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METALLIC STRUCTURES ON DIFFERENTIABLE MANIFOLDS

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Abstract. *In this paper, we study metallic structures, i.e. polynomial structures with the structure polynomial $Q(J) = J^2 - aJ - bI$ on manifolds using the metallic ratio, which is a generalization of the Golden proportion. We investigate for integrability and parallelism conditions of metallic structures. Also, we obtain a metallic Riemannian manifold with respect to the Riemannian metric.*

Keywords: *Metallic ratio; metallic structure; polynomial structure; almost product structure; Riemannian manifold; metallic Riemannian manifold.*

1. INTRODUCTION

The Golden ratio is well known by everyone and this ratio possesses rich application areas such as architecture, painting, design, music, nature, optimization, and perceptual studies. However, in fact, it is probably fair to say that the golden ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

Crasmareanu and Hretcanu [2] researched the geometry of the golden structure on a manifold using a corresponding almost product structure. Ozkan [15], Ozkan, Citlak and Taylan [16] and Ozkan and Yilmaz [17] studied the prolongations of the golden structure. Also, many studies made on golden structures and golden Riemannian manifolds [7, 9, 22, 24].

In this paper, we use the metallic ratio, which is a generalization of the golden proportion. This generalization was introduced in 1999 by Vera W. de Spinadel [23] and named as the metallic means family or metallic proportions. Recently, many authors studied metallic structures [8, 12].

We are inspired by [2] and follow its steps. By this way, we use metallic structure on a differentiable manifold. We are interested in finding the properties of the metallic structure.

The outline of this paper is as follows. In section 2, we give basic definitions about metallic means and metallic means family. In the next section, some relations between the metallic structure and the almost product structure, almost tangent structure, almost complex structure are given. Moreover, we show some properties of the metallic structure. In section 4, we give some examples of the metallic structure. After that, we study a few connections on the metallic structure. In section 6, we search for the integrability and parallelism of the metallic structure with respect to the Schouten and Vrănceanu connections. Section 7 deals with the metallic Riemannian manifold and its properties. At the end of this section, we give an example of the metallic structure on a manifold $\mathbb{R}^2$.

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2. PRELIMINARIES

In this section, we give a brief information about the metallic means family using the Generalized Secondary Fibonacci sequence.

Fibonacci sequence provides the following relation:

$$F(k+1) = F(k) + F(k-1).$$

This relation can be generalized using

$$G(k+1) = aG(k) + bG(k-1), k \geq 1$$

where a, b, $G(0) = c$, and $G(1) = d$ are real numbers.

$$c, d, ad + bc, a(ad + bc) + bd, \ldots$$

are the members of the Generalized Secondary Fibonacci Sequence.

Let a and b be positive integers. The positive root of the equation

$$x^2 - ax - b = 0$$

is called a member of the metallic means family [23]. This root is denoted by

$$\rho_{a,b} = \frac{a+\sqrt{a^2+4b}}{2} \tag{2.1}$$

and it is known as a metallic ratio.

In (2.1), for different values of a and b, we obtain the following ratios:

- If $a = b = 1$, $\rho_{1,1} = \frac{1+\sqrt{5}}{2}$, we get the Golden mean, which is the ratio of two consecutive classical Fibonacci numbers [12],
- If $a = 2$ and $b = 1$, $\rho_{2,1} = 1 + \sqrt{2}$, we get the Silver mean, which is the ratio of two consecutive Pell numbers [6, 12, 18, 19],
- If $a = 3$ and $b = 1$, $\rho_{3,1} = \frac{3+\sqrt{13}}{2}$, we get the Bronze mean [12, 27],
- If $a = 4$ and $b = 1$, $\rho_{4,1} = 2 + \sqrt{5}$, we get the Subtle mean [5, 12],
- If $a = 1$ and $b = 2$, $\rho_{1,2} = 2$, we get the Copper mean [12],
- If $a = 1$ and $b = 3$, $\rho_{1,3} = \frac{1+\sqrt{13}}{2}$, we get the Nickel mean [12].

Throughout this paper, all manifolds are assumed to be of class C^∞ unless otherwise stated.

3. METALLIC STRUCTURES ON MANIFOLDS

Before making the description of the metallic structures on manifolds, we give some definitions in order to state the main result of this paper.

Definition 1 ([12]). *Let M_n be an n −dimensional manifold. A polynomial structure on a manifold M_n is called a metallic structure if it is determined by an $(1,1)$ −tensor field J, which satisfies the equation*

$$J^2 = aJ + bI \tag{3.1}$$

where a, b are positive integers and I is the identity operator on the Lie algebra $\chi(M_n)$ of the vector fields on M_n.

The following proposition gives the main properties of a metallic structure.

Proposition 1 ([12]). *(i) The eigenvalues of J are the metallic ratio $\rho_{a,b}$ and $a - \rho_{a,b}$.*

(ii) A metallic structure J is an isomorphism on the tangent space of the manifold $T_x(M_n)$ for every $x \in M_n$.

(iii) J is invertible and its inverse $J^{-1} = \hat{J}$ satisfies

$$b\hat{J}^2 + a\hat{J} - I = 0.$$

An almost tangent structure T, an almost product structure F, and an almost complex structure C appear in pairs. In other words, $-T$, $-F$ and $-C$ are also an almost tangent structure, an almost product structure, and an almost complex structure, respectively [2].

Proceeding from this point, it can be said that metallic structures appear in pairs:

Proposition 2. *If J is a metallic structure, then $\tilde{J} = aI - J$ is also a metallic structure.*

It is known from [10] that a polynomial structure on a manifold M_n induces a generalized almost product structure F. Therefore, we make a connection between the metallic structure and the almost product structure on M_n.

Proposition 3 ([12]). *If F is an almost product structure on M_n, then*

$$J = \frac{a}{2}I + \left(\frac{2\rho_{a,b} - a}{2}\right)F \tag{3.2}$$

is a metallic structure on M_n.

Contrarily, if J is a metallic structure on M_n, then

$$F = \left(\frac{2}{2\rho_{a,b} - a}J - \frac{a}{2\rho_{a,b} - a}I\right) \tag{3.3}$$

is an almost product structure on M_n.

By the use of the relationship between the almost product structure and the metallic structure in Proposition 3, we can give the following definitions:

Definition 2. *Let* T *be an almost tangent structure on* M_n. *Then*

$$J_t = \frac{a}{2}I + \left(\frac{2\rho_{a,b}-a}{2}\right)T$$

is called a tangent metallic structure on (M_n, T).

This structure verifies the following equation:

$$J_t^2 - aJ_t + \frac{a^2}{4}I = 0.$$

Considering this equation in the real field $\mathbb{R}$, i.e. $x^2 - ax + \frac{a^2}{4} = 0$, we have the tangent real metallic ratio $\rho_t = \frac{a}{2}$.

Definition 3. *Let* C *be an almost complex structure on* M_n. *Then*

$$J_c = \frac{a}{2}I + \left(\frac{2\rho_{a,b}-a}{2}\right)C$$

is called a complex metallic structure on (M_n, C).

The polynomial equation satisfied by J_c is

$$J_c^2 - aJ_c + \frac{a^2+2b}{2}I = 0.$$

For the case of $M_n = \mathbb{R}$, we get the equation

$$x^2 - ax + \frac{a^2+2b}{2} = 0$$

with solutions,

$$x_1 = \frac{a}{2} + \frac{\sqrt{a^2+4b}}{2}i, \quad x_2 = \frac{a}{2} - \frac{\sqrt{a^2+4b}}{2}i.$$

The positive root of equation

$$\rho_c = \frac{a}{2} + \frac{\sqrt{a^2+4b}}{2}i$$

is called a complex metallic ratio.

4. EXAMPLES OF THE METALLIC STRUCTURES

In this section, we give some examples of the metallic structure.

Example 1 (Clifford algebras). *Let $C'(n)$ be the real Clifford algebra of the positive definite form $\sum_{i=1}^{n} (x^i)^2$ of $\mathbb{R}^n$ [14]. According to the Clifford product, the standard base of $\mathbb{R}^n$ satisfies the following relations:*

$$\begin{array}{ll} e_i^2 = 1 & , \quad i = j \\ e_i e_j + e_j e_i = 0 & , \quad i \neq j \end{array} \tag{4.1}$$

Therefore, using $J_i = \frac{1}{2}\left(a + \sqrt{a^2+4b}\, e_i\right)$ and (4.1), we derive a new representation of the Clifford algebra:

$$\begin{cases} J_i, \quad metallic\ structure & , \quad i = j \\ J_i J_j + J_j J_i = a\left(J_i + J_j\right) - \frac{a^2}{2} & , \quad i \neq j. \end{cases}$$

In [14], e_1 and e_2, orthonormal basis vectors of $\mathbb{R}_2^2$, are determined by

$$1 = I_2 \quad , \quad e_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad e_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence we get

$$\begin{aligned} (i) \quad J_1 &= \frac{1}{2}\left(a + \sqrt{a^2+4b}\, e_1\right) = \begin{pmatrix} \frac{a+\sqrt{a^2+4b}}{2} & 0 \\ 0 & \frac{a-\sqrt{a^2+4b}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{a,b} & 0 \\ 0 & a - \rho_{a,b} \end{pmatrix} \\ (ii) \quad J_2 &= \frac{1}{2}\left(a + \sqrt{a^2+4b}\, e_2\right) = \frac{1}{2}\begin{pmatrix} a & \sqrt{a^2+4b} \\ \sqrt{a^2+4b} & a \end{pmatrix}. \end{aligned} \tag{4.2}$$

Example 2 (2D Metallic matrices). *For $J \in \mathbb{R}_n^n$, if J provides the equation*

$$J^2 = aJ + bI_n \tag{4.3}$$

then this matrix is called a metallic matrix where I_n is the identity matrix on $\mathbb{R}_n^n$.

In $\mathbb{R}_2^2$, we obtain a two-parametric family of the metallic structures by solving (4.3),

i) For $r, t \in \mathbb{R}$ and $s \in \mathbb{R} - \{0\}$,

$$\begin{aligned} J_{r,s} &= \begin{pmatrix} r & -\frac{1}{s}\left(r^2 - ar - b\right) \\ s & a - r \end{pmatrix} \quad or \\ J_{s,t} &= \begin{pmatrix} a - t & -\frac{1}{s}\left(t^2 - at - b\right) \\ s & t \end{pmatrix} \quad . \end{aligned} \tag{4.4}$$

ii) For $r = \rho_{a,b}$, $s \in \mathbb{R}$,

$$J_{\rho_{a,b},s} = \begin{pmatrix} \rho_{a,b} & 0 \\ s & a-\rho_{a,b} \end{pmatrix} \quad or \quad J_{a-\rho_{a,b},s} = \begin{pmatrix} a-\rho_{a,b} & 0 \\ s & \rho_{a,b} \end{pmatrix} \quad or$$
$$J_{\rho_{a,b},s} = \begin{pmatrix} \rho_{a,b} & s \\ 0 & a-\rho_{a,b} \end{pmatrix} \quad or \quad J_{a-\rho_{a,b},s} = \begin{pmatrix} a-\rho_{a,b} & s \\ 0 & \rho_{a,b} \end{pmatrix}.$$

iii) For $r = \rho_{a,b}$, $s = 0$,

$$J_{\rho_{a,b},0} = \begin{pmatrix} \rho_{a,b} & 0 \\ 0 & a-\rho_{a,b} \end{pmatrix} \quad or \quad J_{a-\rho_{a,b},s} = \begin{pmatrix} a-\rho_{a,b} & 0 \\ 0 & \rho_{a,b} \end{pmatrix}.$$

Then, from (4.2) and (4.4), we get

$$J_1 = \lim_{s\to 0} J_{\rho_{a,b},s} \quad and \quad J_2 = J_{\frac{a}{2},\frac{\sqrt{a^2+4b}}{2}}.$$

Example 3 (Metallic reflection). *In a Euclidean space* E, *the reflection with respect to a hyperplane* H *with the normal* $v \in E\backslash\{0\}$ *has the formula*

$$r_v(x) = x - \frac{2\langle x,v\rangle}{\langle v,v\rangle} v \quad , x \in E.$$

For $r_v^2 = I_E$, I_E *is the identity on* E *[20].*

We can define a metallic reflection with respect to v *as*

$$J_v = \frac{1}{2}\left(aI_E + \sqrt{a^2+4b}\, r_v\right)$$

where v *is an eigenvector of* J_v *with the corresponding eigenvalue* $a - \rho_{a,b}$. *In [20, p.314], under the condition that* X *is an orthogonal group of* E, *we can give the following:*

r transformation can be stated explicitly as

$$J_v(x) = \rho_{a,b}x - \frac{\sqrt{a^2+4b}\langle x,v\rangle}{\langle v,v\rangle} v.$$

Example 4 (Triple structure in terms of metallic structures). *Let* F *and* T *denote two tensor fields of type* $(1,1)$ *on* M_n. *Using the triple* $(F, T, K = T \circ F)$, *we obtain the following structures:*

1) $F^2 = T^2 = I$ *and* $T \circ F - F \circ T = 0$; *then* $J^2 = I$,

2) $F^2 = T^2 = I$ *and* $T \circ F + F \circ T = 0$; *then* $J^2 = -I$,

3) $F^2 = T^2 = -I$ *and* $T \circ F - F \circ T = 0$; *then* $J^2 = I$,

4) $F^2 = T^2 = -I$ *and* $T \circ F + F \circ T = 0$; *then* $J^2 = -I$.

These structures are called almost hyperproduct (ahp), almost biproduct complex (abpc),almost product bicomplex (apbc), and almost hypercomplex (ahc), respectively [3].

Taking into account (3.2), we get

$$J_F = \frac{a}{2}I + \left(\frac{2\rho_{a,b}-a}{2}\right)F, \quad J_T = \frac{a}{2}I + \left(\frac{2\rho_{a,b}-a}{2}\right)T, \quad J_K = \frac{a}{2}I + \left(\frac{2\rho_{a,b}-a}{2}\right)K.$$

Then, we find a relation between J_F, J_T, J_K *as,*

$$\sqrt{a^2+4b}J_K = 2J_TJ_F - aJ_T - aJ_F + \rho_{a,b}^2 I - bI.$$

Hence, the triple (J_F, J_T, J_K) *is,*

1′) An (ahp)-structure if and only if J_F, J_T *are metallic structures and* $J_FJ_T = J_TJ_F$, *then* J_K *is a metallic structure.*

2′) An (abpc)-structure if and only if J_F, J_T *are metallic structures and* $J_TJ_F + J_FJ_T = a(J_T + J_F) - \frac{1}{2}a^2 I$, *then* J_K *is a complex metallic structure.*

3′) An (apbc)-structure if and only if J_F, J_T *are complex metallic structures and* $J_FJ_T = J_TJ_F$, *then* J_K *is a metallic structure.*

4′) An (ahc)-structure if and only if J_F, J_T *are complex metallic structures and* $J_TJ_F + J_FJ_T = a(J_T + J_F) - \frac{1}{2}a^2 I$ *then,* J_K *is a complex metallic structure.*

Example 5 (Quaternion algebras). *Let* $\widetilde{H}$ *be a quaternion algebra. The base of* $\widetilde{H}$ *is* $\{1, e_1, e_2, e_3\}$, *which verifies*

$$e_1^2 = e_2^2 = e_3^2 = -1$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_3e_1 = -e_1e_3 = e_2, \quad e_2e_3 = -e_3e_2 = e_1,$$

$$q = a_0 1 + a_1e_1 + a_2e_2 + a_3e_3$$

is a quaternion, which can be divided into two parts, such as S_q *is a scalar part and* $\vec{V}_q$ *is a vector part. In this way,* $q = S_q + \vec{V}_q$ *where* $S_q = a_0$ *and* $\vec{V}_q = a_1e_1 + a_2e_2 + a_3e_3$.

The norm of a quaternion is determined by $N_q = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. *Also,* $q_0 = \frac{q}{N_q}$ *is a unit quaternion* $(q \neq 0)$. *A unit form can be written by* $q_0 = cos\alpha + \vec{S}_0 sin\alpha$ *where* $\vec{S}_0$ *is a unit vector verifying the equality* $\vec{S}_0^2 = -1$.

Hence, by the inspire of [25] we can define the metallic biquaternion structure as

$$J_q = \frac{a}{2} + \frac{\sqrt{a^2+4b}i}{2}\vec{S}_0 \quad where \quad \vec{S}_0^2 = -1 and \ i^2 = -1.$$

Similarly, we can define the metallic split quaternion structure as

$$J_q = \frac{a}{2} + \vec{S}_0\frac{\sqrt{a^2+4b}}{2} \quad where \quad \vec{S}_0^2 = 1.$$

5. CONNECTION AS METALLIC STRUCTURE

5.1. CONNECTIONS IN PRINCIPAL FIBRE BUNDLES

Let $P(\pi, M_n, G)$ be a principal fibre bundle where $\pi: P \to M_n$ is a fibre projection and G is a Lie group. $V = \ker\pi_*$ is the vertical distribution on P, H is the complementary distribution of V, i.e. $TP = V \oplus H$, and H is $G-$invariant. The corresponding projectors of V and H are v and h, respectively. So, the tensor field of type $(1,1)$

$$F = v - h$$

is an almost product structure on P. From [2], F defines a connection if and only if the followings are satisfied:

1) $F(X) = X \Leftrightarrow X \in V$

2) $dR_e \circ F_u = F_{ue} \circ dR_e$ for all $e \in G$ and $u \in P$.

By use of the relation between the almost product structure and the metallic structure, we get the following assertion:

Proposition 4. *The metallic structure J on P indicates a connection if and only if the following properties are satisfied:*

1′) $X \in \chi(P)$, where $\chi(P)$ is the Lie algebra of vector fields on P, is an eigenvector of J with respect to the eigenvalue $\rho_{a,b}$ if and only if X is a vertical vector field.
2′) $dR_e \circ J_u = J_{ue} \circ dR_e$ for all $e \in G$ and $u \in P$.

Let $\omega \in \Omega^1(P, g)$ be the connection $1-$form of H and suppose that $\Omega \in \Omega^2(P, g)$ be the curvature form of ω where g is the Lie algebra of G. Then, we get [2],

$$\Omega(X, Y) = -\frac{1}{4}\omega\big(N_F(X, Y)\big) \tag{5.1}$$

where N_F is the Nijenhuis tensor of F.

Proposition 5. *Using (3.2), we obtain*

$$N_F = \frac{4}{(2\rho_{a,b} - a)^2} N_J \tag{5.2}$$

and then we get

$$\Omega(X, Y) = -\frac{1}{(2\rho_{a,b} - a)^2}\,\omega\left(N_J(X, Y)\right).$$

The connection is flat if its curvature form $\Omega \equiv 0$.

Proposition 6. *The connection is flat if and only if the associated metallic structure is integrable, i.e.* $N_J \equiv 0$.

This connection yields a lift $l_\omega: \chi(M_n) \to \chi(P)$ verifying

$$[l_\omega X^*, l_\omega Y^*] - l_\omega[X^*, Y^*] = N_F(l_\omega X^*, l_\omega Y^*) \tag{5.3}$$

for every $X^*, Y^* \in \chi(M_n)$ [2].

Hence, using (5.1) and (5.3):

Proposition 7. *The lift defined by* l_ω *is a morphism if and only if the associated metallic structure is integrable.*

5.2. CONNECTION IN TANGENT BUNDLES

Let $\pi: TM_n \to M_n$ be the tangent bundle of M_n. $V(M_n) = \ker\pi_*$ $(\pi_*: TTM_n \to TM_n)$ is called vertical distribution of M_n. $\left(x^i\right)_{1\leq i\leq n}$ is a local coordinate system on M_n and $(x, y) = \left(x^i, y^i\right)_{1\leq i\leq n}$ is a local coordinate system on TM_n. For an atlas on TM_n with these local coordinates, the tangent structure of TM_n is $T = \frac{\partial}{\partial y^i} \otimes dx^i$, i.e.

$$T\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \quad , \quad T\left(\frac{\partial}{\partial y^i}\right) = 0.$$

$v: \chi(M_n) \to \chi(M_n)$ is a $(1,1)$ tensor field verifying

$$\begin{cases} T \circ v = 0 \\ v \circ T = T \end{cases}$$

which is called the vertical projector.

Definition 4 ([2]). *A complementary distribution* N *to the vertical distribution* $V(M_n)$

$$\chi(M_n) = N \oplus V(M_n) \tag{5.4}$$

is called normalization, horizontal distribution or non-linear connection.

Having in mind that a vertical projector v is $C^\infty(M_n)$ $-$linear with $imv = V(M_n)$, we get:

Proposition 8 ([2]). *With the help of* $kerv = N(v)$, *a vertical vector* v *yields a non-linear connection. Viceversa, if* N *is a non-linear connection, then* h_N *and* v_N *are horizontal and vertical projectors, respectively, with respect to the decomposition (5.4).*

Then we give the following proposition:

Proposition 9 ([2]). *If N is a non-linear connection then v_N is a vertical projector with $N(v_N) = N$.*

Definition 5 ([2]). *If the following relations hold, a $(1,1)$ tensor field Γ is called non-linear connection of an almost product type:*

$$\begin{cases} \Gamma \circ T = -T \\ T \circ \Gamma = T \end{cases}.$$

Let Γ be a nonlinear connection of an almost product type, the following assertions are true:

i) $v_\Gamma = \frac{1}{2}\left(I_{\chi(M_n)} - \Gamma\right)$ is a vertical vector,

ii) While $N(v_\Gamma)$ is the $(+1)$ −eigenspace of Γ, $V(M_n)$ is the (-1) −eigenspace of Γ.

As a result, every vertical projector v yields a non-linear connection of an almost product type as $\Gamma = I_{\chi(M_n)} - 2v$. Using the last relation, we have $\Gamma^2 = I_{\chi(M_n)}$ (Γ tensor field is an almost product structure on M_n).

Hence, we associate this result with the metallic structure.

Proposition 10. *A non-linear connection N on M_n, given by the vertical vector v, can also be defined by a metallic structure $J(= J_\Gamma)$*

$$J = \rho_{a,b} I_{\chi(M_n)} - \sqrt{a^2 + 4b}\, v$$

with N the $\rho_{a,b}$ −eigenspace and $V(M_n)$ the $\left(a - \rho_{a,b}\right)$ −eigenspace.

6. INTEGRABILITY AND PARALLELISM OF METALLIC STRUCTURES

In this section, we give integrability and parallelism conditions of the metallic structure. Recall the Nijenhuis tensor of J,

$$N_J(X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$$

for every $X, Y \in \chi(M_n)$.

Let L and M be two complementary distributions on M_n corresponding to $\rho_{a,b}$ and $a - \rho_{a,b}$, respectively. The corresponding projection operators of L and M are denoted by l and m, which results in

$$\begin{cases} l^2 = l, & m^2 = m, \\ lm = ml = 0, & l + m = I. \end{cases} \tag{6.1}$$

According to the above conditions and based on a straightforward computation from (3.2), we obtain the following equations [12]:

$$\begin{aligned} l &= \frac{1}{2\rho_{a,b}-a}J - \frac{a-\rho_{a,b}}{2\rho_{a,b}-a}I \\ m &= -\frac{1}{2\rho_{a,b}-a}J + \frac{\rho_{a,b}}{2\rho_{a,b}-a}I. \end{aligned} \tag{6.2}$$

Theorem 11. *i) If $N_J = 0$, J is integrable.*
ii) If $m[lX, lY] = 0$, the distribution L is integrable. Similarly, if $l[mX, mY] = 0$, the distribution M is integrable for all vector fields X, Y on $\chi(M_n)$.

Remark 1. *From (3.1) and (6.2),*

$$\begin{aligned} Jl = lJ &= \frac{\rho_{a,b}}{2\rho_{a,b}-a}J + \frac{b}{2\rho_{a,b}-a}I = \rho_{a,b}l \\ Jm = mJ &= -\frac{a-\rho_{a,b}}{2\rho_{a,b}-a}J - \frac{b}{2\rho_{a,b}-a}I = (a-\rho_{a,b})m. \end{aligned} \tag{6.3}$$

Using (6.3), we can easily see that

$$\begin{aligned} l[mX, mY] &= \frac{1}{(2\rho_{a,b}-a)^2} lN_J(mX, mY), \\ m[lX, lY] &= \frac{1}{(2\rho_{a,b}-a)^2} mN_J(lX, lY). \end{aligned}$$

Proposition 11. *Using (5.2), a metallic structure J is integrable if and only if the associated almost product structure F is integrable.*

Proposition 12. *Let $X, Y \in \chi(M_n)$. L is integrable if and only if $mN_J(lX, lY) = 0$ and M is integrable if and only if $lN_J(mX, mY) = 0$. If J is integrable, then the distributions L and M are integrable.*

Now, we give the parallelism conditions of the metallic structures.
Let ∇ be a linear connection on M_n. We associate the pair (J, ∇) with other linear connections, which are Schouten and Vrănceanu connections [1, 13].

i) The Schouten connection,

$$\tilde{\nabla}_X Y = l(\nabla_X lY) + m(\nabla_X mY). \tag{6.4}$$

ii) The Vrănceanu connection,

$$\hat{\nabla}_X Y = l(\nabla_{lX} lY) + m(\nabla_{mX} mY) + l[mX, lY] + m[lX, mY]. \tag{6.5}$$

Proposition 13. *The projectors r, s are parallels in terms of Schouten and Vrănceanu connections for every linear connection ∇ on M. Moreover, J is parallel in terms of Schouten and Vrănceanu connections.*

Proof: For every vector fields X, Y on $\chi(M_n)$ and with the help of (6.1),

$$(\widetilde{\nabla}_X l)Y = \widetilde{\nabla}_X lY - l(\widetilde{\nabla}_X Y) = l(\nabla_X lY) - l(\nabla_X lY) = 0,$$

$$\begin{aligned}(\widehat{\nabla}_X l)Y &= \widehat{\nabla}_X lY - l(\widehat{\nabla}_X Y) \\ &= l(\nabla_{lX} lY) + l[mX, lY] - l(\nabla_{lX} lY) - l[mX, lY] \\ &= 0.\end{aligned}$$

Therefore, l is parallel relevant to $\widetilde{\nabla}$ and $\widehat{\nabla}$.

Similarly, the projector m is parallel with respect to the Schouten and Vrănceanu connections. From (6.3), J is parallel in terms of Schouten and Vrănceanu connections.

Definition 6 ([4]). *If $\nabla_X Y \in L$ where $X \in \chi(M_n)$ and $Y \in L$, the distribution L is called parallel with respect to linear connection ∇.*

Definition 7 ([4]). *If $(\Delta J)(X, Y) \in L$ where*

$$(\Delta J)(X, Y) = J\nabla_X Y - J\nabla_Y X - \nabla_{JX} Y + \nabla_Y (JX) \tag{6.6}$$

for $X \in L$, $Y \in \chi(M_n)$, the distribution L is called $\nabla - half\ parallel$.

Definition 8 ([4]). *If $(\Delta J)(X, Y) \in M$ where $X \in L$, $Y \in \chi(M_n)$, the distribution L is called $\nabla - anti\ half\ parallel$.*

Proposition 14. *The distributions L and M are parallel in terms of Schouten and Vrănceanu connections for the linear connection ∇ on M_n.*

Proof: Let $X \in \chi(M_n)$ and $Y \in L$. $mY = 0$ and $lY = Y$, using (6.4) and (6.5) we obtain

$$\widetilde{\nabla}_X Y = l(\nabla_X Y) \in L,$$

$$\widehat{\nabla}_X Y = l(\nabla_{lX} Y) + l[mX, Y] \in L.$$

Thus, L is parallel with respect to $\widetilde{\nabla}$ and $\widehat{\nabla}$.

In the same manner, we can see that M satisfies the similar relations.

Proposition 15. *The distributions L and M are parallel with respect to ∇ linear connection if and only if ∇ and $\widetilde{\nabla}$ are equal.*

Proof: If L, M are $\nabla -$ parallel then $\nabla_X(lY) \in L$ and $\nabla_X(mY) \in M$ where $X, Y \in \chi(M_n)$. For that reason

$$\nabla_X(lY) = l\nabla_X(lY) \quad \text{and} \quad \nabla_X(mY) = m\nabla_X(mY).$$

Since, $l + m = I$ and from (6.4),

$$\nabla_X Y = l\nabla_X(lY) + m\nabla_X(mY) = \widetilde{\nabla}_X Y.$$

Therefore $\nabla = \widetilde{\nabla}$.

The converse can be shown easily.

Proposition 16. *If* $[lX, mY] \in L$ *where* $X \in L$, $Y \in \chi(M_n)$, *the distribution* L *is half parallel with respect to the Vrănceanu connection.*

Proof: In the view of the equation (6.6) for $\widehat{\nabla}$, we get

$$m(\Delta J)(X,Y) = mJ\widehat{\nabla}_X Y - mJ\widehat{\nabla}_Y X - m\widehat{\nabla}_{JX} Y + m\widehat{\nabla}_Y (JX)$$

where $X \in L$, $Y \in \chi(M_n)$.

As a result, by (6.3) and (6.5), we have

$$m(\Delta J)(X,Y) = \left(a - 2\rho_{a,b}\right) m[lX, mY]$$

which proves the proposition.

Similarly, we have the following proposition for distribution M:

Proposition 17. *If* $[mX, lY] \in M$ *where* $X \in M$, $Y \in \chi(M_n)$, *the distribution* M *is half parallel with respect to the Vrănceanu connection.*

Proposition 18. *The distributions* L *and* M *are anti half parallel with respect to Vrănceanu connection.*

Proof: Taking account of the equation (6.6) for $\widehat{\nabla}$, we get

$$l(\Delta J)(X,Y) = lJ\widehat{\nabla}_X Y - lJ\widehat{\nabla}_Y X - l\widehat{\nabla}_{JX} Y + l\widehat{\nabla}_Y (JX)$$

where $X \in L$, $Y \in \chi(M_n)$.

Using (6.3) and (6.5), we get

$$l(\Delta J)(X,Y) = \left(2\rho_{a,b} - a\right) l[mX, lY].$$

Because $mX = 0$, we get $l(\Delta J)(X,Y) = 0$. Thus, $(\Delta J)(X,Y) \in M$.

In the same way, we can show that M is anti-half parallel with respect to the Vrănceanu connection.

7. METALLIC RIEMANNIAN MANIFOLDS

In this section, we obtain a metallic Riemannian manifold with respect to the Riemannian metric and we give some properties of the metallic Riemannian manifold.

Definition 9 ([11, 21, 26]). *Let* F *be an almost product structure on* M_n *and* g *be a Riemannian metric (or a semi-Riemannian metric) given by*

$$g\big(F(X), F(Y)\big) = g(X,Y) \quad , \quad \forall X, Y \in \chi(M_n)$$

or equivalently, F be a g-symmetric endomorphism, such as

$$g\big(F(X),F(Y)\big)=g(X,Y).$$

So, the pair (g,F) is called a Riemannian almost product structure (or a semi-Riemannian almost product structure).

By the help of (3.2) and (3.3), the following proposition can be given:

Proposition 19. *The almost product structure F is a $g-$symmetric endomorphism if and only if the metallic structure J is also a $g-$symmetric endomorphism.*

Definition 10 ([12]). *Let g be a Riemannian metric (or a semi-Riemannian metric) on M_n, such as*

$$g(J(X),Y)=g\big(X,J(Y)\big),\quad \forall X,Y\in\chi(M_n).$$

Then, (g,J) is called a metallic Riemannian structure (or a metallic semi-Riemannian structure). Also, (M_n,g,J) is named a metallic Riemannian manifold (or a metallic semi-Riemannian manifold).

Corollary 1. *On a metallic Riemannian manifold,*
a) The projectors l,m are $g-$symmetric, i.e.

$$g(l(X),Y)=g\big(X,l(Y)\big),\ g(m(X),Y)=g\big(X,m(Y)\big).$$

b) The distributions L, M are $g-$orthogonal, i.e.

$$g(l(X),m(Y))=0.$$

c) The metallic structure is N_J-symmetric, i.e.

$$N_J(J(X),Y)=N_J\big(X,J(Y)\big).$$

A Riemannian almost product structure is a locally product structure if F is parallel with respect to the Levi-Civita connection $\breve{\nabla}$ of g, i.e. $\breve{\nabla}F=0$ and if ∇ is a symmetric linear connection, then the Nijenhuis tensor of F satisfies

$$N_F(X,Y)=(\nabla_{FX}F)Y-(\nabla_{FY}F)X-F(\nabla_XF)Y+F(\nabla_YF)X.$$

If (M_n,g,J) is a locally product metallic Riemannian manifold, then the metallic structure J is integrable.

Theorem 2. *The set of linear connections ∇ for which $\nabla J=0$ is*

$$\begin{aligned}\nabla_XY \;=\;& \tfrac{1}{a^2+4b}\left[(a^2+2b)\overline{\nabla}_XY+2J(\overline{\nabla}_XJY)-aJ(\overline{\nabla}_XY)-a(\overline{\nabla}_XJY)\right]\\ &+O_FQ(X,Y)\end{aligned}$$

where $\bar{\nabla}$ is an arbitrary fixed linear connection and Q is a $(1,2)$ −tensor field for which $O_F Q$ is an associated Obata operator

$$O_F Q(X,Y) = \frac{1}{2}[Q(X,Y) + FQ(X,FY)]$$

for the corresponding almost product structure (3.2).

We conclude with the following example for the metallic structure.

Example 6.

$$l = \frac{x^2}{x^2+y^2}\frac{\partial}{\partial x} \otimes dx + \frac{xy}{x^2+y^2}\frac{\partial}{\partial x} \otimes dy + \frac{xy}{x^2+y^2}\frac{\partial}{\partial y} \otimes dx + \frac{y^2}{x^2+y^2}\frac{\partial}{\partial y} \otimes dy$$

$$m = \frac{y^2}{x^2+y^2}\frac{\partial}{\partial x} \otimes dx - \frac{xy}{x^2+y^2}\frac{\partial}{\partial x} \otimes dy - \frac{xy}{x^2+y^2}\frac{\partial}{\partial y} \otimes dx + \frac{x^2}{x^2+y^2}\frac{\partial}{\partial y} \otimes dy$$

are projection operators in $\mathbb{R}^2$ which satisfy the conditions (6.1).

$$L = Sp\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\} \quad and \quad M = Sp\left\{y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right\}$$

are complementary distributions corresponding to the projection operators l and m, respectively. The distributions L, M are orthogonal with respect to the Euclidean metric of $\mathbb{R}^2$. Furthermore, these distributions are connected to the metallic structure

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\rho_{a,b}x^2+(a-\rho_{a,b})y^2}{x^2+y^2}\frac{\partial}{\partial x} + \frac{(2\rho_{a,b}-a)xy}{x^2+y^2}\frac{\partial}{\partial y},$$

$$J\left(\frac{\partial}{\partial y}\right) = \frac{(2\rho_{a,b}-a)xy}{x^2+y^2}\frac{\partial}{\partial x} + \frac{(a-\rho_{a,b})x^2+\rho_{a,b}y^2}{x^2+y^2}\frac{\partial}{\partial y},$$

which is integrable because $N_J\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = 0$.

REFERENCES

[1] Bejancu, A., Farran, H.R., *Foliations and geometric* structures - Mathematics and Its Applications, **Vol.580**, Springer, 2006.
[2] Crasmareanu, M., Hretcanu, C. E., *Chaos, Solitons & Fractals*, **38**, 1229, 2008.
[3] Cruceanu, V., *An. St. Univ. I. Cuza, Iasi, Mat.*, **52**(1), 5, 2006.
[4] Das, L.S., Nikic, J., *Nivas R., Diff. Geom. Dyn. Syst.*, **8**, 82, 2006.
[5] El Naschie, M.S., *Chaos, Solitons & Fractals*, **9**(8), 1445, 1998.
[6] Falcon S., Plaza A., *Chaos, Solitons & Fractals*, **38**(4), 993, 2008.
[7] Gezer, A., Cengiz, N., Salimov, A., *Turk J. Math.*, **37**(4), 693, 2013.
[8] Gezer, A., Karaman, C., *Turk J. Math.*, **39**(6), 954, 2015.
[9] Gezer, A., Karaman, C., *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, **86**(1), 41, 2016.
[10] Goldberg, S.I., Petridis, N. C., *Kodai Math Sem. Rep.*, **25**, 111, 1973.

[11] Gray, A., *J. Math. Mech.*, **16**, 715, 1967.
[12] Htercanu, C.E., Crasmareanu, M., *Rev. Un. Mat. Argentina*, **54**(2), 15, 2013.
[13] Ianus, S., *Kodai Math. Sem. Rep.*, **23**, 305, 1971.
[14] Lounesto, P., *Clifford Algebras and Spinors*, Cambridge University Press, United Kingdom, 2001.
[15] Ozkan, M., *Differ. Geom. Dyn. Syst.*, **16**, 227, 2014.
[16] Ozkan, M., Citlak, A.A., Taylan, E., *GU J Sci*, **28**(2), 253, 2015.
[17] Ozkan, M., Yilmaz, F., *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **65**(1), 35, 2016.
[18] Ozkan, M., Peltek, B., *Int. Electron. J. Geom.*, **9**(2), 59, 2016.
[19] Ozkan, M., Taylan, E., Citlak, A.A., *Journal of Science and Arts*, **2**(39), 223, 2017.
[20] Procesi C., *Lie groups: An approach through invariants and representations*., Universitext, Springer, 2007.
[21] Pripoae G. T., *Classification of semi-Riemannian almost product structure*, BSG Proc. 11, Geometry Balkan Press, Bucharest, 243, 2004.
[22] Savas M., Ozkan, M., Iscan, M., *Journal of Science and Arts*, **2**(35), 89, 2016.
[23] Spinadel, V. W. de, *Vis. Math*., **1**, 3, 1999.
[24] Şahin, B., Akyol, M.A., *Math. Commun*., **19**, 1, 2014.
[25] Yardımci, E.H., Yayli, Y., *Int. Electron. J. Pure Appl. Math.*, **7**, 109, 2014.
[26] Yano, K., Kon, M., *Structures on manifolds*, Series in Pure Mathematics, Vol. 3, World Scientific, Singapore, 1984.
[27] Yilmaz, F., Ozkan, M., *Bronze structure*, 14th Mathematics Symposium, Niğde, Turkey, 109, 2015.