ORIGINAL PAPER

A NUMERICAL APPROACH FOR SOLVING PANTOGRAPH–TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MIXED DELAYS USING DICKSON POLYNOMIALS OF THE SECOND KIND

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Abstract. *In this study, a hybrid matrix-collocation method based on Dickson polynomials of the second kind along with Taylor polynomials is proposed to solve pantograph type functional differential equations with mixed delays under the initial conditions. The parameter-α in Dickson polynomials is interpreted for obtaining the optimum solutions. An error estimation related with the residual function and the mean-value theorem is implemented and also some illustrative examples are presented. It is observed that the proposed method is easy to be applied.*

Keywords: *Pantograph–type functional equations; Delay differential equations; Dickson polynomials; Matrix-collocation method.*

1. INTRODUCTION

In this study, we consider the high-order linear pantograph type functional differential equation with mixed delays (advanced, proportional or neutral delays) and variable coefficients [1-16]:

$$\sum_{k=0}^{m} P_k(x)\,y^{(k)}(x)+\sum_{r=0}^{m_1}\sum_{j=0}^{m_2} Q_{rj}(x)\,y^{(r)}\left(\lambda_j x+\mu_j\right)=g(x),\ m\geq m_1,\ a\leq x\leq b \tag{1}$$

subject to the initial conditions

$$y^{(k)}(a)=\alpha_k,\ k=0,1,2\ldots,m-1 \tag{2}$$

where $P_k(x)$, $Q_{rj}(x)$, $g(x)$ are known analytic functions defined on the interval $a\leq x\leq b$, also λ_j, μ_j and α_k are appropriate contants; $y(x)$ is an unknown function. On the other hand, if we take m=1, m_1=0, $P_1(x)$=1 and $\mu_j=0$, Eq. (1) is reduced to the multi-pantograph equation (P_0 is a constant) [11, 12]:

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$$y'(x)+P_0y(x)+\sum_{j=0}^{m_2}Q_{0j}(x)y(\lambda_j x)=g(x),$$

where $0<\lambda_0<\lambda_1<\ldots<\lambda_{m_2}<1$.

The functional differential equation (1) is a form of differential, differential-difference, delay differential and pantograph equations. These equations play an important role in many applied areas such as mathematics, engineering, electrodynamics, oscillation theory and etc. [1-22, 35-37]. Also, the behaviors of the analytic and numerical solutions of the pantograph type functional equation (1) have been investigated by many authors [1-16].

In recent years, many studies have been performed to obtain the numerical solution of functional differential equations. So, in order to find the approximate solutions of some types of Eq. (1), since the beginning of 1994; Taylor, Laguerre, Bessel, hybrid Euler-Taylor, Bernoulli, Chelyshkov and Dickson (first kind) matrix-collocation methods have been employed by Sezer et al. [3, 12-14, 17-22]. The other numerical methods are variational iteration [1, 2], homotopy perturbation [4, 15], Runge-Kutta [5, 11, 16], Adomian decomposition [6], collocation [9], interpolation [36] and one-leg θ methods [37].

Our aim in this study is to employ a novel matrix-collocation method, which is based on the Dickson polynomials of the second kind [23-33] along with the collocation points, to find the approximate solution of the problem (1)-(2) in the truncated Dickson series form

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}y_nE_n(x,\alpha) \tag{3}$$

where $E_n(x,\alpha)$ is the Dickson polynomial of the second kind with a parameter $\alpha\in R$ defined by

$$E_n(x,\alpha)=\sum_{p=0}^{\lfloor n/2\rfloor}\binom{n-p}{p}(-\alpha)^p\, x^{(n-2p)} \tag{4}$$

where $\lfloor\bullet\rfloor$ is the floor function [23, 24]. Some properties of the Dickson polynomials of the second kind are as follows:

- The second kind Dickson polynomials for n=0,1,2,... satisfy the recurrance relation [25]:

$$E_{n+2}(x,\alpha)=xE_{n+1}(x,\alpha)-\alpha E_n(x,\alpha),\ n\geq 0 \tag{5}$$

where the inital polynomials $E_0(x,\alpha)=1$ and $E_1(x,\alpha)=x$.

- $E_n(x,\alpha)$ has the generating function [25]:

$$\sum_{n=0}^{\infty}E_n(x,\alpha)z^n=\frac{1}{1-xz+\alpha z^2}.$$

- The polynomials $E_n(x,\alpha)$; n=0,1,2,… satisfy the second order differential equation [25]:

$$(x^2-4\alpha)E_n''(x,\alpha)+3xE_n'(x,\alpha)-n(n+2)E_n(x,\alpha)=0.$$

- By using the relation (4) or (5), the first five Dickson polynomials $E_n(x,\alpha)$ are given by [23-25, 33]

$E_0(x,\alpha)=1$
$E_1(x,\alpha)=x$
$E_2(x,\alpha)=x^2-\alpha$
$E_3(x,\alpha)=x^3-2\alpha x$
$E_4(x,\alpha)=x^4-3\alpha x^2+\alpha^2.$

Dickson polynomials were introduced by Dickson over finite fields [23]; Brewer restudied them [24]. These polynomials are widely used in mathematics, integer rings, finite fields, cryptography, algebraic and number theory, combinatorial designs, communication and storage [20-33]. Also, $E_n(x,\alpha)$ is closely related to the Chebyshev polynomials of the second kind $U_n(x)$, so their connection is $E_n(2x,1)=U_n(x)$ [25,33].

2. MATERIALS AND METHODS

2.1 MATRIX PROPERTIES OF DICKSON POLYNOMIALS OF THE SECOND KIND

In this section, we present the matrix forms of the solution function $y(x)$ defined by (3), the functional expression $y(\lambda_j x+\mu_j)$ and their derivatives in Eq. (1). These important properties will enable us to solve the functional differential equation (1).

Our purpose, we can first write the truncated Dickson series (3) in the matrix form, for n=0, 1, 2, ..., *N*,

$$y(x)\cong y_N(x)=\boldsymbol{E}(x,\alpha)\boldsymbol{Y}\,, \tag{6}$$

where

$$\boldsymbol{E}(x,\alpha)=\begin{bmatrix}E_0(x,\alpha) & E_1(x,\alpha) & \cdots & E_N(x,\alpha)\end{bmatrix} \text{ and } \boldsymbol{Y}=\begin{bmatrix}y_0 & y_1 & \cdots & y_N\end{bmatrix}^T.$$

Also, by using the Dickson polynomials $E_n(x,\alpha)$ defined by (4) or (5), the matrix $\boldsymbol{E}(x,\alpha)$ can be written as follows:

$$\boldsymbol{E}(x,\alpha)=\boldsymbol{X}(x)\boldsymbol{C}(\alpha), \tag{7}$$

where

$$\boldsymbol{X}(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix},\ \boldsymbol{E}^T(x,\alpha)=\boldsymbol{C}^T(\alpha)\boldsymbol{X}^T(x) \text{ and}$$

$$
\boldsymbol{C}(\alpha)=\begin{bmatrix}
\binom{0}{0}(-\alpha)^0 & 0 & \binom{1}{1}(-\alpha)^1 & 0 & \binom{2}{2}(-\alpha)^2 & . \;.\; . & \binom{\frac{N}{2}}{\frac{N}{2}}(-\alpha)^{N/2} & 0 \\
0 & \binom{1}{0}(-\alpha)^0 & 0 & \binom{2}{1}(-\alpha)^1 & 0 & . \;.\; . & 0 & \binom{\frac{N+1}{2}}{\frac{N-1}{2}}(-\alpha)^{\frac{N-1}{2}} \\
0 & 0 & \binom{2}{0}(-\alpha)^0 & 0 & \binom{3}{1}(-\alpha)^0 & . \;.\; . & \binom{\frac{N+2}{2}}{\frac{N-2}{2}}(-\alpha)^{\frac{N-2}{2}} & 0 \\
0 & 0 & 0 & \binom{3}{0}(-\alpha)^0 & 0 & . \;.\; . & 0 & \binom{\frac{N+3}{2}}{\frac{N-3}{2}}(\alpha)^{\frac{N-3}{2}} \\
0 & 0 & 0 & 0 & \binom{4}{0}(-\alpha)^0 & . \;.\; . & \binom{\frac{N+4}{2}}{\frac{N-4}{2}}(-\alpha)^{\frac{N-4}{2}} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & . \;\ddots\; . & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & . \;.\; . & \underbrace{\binom{N}{0}(-\alpha)^0}_{\text{Even } N} & \underbrace{\binom{N}{0}(-\alpha)^0}_{\text{Odd } N}
\end{bmatrix}.
$$

By the matix relations (6) and (7), we get

$$y_N(x)=\boldsymbol{X}(x)\boldsymbol{C}(\alpha)\boldsymbol{Y}\,. \tag{8}$$

On the other hand, it is well-known from [3,12-14,17-22] that the relation between the matrix $\boldsymbol{X}(x)$ and its derivative $\boldsymbol{X}^{(k)}(x)$ can be formed as

$$\boldsymbol{X}^{(k)}(x)=\boldsymbol{X}(x)\boldsymbol{B}^k\,, \tag{9}$$

where

$$
\boldsymbol{B}=\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & N \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

By using the matrix relations (8) and (9), we obtain

$$y_N^{(k)}(x)=\boldsymbol{X}^{(k)}(x)\boldsymbol{C}(\alpha)\boldsymbol{Y}=\boldsymbol{X}(x)\boldsymbol{B}^k\boldsymbol{C}(\alpha)\boldsymbol{Y}\,,\; k=0,1,\ldots,m\,. \tag{10}$$

By substituting $x\to\lambda_j x+\mu_j$ into (10) for $r=0,1,\ldots,m_1$, we have

$$y_N^{(r)}\left(\lambda_j x+\mu_j\right)=\boldsymbol{X}\left(\lambda_j x+\mu_j\right)\boldsymbol{B}^r\boldsymbol{C}(\alpha)\boldsymbol{Y}=\boldsymbol{X}(x)\boldsymbol{S}\left(\lambda_j,\mu_j\right)\boldsymbol{B}^r\boldsymbol{C}(\alpha)\boldsymbol{Y}\,, \tag{11}$$

where $\boldsymbol{B}^0$ is a unit matrix and

$$\boldsymbol{S}\left(\lambda_j,\mu_j\right)=\begin{bmatrix} \binom{0}{0}\lambda_j^0\mu_j^0 & \binom{1}{0}\lambda_j^0\mu_j^1 & \binom{2}{0}\lambda_j^0\mu_j^2 & \cdots & \binom{N}{0}\lambda_j^0\mu_j^N \\ 0 & \binom{1}{1}\lambda_j^1\mu_j^0 & \binom{2}{1}\lambda_j^1\mu_j^1 & \cdots & \binom{N}{1}\lambda_j^1\mu_j^{N-1} \\ 0 & 0 & \binom{2}{2}\lambda_j^2\mu_j^0 & \cdots & \binom{N}{2}\lambda_j^2\mu_j^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \binom{N}{N}\lambda_j^N\mu_j^0 \end{bmatrix}.$$

Note that the matrix $\boldsymbol{X}\left(\lambda_j x+\mu_j\right)$ can be written as [17-20]:

$$\boldsymbol{X}\left(\lambda_j x+\mu_j\right)=\boldsymbol{X}\left(x\right)\boldsymbol{S}\left(\lambda_j,\mu_j\right).$$

2.2 MATRIX-COLLOCATION METHOD

In this section, we construct a hybrid matrix-collocation method to solve the pantograph type functional differential equation (1) in terms of the Dickson polynomials of the second kind.

We first obtain the following matrix equation, by substituting the matrix relations (10) and (11) into Eq. (1):

$$\left(\sum_{k=0}^{m}\boldsymbol{P}_k\left(x\right)\boldsymbol{X}\left(x\right)\boldsymbol{B}^k+\sum_{r=0}^{m_1}\sum_{j=0}^{m_2}\boldsymbol{Q}_{rj}\left(x\right)\boldsymbol{X}\left(x\right)\boldsymbol{S}\left(\lambda_j,\mu_j\right)\boldsymbol{B}^r\right)\boldsymbol{C}\left(\alpha\right)\boldsymbol{Y}=g\left(x\right) \tag{12}$$

Then, by placing the collocation points defined by

$$x_i=a+\left(\frac{b-a}{N}\right)i,\ i=0,1,2,\ldots,N,\ a=x_0<x_1<\ldots<x_N=b$$

into Eq. (12), we obtain

$$\left(\sum_{k=0}^{m}\boldsymbol{P}_k\left(x_i\right)\boldsymbol{X}\left(x_i\right)\boldsymbol{B}^k+\sum_{r=0}^{m_1}\sum_{j=0}^{m_2}\boldsymbol{Q}_{rj}\left(x_i\right)\boldsymbol{X}\left(x_i\right)\boldsymbol{S}\left(\lambda_j,\mu_j\right)\boldsymbol{B}^r\right)\boldsymbol{C}\left(\alpha\right)\boldsymbol{Y}=g\left(x_i\right)$$

or briefly the compact form

$$\left(\sum_{k=0}^{m}\boldsymbol{P}_k\boldsymbol{X}\boldsymbol{B}^k+\sum_{r=0}^{m_1}\sum_{j=0}^{m_2}\boldsymbol{Q}_{rj}\boldsymbol{X}\boldsymbol{S}\left(\lambda_j,\mu_j\right)\boldsymbol{B}^r\right)\boldsymbol{C}\left(\alpha\right)\boldsymbol{Y}=\boldsymbol{G} \tag{13}$$

where

$$\boldsymbol{P}_{kj}=\begin{bmatrix} P_{kj}\left(x_0\right) & 0 & \cdots & 0 \\ 0 & P_{kj}\left(x_1\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{kj}\left(x_N\right) \end{bmatrix},\ \boldsymbol{Q}_{rj}=\begin{bmatrix} Q_{rj}\left(x_0\right) & 0 & \cdots & 0 \\ 0 & Q_{rj}\left(x_1\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{rj}\left(x_N\right) \end{bmatrix},$$

$$\boldsymbol{X}=\begin{bmatrix}\boldsymbol{X}(x_0)\\ \boldsymbol{X}(x_1)\\ \vdots\\ \boldsymbol{X}(x_N)\end{bmatrix}=\begin{bmatrix}1 & x_0 & \cdots & x_0^N\\ 1 & x_1 & \cdots & x_1^N\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_N & \cdots & x_N^N\end{bmatrix},\ \boldsymbol{G}=\begin{bmatrix}g(x_0) & g(x_1) & \cdots & g(x_N)\end{bmatrix}^T.$$

Here, the compact form (13) is the fundamental matrix equation for Eq. (1) and it can also be written as

$$\boldsymbol{WY}=\boldsymbol{G} \text{ or } [\boldsymbol{W}\ ;\ \boldsymbol{G}], \tag{14}$$

where

$$\boldsymbol{W}=[w_{pq}]=\left(\sum_{k=0}^{m}\boldsymbol{P}_k\boldsymbol{XB}^k+\sum_{r=0}^{m_1}\sum_{j=0}^{m_2}\boldsymbol{Q}_{rj}\boldsymbol{XS}(\lambda_j,\mu_j)\boldsymbol{B}^r\right)\boldsymbol{C}(\alpha),\ p,q=0,1,\ldots,N.$$

Also, the matrix equation (13) is equivalent to a system of (N+1) algebraic equations for the unknown coefficients $\{y_0,y_1,\ldots,y_N\}$. By means of the relation (10), we can now obtain the corresponding matrix form for the initial conditions (2) as

$$\boldsymbol{X}(a)\boldsymbol{B}^k\boldsymbol{C}(\alpha)\boldsymbol{Y}=\alpha_k \text{ or } \boldsymbol{U}_k\boldsymbol{Y}=\alpha_i \Rightarrow [\boldsymbol{U}_k\ ;\ \alpha_k],\ k=0,1,\ldots,m-1 \tag{15}$$

where

$$\boldsymbol{U}_k=\boldsymbol{X}(a)\boldsymbol{B}^k\boldsymbol{C}(\alpha)=[u_{k0}\quad u_{k1}\quad \ldots\quad u_{kN}].$$

Eventually, by substituting the row matrices (15) into the last (or any) m rows of the matrix system (14) and then we have the augmented matrix

$$[\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{G}}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1)\\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots\\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & g(x_{N-m})\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \alpha_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \alpha_1\\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots\\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \alpha_{m-1}\end{bmatrix}. \tag{16}$$

If $\operatorname{rank}\tilde{\boldsymbol{W}}=\operatorname{rank}[\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{G}}]=N+1$, then we can write

$$\boldsymbol{Y}=(\tilde{\boldsymbol{W}})^{-1}\tilde{\boldsymbol{G}}. \tag{17}$$

Thus, the coefficients matrix $\boldsymbol{Y}$ of the augmented matrix (16) is uniquely determined by Eq. (17); then the problem (1)-(2) has a unique solution. However, if $\operatorname{rank}\tilde{\boldsymbol{W}} \neq \operatorname{rank}[\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{G}}]$, then the augmented matrix (16) has no solution.

3. ERROR ESTIMATION BASED ON RESIDUAL FUNCTION: CONTROL OF SOLUTIONS

We can control the precision of the obtained solutions, since the finite Dickson serie (3) is an approximate solution of Eq. (1). When $y_N(x)$ is substituted into Eq. (1), we obtain the residual function for $x \in [a,b]$:

$$R_N(x) = \sum_{k=0}^{m} P_k(x) y_N^{(k)}(x) + \sum_{r=0}^{m_1} \sum_{j=0}^{m_2} Q_{rj}(x) y_N^{(r)}(\lambda_j x + \mu_j) - g(x) \cong 0 . \qquad (18)$$

Residual correction and its theory can be found in [34,35]. Residual error estimation has been employed by some authors [3,17-21]. Now, we describe residual error estimation based on the residual function for the Dickson polynomials of the second kind. The error function $e_N(x)$ is defined by

$$e_N(x) = y(x) - y_N(x) , \qquad (19)$$

where $y(x)$ is the exact solution of Eq. (1). By Eqs. (18) and (19), the error equation is

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x) , \qquad (20)$$

subject to the initial conditions

$$e_N^{(k)}(a) = 0 ,\; k = 0,1,\ldots,m-1. \qquad (21)$$

Eqs. (20) and (21) constitute the error problem solved with the same procedure in Section 2. Thus,

$$e_{N,M}(x) = \sum_{n=0}^{M} y_n^* E_n(x,\alpha) \;,\; (M > N).$$

Here, $e_{N,M}(x)$ is the estimated error function. Thus, the corrected solution is $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$ and the corrected error function is also defined by $E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y(x) - y_{N,M}(x)$.

On the other hand, by using the residual function $R_N(x)$ and the mean value of $|R_N(x)|$ on the interval [a,b], the precision of the solution can be analyzed and also the error bound can be calculated [17,18]. When N is sufficiently large enough, as $R_N(x) \to 0$, the error decreases. By means of the mean-value theorem, we can find the upper bound error $\bar{R}_N$ as follows [17, 18]:

i) $\left| \int_a^b R_N(x)\,dx \right| \le \int_a^b |R_N(x)|\,dx ,$

ii) $\int_a^b R_N(x)\,dx = (b-a) R_N(x_0) ,\; x_0 \in [a,b]$

iii) $\left|\int_a^b R_N(x)\,dx\right| = (b-a)\left|R_N(x_0)\right|$.

By (i) and (iii), we get

$$(b-a)\left|R_N(x_0)\right| \le \int_a^b \left|R_N(x)\right| dx \Rightarrow \left|R_N(x_0)\right| \le \frac{\int_a^b \left|R_N(x)\right| dx}{b-a} = \bar{R}_N .$$

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we solve some illustrative examples, by applying the present method to Eq. (1). The behavior of the approximate solutions is investigated by means of the parameter-α. The comparisons are made in tables and figures to show the efficiency and validity of the present method. Additionally, in order to compare the numerical results, we employ a distinctive error computation δ_N, which is defined by

$$\delta_N = \sqrt{\frac{1}{N}\sum_{i=0}^{N} e_N^2(x_i)}, \tag{22}$$

where $x_0, x_1, \ldots, x_N$ are the collocation points [36]. By means of (22) and the residual error estimation, we can also employ the corrected error $\delta_{N,M}$, which is as follows:

$$\delta_{N,M} = \sqrt{\frac{1}{M}\sum_{i=0}^{M} E_{N,M}^2(x_i)},$$

where $E_{N,M}(x)$ is the corrected error function.

Example 4.1 Consider the second order linear differential equation with variable coefficients

$$y''(x) - xy'(x) + y(x) = 3 - x^2,\ 0 \le x \le 2$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$. The exact solution of this problems is $y(x) = x^2 + 1$. Here,

$$P_0(x) = P_2(x) = 1,\ P_1(x) = -x,\ g(x) = 3 - x^2.$$

We seek the approximate solution with truncated Dickson polynomials for N=2:

$$y(x) = \sum_{n=0}^{2} y_n E_n(x,\alpha).$$

The fundamental matrix equation of this problem is

$$\left\{\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0 + \boldsymbol{P}_1\boldsymbol{X}\boldsymbol{B}^1 + \boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2\right\}\boldsymbol{C}(\alpha)\boldsymbol{Y} = \boldsymbol{G}$$

When this is solved, the matrix system is obtained as

$$[\boldsymbol{W};\ \boldsymbol{G}] = \begin{bmatrix} 1 & 0 & 2-\alpha & ; & 3 \\ 1 & 0 & 1-\alpha & ; & 2 \\ 1 & 0 & -2-\alpha & ; & -1 \end{bmatrix}.$$

The matrix forms of the initial conditions are

$$[U_0;\ \alpha_0] = [1 \quad 0 \quad -\alpha \quad ; \quad 1] \text{ and } [U_1;\ \alpha_1] = [0 \quad 1 \quad 0 \quad ; \quad 0].$$

If we write these conditions into the matrix system, then we find the augmented matrix

$$[\tilde{\boldsymbol{W}};\ \tilde{\boldsymbol{G}}] = \begin{bmatrix} 1 & 0 & 2-\alpha & ; & 3 \\ 1 & 0 & -\alpha & ; & 1 \\ 0 & 1 & 0 & ; & 0 \end{bmatrix}$$

By solving this matrix, we get the coefficient matrix

$$\boldsymbol{Y} = \begin{bmatrix} 1+\alpha \\ 0 \\ 1 \end{bmatrix}$$

and thus

$$y(x) = [1 \quad x \quad x^2-\alpha] \begin{bmatrix} 1+\alpha \\ 0 \\ 1 \end{bmatrix}.$$

This yields the exact solution of the problem.

Example 4.2 [2] Consider the first-order neutral differential equation with proportional delay

$$y'(x) - \frac{1}{2} y'\left(\frac{x}{2}\right) - \frac{1}{2} y\left(\frac{x}{2}\right) + y(x) = 0,\ 0 \le x \le 1$$

subject to the initial condition $y(0)=1$. The exact solution of this problem is $y(x)=e^{-x}$. We solve this problem for N={2, 5}; M=6. Thus, we find the approximate solutions $y_2(x)\,|_{\alpha=1}$, $y_5(x)\,|_{\alpha=1}$ and the corrected approximate solution $y_{5,6}(x)\,|_{\alpha=1}$:

$$y_2(x)\,|_{\alpha=1} = 1 - x + 0.387097x^2,$$

$$y_5(x)\,|_{\alpha=1} = 1 - x + 0.499859x^2 - 0.165780x^3 + 0.039387x^4 - 0.005601x^5 \text{ and}$$

$$y_{5,6}(x)\,|_{\alpha=1} = 1 - x + 0.5x^2 - 0.166590x^3 + 0.041394x^4 - 0.007834x^5 + 0.000920x^6.$$

These approximate solutions are compared with the exact solution in Table 1 and Fig. 1. As seen from Table 1 and Fig. 1, our solutions are in harmony with the exact solution and also the errors δ_N are decreased by the values of N and residual error estimation. Notice that when $\alpha = 5\times10^3$, our solution deviates from its normal way; this situation can be observed in Table 1 and Fig. 1.

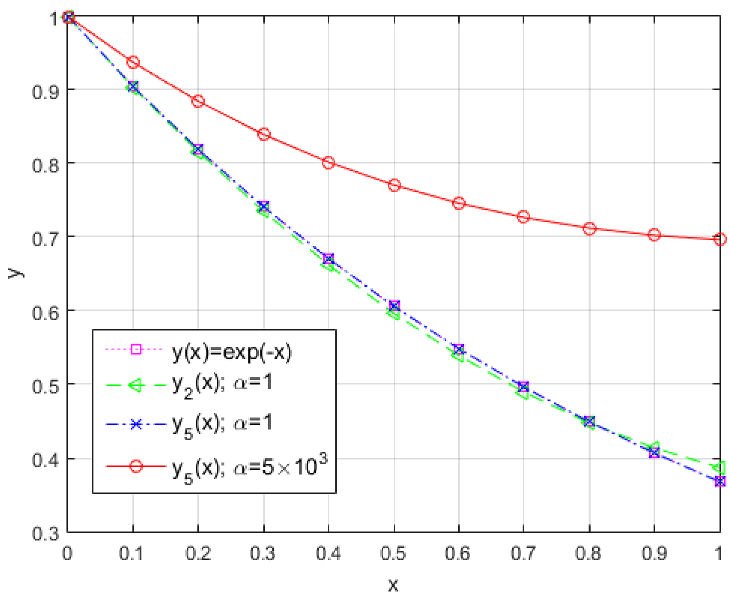

Figure 1. Comparison of the exact and the approximate solutions of Example 4.2 for different α.

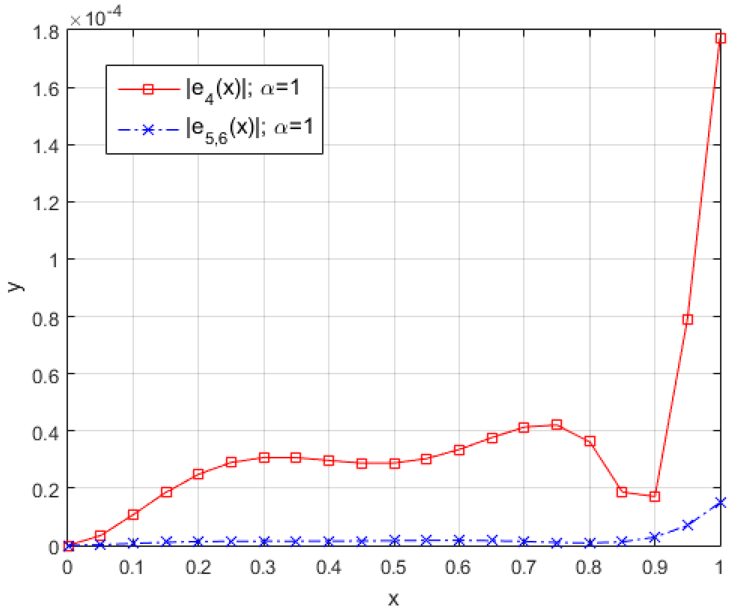

Figure 2. Comparison of the actual and estimated absolute errors of Example 4.2 for *N*=4 and 5; *M*=6.

In Table 2, the present results are compared with those of One-leg θ with $\theta = 0.8$ and *h*=0.01 [2, 37], two stage order Runge-Kutta [2, 16] and variational iteration methods (VIM) (*n*=8) [2]. It is seen from Table 2 that our results are better than those of the mentioned methods. Fig. 2 shows that the estimated absolute errors are more consistent than the actual absolute errors. The upper error bounds $\bar{R}_N$ are calculated as follows:

$$\left\{\bar{R}_2,\ \bar{R}_3,\ \bar{R}_4,\ \bar{R}_5,\ \bar{R}_{5,6}\right\} = \left\{4.23e-02,\ 3.80e-03,\ 2.71e-04,\ 1.62e-05,\ 8.40e-07\right\}.$$

Table 1. Comparison of the exact, approximate, corrected approximate solutions and errors δ_N for Example 4.2.

x_i	$y(x_i)=e^{-x_i}$	$y_2(x_i)$ $\alpha=1$	$y_5(x_i)$ $\alpha=1$	$y_5(x_i)$ $\alpha=5\times10^3$	$y_{5,6}(x_i)$ $\alpha=1$
0.0	1	1	1	1	1
0.2	0.81873075	0.815483871	0.81872935	0.88443758	0.81873069
0.4	0.67032005	0.661935484	0.67031847	0.80173075	0.67031997
0.6	0.54881164	0.539354839	0.54880980	0.74591769	0.54881156
0.8	0.44932897	0.447741936	0.44932802	0.71211892	0.44932887
1.0	0.36787944	0.387096774	0.36786509	0.69632218	0.36788006
δ_N	-	$1.52e-02$	$6.55e-06$	$2.18e-01$	$2.62e-07$

Table 2. Comparison of the actual and corrected absolute errors for Example 4.2.

x_i	$\lvert e_5(x_i)\rvert$ $\alpha=1$	$\lvert E_{5,6}(x_i)\rvert$ $\alpha=1$	**One-leg θ met. [2,37]**	**Runge-Kutta met. [2,16]**	**VIM [2] *n*=8**
0.1	$7.26e-07$	$4.00e-08$	$2.57e-03$	$4.55e-04$	$3.72e-04$
0.2	$1.41e-06$	$6.65e-08$	$8.86e-03$	$8.24e-04$	$7.08e-04$
0.3	$1.55e-06$	$7.14e-08$	$1.72e-02$	$1.12e-03$	$1.01e-03$
0.4	$1.57e-06$	$7.77e-08$	$2.66e-02$	$1.35e-03$	$1.29e-03$
0.5	$1.75e-06$	$8.10e-08$	$3.63e-02$	$1.52e-03$	$1.54e-03$
0.6	$1.84e-06$	$7.24e-08$	$4.58e-02$	$1.66e-03$	$1.76e-03$
0.7	$1.45e-06$	$6.92e-08$	$5.47e-02$	$1.75e-03$	$1.97e-03$
0.8	$9.42e-07$	$9.46e-08$	$6.29e-02$	$1.81e-03$	$2.15e-03$
0.9	$2.95e-06$	$4.29e-08$	$7.02e-02$	$1.84e-03$	$2.32e-03$
1.0	$1.44e-05$	$6.18e-07$	$7.66e-02$	$1.85e-03$	$2.47e-03$

Example 4.3 Consider the second-order pantograph type functional differential equation with mixed delays

$$y''(x)+x^2y'(x+1)-(x+1)y\left(\frac{x}{2}\right)=g(x),\ 0\le x\le 1$$

subject to the initial conditions $y(0)=1$ and $y'(0)=0$. The exact solution of this problem is $y(x)=\cos(x)$. Here,

$$g(x)=(-1-x)\cos\left(\frac{x}{2}\right)-\cos(x)-x^2\sin(x+1).$$

We find the approximate solutions for N={2, 3, 7, 10} and M=8. The approximate solutions are improved by means of N and residual error estimation as seen in Table 3. For different α, the approximate solutions $y_2(x)$ and $y_7(x)$ are plotted along with the exact solution in Fig. 3. In addition, in Fig. 4, the behavior of the approximate solution $y_{10}(x)$

obtained on the interval [0, 1] is compared with the exact solution on [0,5]. The upper error bounds $\bar{R}_N$ are calculated as follows:

$$\left\{\bar{R}_3,\ \bar{R}_5,\ \bar{R}_7,\ \bar{R}_{10}\right\}=\left\{2.06e-01,\ 3.00e-03,\ 1.62e-05,\ 5.60e-07\right\}.$$

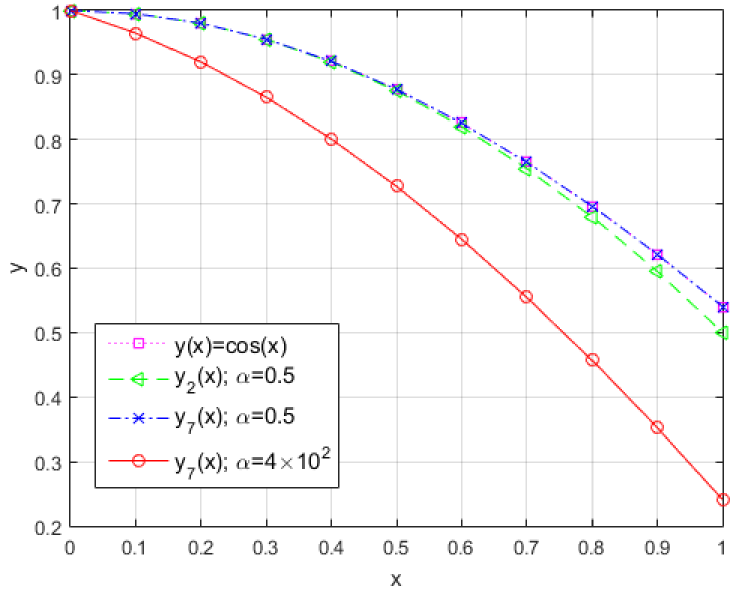

Figure 3. Comparison of the exact and the approximate solutions of Example 4.3 for different α.

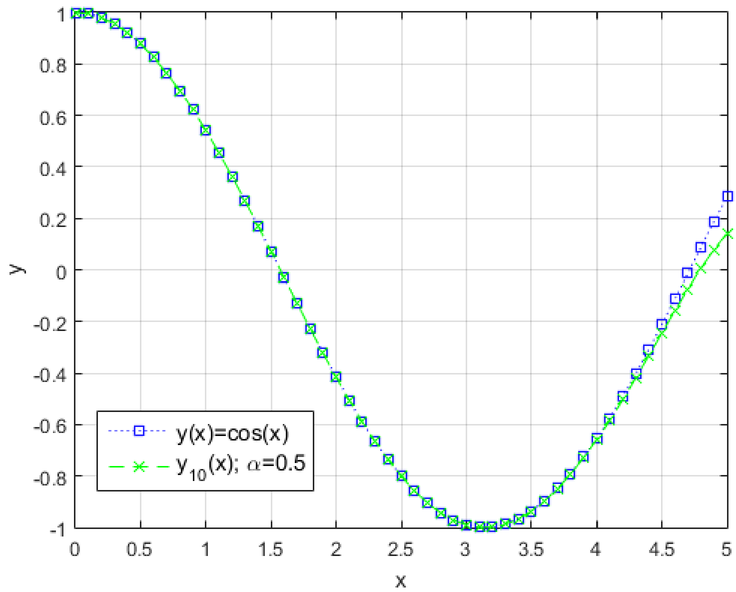

Figure 4. The behavior of the exact and the approximate solutions on [0,5] for Example 4.3.

Table 3. Comparison of the actual, estimated, corrected absolute errors and δ_N for Example 4.3.

x_i	$\|e_3(x_i)\|$ $\alpha=0.5$	$\|e_7(x_i)\|$ $\alpha=0.5$	$\|e_{7,8}(x_i)\|$ $\alpha=0.5$	$\|E_{7,8}(x_i)\|$ $\alpha=0.5$
0.0	0	$2.20e-16$	$2.71e-20$	$2.22e-16$
0.2	$2.28e-04$	$8.69e-09$	$8.71e-09$	$1.53e-11$
0.4	$1.29e-03$	$4.85e-08$	$5.52e-08$	$6.63e-09$
0.6	$2.61e-03$	$1.05e-08$	$6.16e-08$	$5.11e-08$
0.8	$2.13e-03$	$8.44e-07$	$6.82e-07$	$1.63e-07$
1.0	$3.51e-03$	$3.62e-06$	$3.37e-06$	$2.46e-07$
δ_N	$2.64e-03$	$1.47e-06$	-	$1.26e-07$

5. CONCLUSIONS

In this study, a matrix-collocation method based on the second kind Dickson polynomials has been introduced to solve high-order linear pantograph type functional differential equation with mixed delays under the initial conditions. An error estimation based on residual function has been implemented to improve the accuracy of the approximate solutions. As seen from tables and figures, this error estimation is very useful. The parameter-α can be taken on [0,1] for obtaining optimal approximate solutions. As N is increased, the upper error bounds $\bar{R}_N$ are decreased as seen in Examples 4.2 and 4.3. In addition, we have compared the present numerical results with those obtained by the existed methods in Table 2 and the obtained approximate solutions are consistent with the exact solutions as seen in Figs. 1, 3 and 4. It can be seen from the comparisons and error estimations that this method is very consistent and reliable. This method can also be developed for other well-known problems, such as integro-differential equations.

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