ORIGINAL PAPER

GAUSSIAN BALANCING NUMBERS AND GAUSSIAN LUCAS-BALANCING NUMBERS

DURSUN TASCI[1]

Manuscript received: 05.01.2018; Accepted paper: 23.05.2018;
Published online: 30.09.2018.

Abstract. *In this study we define Gaussian balancing numbers and Gaussian Lucas-balancing numbers. Then we obtain Binet-like formulas, generating functions and some identities related with Gaussian balancing numbers and Gaussian Lucas-balancing numbers. Moreover, we give the new properties of Gaussian balancing numbers and Gaussian Lucas-balancing numbers in relation with balancing matrix formula.*

Keywords: *Balancing and Lucas-balancing numbers, Gaussian balancing numbers, Gaussian Lucas-balancing numbers.*

1. INTRODUCTION

Horadam [3] introduced the concept the complex Fibonacci numbers as the Gaussian Fibonacci numbers. Behera and Panda [2] introduced the concept of balancing numbers. They defined a balancing number n as a solution of Diophantine equation. A positive integer n is called a balancing number if $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$ for some natural number r. Here r is called the balancer corresponding to the balancing number n. For example 6 and 35 are balancing number with balancers 2 and 14. Again some authors proved that the balancing numbers fulfil the following recurrence relation $B_{n+1}=6B_n-B_{n-1}$, $n\geq 1$ where $B_0=1$ and $B_1=6$. Panda [6] studied several fascinating properties of balancing numbers calling the positive square root of $8x^2+1$, a Lucas- balancing number for each balancing number x. All balancing number x and corresponding Lucas-balancing numbers y are positive integer solutions of Diophantine equation $8x^2+1=y^2$. Balancing and Lucas-balancing numbers share the same linear recurrence $x_{n+1}=6x_n-x_{n-1}$, while initial values of balancing numbers are $x_0=0, x_1=1$ and for Lucas-balancing numbers $x_0=1, x_1=3$. Alvarado et. Al. [1], Liptai [4,5] and Szalay [10] studied certain Diophantine equations relating to balancing numbers. Ray [7,8,9] studied some Diophantine equations involving balancing and Lucas-balancing numbers. Tasci [11,12] studied Gaussian Padovan and Gaussian Pell-Padovan sequaences, and Complex Fibonacci p-numbers.

We denote the n^{th} balancing and Lucas-balancing numbers by B_n and C_n, respectively. The sequences $\{B_n\}$ and $\{C_n\}$ satisfy the recurrence relations

$$B_{n+1}=6B_n-B_{n-1}\ \ (n\geq 1), B_0=0, B_1=1$$

and

$$C_{n+1}=6C_n-C_{n-1}\ \ (n\geq 1), C_0=1, C_1=3.$$

[1] Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey. E-mail: dtasci@gazi.edu.tr.

We note that Binet-like formulas for balancing and Lucas-balancing numbers are

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}} \text{ and } C_n = \frac{\alpha^n + \beta^n}{2},$$

respectively. We remark that $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$.

2. GAUSSIAN BALANCING NUMBERS

Definition 2.1 The Gaussian balancing sequence is the sequence of complex numbers GB_n defined by the initial values $GB_0 = i, GB_1 = 1$ and the recurrence relation

$$GB_{n+1} = 6GB_n - GB_{n-1} \text{ for all } n \geq 1.$$

The first few values of GB_n are $i, 1, 6 - i, 35 - 6i, 204 - 35i, \ldots$.
The following theorem is related with the generating function of the Gaussian balancing sequence.

Theorem 2.1 The generating function of the Gaussian balancing sequence is

$$f(x) = \frac{i + (1 - 6i)x}{1 - 6x + x^2}.$$

Proof: Let

$$f(x) = \sum_{n=0}^{\infty} GB_n x^n = GB_0 + GB_1 x + GB_2 x^2 + \cdots + GB_n x^n + \cdots.$$

Since

$$6xf(x) = 6\sum_{n=0}^{\infty} GB_n x^{n+1} = 6GB_0 x + 6GB_1 x^2 + 6GB_2 x^3 + \cdots + 6GB_{n-1} x^n + \cdots$$

and

$$x^2 f(x) = \sum_{n=0}^{\infty} GB_n x^{n+2} = GB_0 x^2 + GB_1 x^3 + GB_2 x^4 + \cdots + GB_{n-2} x^n + \cdots.$$

So we write

$$(1 - 6x + x^2)f(x) = GB_0 + (GB_1 - 6GB_0)x + (GB_2 - 6GB_1 + GB_0)x^2 + \cdots + GB_n - 6GB_{n-1} + GB_{n-2})x^n + \cdots.$$

Considering $GB_0 = i, GB_1 = 1$ and $GB_n = 6GB_{n-1} - GB_{n-2}$, we have

$$(1 - 6x + x^2)f(x) = i + (1 - 6i)x$$

or

$$f(x) = \frac{i + (1 - 6i)x}{1 - 6x + x^2}$$

So the theorem is proved.

Theorem 2.2. The Binet-like formula for the Gaussian balancing numbers is

$$GB_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} - i\frac{\beta\alpha^n - \alpha\beta^n}{\alpha - \beta},$$

where $\propto = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$ are the roots of the equation $x^2 - 6x + 1 = 0$.

Proof: The general term of Gaussian balancing numbers can be expressed in the following form

$$GB_n = c_1\alpha^n + c_2\beta^n,$$

where c_1 and c_2 are coefficients. Using the values n = 0,1 we found

$$c_1 = \frac{1 - i\beta}{\alpha - \beta} \text{ and } c_2 = \frac{i\alpha - 1}{\alpha - \beta}.$$

Considering the values c_1, c_2 and making some calculations, we obtain

$$GB_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} - i\frac{\beta\alpha^n - \alpha\beta^n}{\alpha - \beta}$$

So the theorem is proved.

Theorem 2.3.

$$\sum_{i=0}^{n} GB_i = \frac{1}{4}(5GB_n - GB_{n-1} + 5i - 1).$$

Proof: By the definition of Gaussian balancing sequence recurrence relation

$$GB_n = 6GB_{n-1} - GB_{n-2}$$

we have

$$\begin{gathered}
GB_0 = 6GB_{-1} - GB_{-2} \\
GB_1 = 6GB_0 - GB_{-1} \\
GB_2 = 6GB_1 - GB_0 \\
\vdots \\
GB_{n-2} = 6GB_{n-3} - GB_{n-4} \\
GB_{n-1} = 6GB_{n-2} - GB_{n-3} \\
GB_n = 6GB_{n-1} - GB_{n-2}.
\end{gathered}$$

Thus we obtain

$$\sum_{i=0}^{n} GB_i = \frac{1}{4}(5GB_n - GB_{n-1} - 5GB_{-1} + GB_{-2}).$$

Now considering $GB_{-1} = 6i - 1, GB_{-2} = 35i - 6$ we write

$$\sum_{i=0}^{n} GB_i = \frac{1}{4}(5GB_n - GB_{n-1} + 5i - 1).$$

So the theorem is proved.

Now we investigate the new properties of Gaussian balancing numbers in relation with balancing matrix formula. We consider the following matrices:

$$Q=\begin{bmatrix}6 & -1\\ 1 & 0\end{bmatrix}, K=\begin{bmatrix}6-i & 1\\ 1 & i\end{bmatrix} \text{ and } P_n=\begin{bmatrix}GB_{n+2} & GB_{n+1}\\ GB_{n+1} & GB_n\end{bmatrix}.$$

Theorem 2.4. For all $n \in Z^+$

$$Q^n K = P_n \tag{2.1}$$

Proof: The proof is easily seen that using the induction on n. We remark that if

$$Q=\begin{bmatrix}6 & -1\\ 1 & 0\end{bmatrix}$$

then for all $n \in Z^+$

$$Q^n=\begin{bmatrix}B_{n+1} & -B_n\\ B_n & -B_{n-1}\end{bmatrix}.$$

Theorem 2.5. (Cassini formula for Gaussian balancing numbers)

$$GB_{n+2}GB_n - GB_{n+1}^2 = 6i.$$

Proof: From (2.1) we have

$$\det(Q^n K) = \det(P_n)$$

or

$$\det(Q)^n \det(K) = \det(P_n).$$

On the other hand, since det(Q)=1, det(K)=$6i$ and $\det(P_n) = GB_{n+2}GB_n - GB_{n+1}^2$, we obtain

$$GB_{n+2}GB_n - GB_{n+1}^2 = 6i.$$

So the theorem is proved.

Theorem 2.6. For all $n \in Z^+$

$$\begin{bmatrix}6 & -1\\ 1 & 0\end{bmatrix}^n \begin{bmatrix}1\\ i\end{bmatrix} = \begin{bmatrix}GB_{n+1}\\ GB_n\end{bmatrix}.$$

Proof: The proof can be seen by mathematical induction on n.

3. GAUSSIAN LUCAS-BALANCING NUMBERS

Definition 3.1. The Gaussian Lucas –balancing sequence is defined by recurrence relation

$$GC_{n+1} = 6GC_n - GC_{n-1} \text{ for all } n \geq 1,$$

and initial values are $GC_0 = 1-3i, GC_1 = 3-i$.

The first few values of GC_n are $1-3i$,$3-i, 17-3i, 99-17i, 577-99i,$ $3363-577i, \ldots$.

Theorem 3.1. The generating function of Gaussian Lucas-balancing sequence is

$$g(x) = \frac{1-3i+(-3+17i)x}{1-6x+x^2}.$$

Proof: Let

$$g(x) = \sum_{n=0}^{\infty} GC_n x^n$$

be the generating function of the Gaussian Lucas-balancing sequence. In this case, we have

$$6xg(x) = 6\sum_{n=0}^{\infty} GC_n x^{n+1} = 6GC_0x + 6GC_1x^2 + 6GC_2x^3 + \cdots + 6GC_{n-1}x^n + \cdots$$

and

$$x^2g(x) = \sum_{n=0}^{\infty} GC_n x^{n+2} = GC_0x^2 + GC_1x^3 + GC_2x^4 + \cdots + GC_{n-2}x^n + \cdots.$$

So we obtain

$$(1-6x+x^2)g(x) = GC_0 + (GC_1 - 6GC_0)x + (GC_2 - 6GC_1 + GC_0)x^2 + \cdots + GC_n - 6GC_{n-1} + GC_{n-2})x^n + \cdots.$$

Since $GC_0 = 1-3i, GC_1 = 3-i$ and $GC_n = 6GC_{n-1} - GC_{n-2}$, we write

$$g(x) = \frac{1-3i+(-3+17i)x}{1-6x+x^2}$$

which is desired.

Theorem 3.2. The Binet-like formula of Gaussian Lucas-balancing is

$$GC_n = \frac{(\beta-3)\alpha^n + (3-\alpha)\beta^n}{\beta-\alpha} + i\,\frac{(1-3\beta)\alpha^n + (3\alpha-1)\beta^n}{\beta-\alpha}. \quad (3.1)$$

Proof: From the theory of difference equations we know the general term of Gaussian Lucas-balancing numbers can be expressed in the following form

$$GC_n = c\alpha^n + d\beta^n.$$

Using the values n = 0,1

$$c = \frac{(\beta-3)+i(1-3\beta)}{\beta-\alpha}, d = \frac{(3-\alpha)+i(3\alpha-1)}{\beta-\alpha}$$

can be found. So we obtain (3.1) and the proof is complete.

Theorem 3.3

$$\sum_{i=0}^{n} GC_i = \frac{1}{4}(5GC_n - GC_{n-1} + 2 - 14i).$$

Proof: The proof of Theorem 3.3 is similar to the proof of Theorem 2.3.

Theorem 3.4.

i) $GB_{n+1} - GB_{n-1} = 2GC_n$
ii) $GC_{n+1} - GC_{n-1} = 16GB_n$
iii) $GC_n^2 - 8GB_n^2 = -6i.$

Proof: The proof is easily seen that considering the Binet-like formulas of Gaussian balancing number and Gaussian Lucas-balancing numbers.

Theorem 3.5. Let

$$Q = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}, S = \begin{bmatrix} 17-3i & 3-i \\ 3-i & 1-3i \end{bmatrix} \text{ and } R_n = \begin{bmatrix} GC_{n+2} & GC_{n+1} \\ GC_{n+1} & GC_n \end{bmatrix}.$$

Then for all $n \in Z^+$

$$Q^n S = R_n \ .$$

Proof: The proof can be seen by mathematical induction on n.

Theorem 3.6. (Cassini formula for Gaussian Lucas-balancing numbers)

$$GC_{n+1}GC_{n-1} - GC^2{}_n = -48i.$$

Proof: (Mathematical induction on n.)

Theorem 3.7. For all $n \in Z^+$

$$\begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 3-i \\ 1-3i \end{bmatrix} = \begin{bmatrix} GC_{n+1} \\ GC_n \end{bmatrix}.$$

Proof: Use induction on n.

REFERENCES

[1] Alvarado S., Dujella A. , Luca F., *Integers*, **12,** 1127, 2012.
[2] Behera, A., Panda, G.K., *Fibonacci Quarterly*, **37**, 98, 1999.
[3] Horadam, A.F., *American Math. Monthly*, **70**, 289, 1963.
[4] Liptai, K., *Fibonacci Quarterly*, **42**, 330, 2004.
[5] Liptai, K., *Acta Math. Univ. Ostrav.*, **14**, 43, 2006.
[6] Panda, G.K., Some fascinating properties of balancing numbers, *Proceeding of the eleventh international conference on Fibonacci numbers and their appl., Congr. Numer.*, **194**, 185, 2009.
[7] Ray, P.K., *Int. J. Pure Appl.Math.*, **85**, 487, 2013.
[8] Ray, P.K., *Acta et Commentationes Univ. Tartuensis de Mathematica*, **20**, 165, 2016.
[9] Ray, P.K., *Math. Rep. (bucur)*, **17**, 225, 2016.
[10] Szalay, L., *Ann. Math. Inform*, **34**, 77, 2007.
[11] Tasci, D., *Commun. Fac. Sci. Uni. Ank. Ser. A.*, **1**, 67, 2018.
[12] Tasci, D., *Communications in Mathematics and Appl.*, **4**, 213, 2013.