T\bar{x}_k} \\
& + q_1\left[\left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk}\right]\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_k{}^T\bar{x}_k} \\
& + \sum_{s\in N_j\cap N_k} q_1\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj}\, w_t\right) + \left(\sum_{t\in N_j\cap N_k} w_{jk} w_{kt}\right) w_{sk}\right]\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_k{}^T\bar{x}_k} \\
& + q_1\left(\sum_{t\in N_j\cap N_k}\left(w_j w_{jt} + w_{tj} w_t\right) w_{tk}\right)\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_t{}^T\bar{x}_t} \\
& + q_1\left[\left(\sum_{t\in N_j\cap N_k} w_{jt}\, w_{tk}\right) w_k\right]\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_t{}^T\bar{x}_t} \\
& + q_1\left(\sum_{t\in N_j, s\in N_k} w_{jt}\, w_{ts} w_{sk}\right)\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_t{}^T\bar{x}_t} + q_1\left(\sum_{\substack{t\in N_j, s\in N_k\\ t\neq k, s\neq j}} w_{jt} w_{ts} w_{sk}\right)\sqrt{\bar{x}_j{}^T\bar{x}_j}\sqrt{\bar{x}_t{}^T\bar{x}_t}
\end{aligned}
$$

thus we obtain

$$
\begin{aligned}
q_1^3 \le\ & q_1\left(w_j^2 w_j\right) + \sum_{k\in N_j} q_1\left(w_{jk}^2 w_j\right) + q_1\left[\sum_{k\in N_j}\left(\sum_{k,t\in N_j} w_j w_{jk} + w_{kj}\, w_k + w_{jt} w_{tk}\right) w_{jk}\right] \\
& + q_1\left[\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k\right] + q_1\left[\left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk}\right] \\
& + \sum_{s\in N_j\cap N_k} q_1\left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj}\, w_t\right) + \left(\sum_{t\in N_j\cap N_k} w_{jk} w_{kt}\right) w_{sk}\right] \\
& + q_1\left(\sum_{t\in N_j\cap N_k}\left(w_j w_{jt} + w_{tj} w_t\right) w_{tk}\right) + q_1\left[\left(\sum_{t\in N_j\cap N_k} w_{jt}\, w_{tk}\right) w_k\right] \\
& + q_1\left(\sum\nolimits_{t\in N_j, s\in N_k} w_{jt}\, w_{ts} w_{sk}\right) + q_1\left(\sum\nolimits_{\substack{t\in N_j, s\in N_k\\ t\neq k, s\neq j}} w_{jt} w_{ts} w_{sk}\right) \qquad (4)
\end{aligned}
$$

From (3) and (4) we get

$$
q_1 \leq \max_{i\sim j} \sqrt[6]{
\begin{gathered}
\left(
\begin{gathered}
q_1(w_i^2 w_i) + \sum_{k\in N_i} q_1(w_{ik}^2 w_i) + \sum_{k\in N_i} q_1 \left[\left(\sum_{k,t\in N_i} w_i\, w_{ik} + w_{ki} w_k + w_{it} w_{tk}\right) w_{ik}\right] \\
+ q_1\left[\left(w_i w_{ij} + w_{ji} w_j\right) w_j\right] + \sum_{k\in N_i\cap N_j} q_1 \left(w_{ik} w_{kj} w_j\right) + q_1\left(w_i^2 w_{ij}\right) + \sum_{k\in N_i} q_1\left(w_{ik}^2 w_{ij}\right) \\
+ \sum_{s\in N_i\cap N_j} q_1 \left[\left(\sum_{k\in N_i\cap N_j} w_i\, w_{ik} + w_{ki} w_k\right) + \left(\sum_{t\in N_i\cap N_j} w_{ij} w_{jt}\right) w_{sj}\right] \\
+ q_1 \left(\sum_{k\in N_i\cap N_j} (w_i w_{ik} + w_{ki} w_k) w_{kj}\right) + q_1 \left[\left(\sum_{k\in N_i\cap N_j} w_{ik} w_{kj}\right) w_j\right] \\
+ q_1 \left(\sum_{k\in N_i, t\in N_j} w_{ik} w_{kt} w_{tj}\right) + q_1 \left(\sum_{\substack{k\in N_i, t\in N_j, \\ k\neq j, t\neq i}} w_{ik} w_{kt}\, w_{tj}\right)
\end{gathered}
\right) \\
\left(
\begin{gathered}
q_1\left(w_j^2 w_j\right) + \sum_{k\in N_j} q_1\left(w_{jk}^2 w_j\right) + q_1 \left[\sum_{k\in N_j} \left(\sum_{k,t\in N_j} w_j\, w_{jk} + w_{kj} w_k + w_{jt} w_{tk}\right) w_{jk}\right] \\
+ q_1 \left[\left(w_j w_{jk} + w_{kj} w_k + \sum_{t\in N_j\cap N_k} w_{jt}\, w_{tk}\right) w_k\right] + q_1 \left[\left(w_j^2 + \sum_{t\in N_j} w_{jt}^2\right) w_{jk}\right] \\
+ \sum_{s\in N_j\cap N_k} q_1 \left[\left(\sum_{t\in N_j\cap N_k} w_j w_{jt} + w_{tj} w_t\right) + \left(\sum_{t\in N_j\cap N_k} w_{jk}\, w_{kt}\right) w_{sk}\right] \\
+ q_1 \left(\sum_{t\in N_j\cap N_k} \left(w_j w_{jt} + w_{tj} w_t\right) w_{tk}\right) + q_1 \left[\left(\sum_{t\in N_j\cap N_k} w_{jt} w_{tk}\right) w_k\right] \\
+ q_1 \left(\sum_{t\in N_j, s\in N_k} w_{jt} w_{ts} w_{sk}\right) + q_1 \left(\sum_{\substack{t\in N_j, s\in N_k, \\ k\neq t, s\neq j}} w_{jt} w_{ts}\, w_{sk}\right)
\end{gathered}
\right)
\end{gathered}
}
$$

4. CONCLUSION

To summarize; we have introduced weighted graphs, where the weights of the edges are positive definite matrices of the same order in this paper. Then, we have give some bound and found different upper bounds for the signless Laplacian radius of weighted graphs. We have obtained some results by characterizing these upper bounds.

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