ORIGINAL PAPER

BEHAVIOUR OF SOLUTIONS FOR A SYSTEM OF TWO HIGHER-ORDER DIFFERENCE EQUATIONS

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Abstract. *In this paper, we investigate the global behavior of the positive solutions of the system of difference equations*

$$u_{n+1}=\frac{au_{n-k}}{b+c\prod_{i=0}^{k}v_{n-i}^{r}},\ v_{n+1}=\frac{dv_{n-k}}{e+f\prod_{i=0}^{k}u_{n-i}^{r}},\ n\in\mathbb{N}_0,$$

where the initial conditions u_{-i}, v_{-i}, $(i=0,\dots,k)$, *and the parameters* a, b, c, d, e, f, r *are positive real numbers, by extending some recent results in the literature. Also, we estimate the rate of convergence of a solution that converges to the zero equilibrium point of the above mentioned system.*

Keywords: *System of difference equations, Stability, Global behavior, Periodic solution, Rate of convergence.*

1. INTRODUCTION

Theory of difference equations have gained a great importance for several decades. Most of the recent applications of this theory have appeared in many scientific areas such as biology, physics, engineering, economics. Particularly, rational difference equations and their systems of higher order have great importance in applications. It is very worthy to examine the behavior of solutions of a system of higher-order rational difference equations and to discuss the stability character of their equilibrium points. Recently, many researchers have investigated global behavior of solutions of difference equations or systems and have suggested some diverse methods for the qualitative behavior of the their solutions. For example, Shojaei et al.[18] investigated the general solution, the local and the global asymptotic stability of equilibrium points and period three cycles of the third order rational difference equation

$$x_{n+1}=\frac{\alpha x_{n-2}}{\beta+\gamma x_n x_{n-1}x_{n-2}},\ n\in\mathbb{N}_0,$$

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where the parameters α, β, γ and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. Dehghan et al. [4] investigated the stability, the periodic character and the boundedness nature of solutions of the following difference equation

$$y_{n+1} = \frac{\alpha y_{n-2}}{\beta + \gamma y_n^k y_{n-1}^k y_{n-2}^k}, \; n \in \mathbb{N}_0,$$

where the parameters α, β, γ and the initial conditions y_{-2}, y_{-1}, y_0 are positive real numbers and $k \geq 2$ is a fixed integer. Clark et al. [2,3] investigated the system of rational difference equations

$$x_{n+1} = \frac{x_n}{a + cy_n}, \; y_{n+1} = \frac{y_n}{b + dx_n}, \; n \in \mathbb{N}_0,$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers. Yang et al. [29] investigated global behavior of the system of difference equations

$$x_{n+1} = \frac{Ax_n}{1 + y_n^p}, \; y_{n+1} = \frac{By_n}{1 + x_n^p}, \; n \in \mathbb{N}_0,$$

where the parameters A, B, p and the initial conditions x_0, y_0 are nonnegative real numbers. Zhang et al. [36] investigated the solutions, stability character, and asymptotic behavior of the system of difference equations

$$x_{n+1} = \frac{x_{n-k}}{q + \prod_{i=0}^{k} y_{n-i}}, \; y_{n+1} = \frac{y_{n-k}}{p + \prod_{i=0}^{k} x_{n-i}}, \; n \in \mathbb{N}_0,$$

where the parameters p, q and the initial conditions $x_{-i}, y_{-i}, (i = 0, 1, \ldots, k)$, are nonnegative real numbers. For more works related to difference equations and their systems, see references [1, 5-7, 9-14, 16, 19-24, 26-28, 30-35]. In the present paper, we investigate the global behavior of the positive solutions of the system of difference equations

$$u_{n+1} = \frac{au_{n-k}}{b + c\prod_{i=0}^{k} v_{n-i}^r}, \; v_{n+1} = \frac{dv_{n-k}}{e + f\prod_{i=0}^{k} u_{n-i}^r}, \; n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions u_{-i}, v_{-i} $(i = 0, 1, \ldots, k)$ and the parameters a, b, c, d, e, f, r are positive real numbers. Also, we estimate the rate of convergence of a solution that converges to the zero equilibrium point of the above mentioned system.

Note that system (1.1) can be written as

$$x_{n+1} = \frac{\alpha x_{n-k}}{1 + \prod_{i=0}^{k} y_{n-i}^r}, \; y_{n+1} = \frac{\beta y_{n-k}}{1 + \prod_{i=0}^{k} x_{n-i}^r}, \; n \in \mathbb{N}_0, \tag{1.2}$$

if the change of variables $u_n = \left(\frac{e}{f}\right)^{1/r(k+1)} x_n,\ v_n = \left(\frac{b}{c}\right)^{1/r(k+1)} y_n$ is applied to the system such that $\alpha = \dfrac{a}{b}$ and $\beta = \dfrac{d}{e}$. So, we will consider system (1.2) instead of system (1.1) from now.

2. PRELIMINARIES

Let I, J be some intervals of real numbers and $f : I^{k+1} \times J^{k+1} \to I$, $g : I^{k+1} \times J^{k+1} \to J$ be continuously differentiable functions. Then, for every initial conditions $(x_{-i}, y_{-i}) \in I \times J$, $(i = 0,1,\ldots,k)$, the system of difference equations

$$\begin{cases} x_{n+1} = f\left(x_n,\ldots,x_{n-k},y_n,\ldots,y_{n-k}\right) \\ y_{n+1} = g\left(x_n,\ldots,x_{n-k},y_n,\ldots,y_{n-k}\right) \end{cases}, \ n \in \mathbb{N}_0, \tag{2.1}$$

has the unique solution $\{x_n, y_n\}_{n=-k}^{\infty}$. Also, an equilibrium point of system (2.1) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f\left(\bar{x},\ldots,\bar{x},\bar{y},\ldots,\bar{y}\right),$$
$$\bar{y} = g\left(\bar{x},\ldots,\bar{x},\bar{y},\ldots,\bar{y}\right).$$

We rewrite system (2.1) in the vector form

$$X_{n+1} = F\left(X_n\right),\ n \in \mathbb{N}_0, \tag{2.2}$$

where $X_n = \left(x_n,\ldots,x_{n-k},y_n,\ldots,y_{n-k}\right)^T$, F is a vector map such that $F : I^{k+1} \times J^{k+1} \to I^{k+1} \times J^{k+1}$ and

$$F\left(\begin{pmatrix} z_0 \\ \vdots \\ z_k \\ t_0 \\ \vdots \\ t_k \end{pmatrix}\right) = \begin{pmatrix} f\left(z_0,\ldots,z_k,t_0,\ldots,t_k\right) \\ \vdots \\ z_{k-1} \\ g\left(z_0,\ldots,z_k,t_0,\ldots,t_k\right) \\ \vdots \\ t_{k-1} \end{pmatrix}.$$

It is clear that if an equilibrium point of system (2.1) is $(\bar{x}, \bar{y})$, then the corresponding equilibrium point of system (2.2) is the point $\bar{X} = \left(\bar{x},\ldots,\bar{x},\bar{y},\ldots,\bar{y}\right)^T$.

In this study, we denote by $\|.\|$ any convenient vector norm and the corresponding matrix norm. Also, we denote by $X_0 \in I^{k+1} \times J^{k+1}$ a initial condition of system (2.2).

Definition 2.1. *Let* $\overline{X}$ *be an equilibrium point of system* (2.2).

i) *The equilibrium point* $\overline{X}$ *is called stable if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\left\| X_0 - \overline{X} \right\| < \delta$ *implies* $\left\| X_n - \overline{X} \right\| < \varepsilon$, *for all* $n \geq 0$. *Otherwise the equilibrium point* $\overline{X}$ *is called unstable.*

ii) *The equilibrium point* $\overline{X}$ *is called local asymptotically stable if it is stable and there exists* $\gamma > 0$ *such that* $\left\| X_0 - \overline{X} \right\| < \gamma$ *and* $X_n \to \overline{X}$ *as* $n \to \infty$.

iii) *The equilibrium point* $\overline{X}$ *is called a global attractor if* $X_n \to \overline{X}$ *as* $n \to \infty$.

iv) *The equilibrium point* $\overline{X}$ *is called globally asymptotically stable if it is both local asymptotically stable and global attractor.*

The linearized system of system (2.2) evaluated at the equilibrium point $\overline{X}$ is

$$Z_{n+1} = J_F Z_n, \ n \in \mathbb{N}_0, \tag{2.3}$$

where J_F is the Jacobian matrix of the map F at the equilibrium point $\overline{X}$. The characteristic polynomial of system (2.3) about the equilibrium point $\overline{X}$ is

$$P(\lambda) = a_0 \lambda^{2(k+1)} + a_1^{2k+1} \lambda + \cdots + a_{2k+1} \lambda + a_{2(k+1)}, \tag{2.4}$$

with real coefficients and $a_0 > 0$.

Theorem 2.2. *[15] Assume that* $\overline{X}$ *is a equilibrium point of system* (2.2), *i.e.,* $\overline{X} = F(\overline{X})$. *If all eigenvalues of the Jacobian matrix* J_F *evaluated at* $\overline{X}$ *lie in the open unit disk* $|\lambda| < 1$, *then* $\overline{X}$ *is locally asymptotically stable. If one of them has a modulus greater than one, then* $\overline{X}$ *is unstable.*

Theorem 2.3. (Schur-Cohn Criterion) *[8] The zeros of the characteristic polynomial* (2.4) *lie in the unit disk if and only if the followings hold:*

i) $P(1) > 0$,

ii) $(-1)^{2(k+1)} P(-1) > 0$,

iii) *the* $2k \times 2k$ *matrices*

$$B_{2k}^{\pm} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ p_1 & 1 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ p_{2k-2} & & & 1 & 0 \\ p_{2k-1} & p_{2k-2} & \cdots & p_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & p_{2k+1} \\ 0 & 0 & \cdots & p_{2k+1} & p_{2k} \\ \vdots & \vdots & & & \vdots \\ 0 & p_{2k+1} & & & p_3 \\ p_{2k+1} & p_{2k} & \cdots & p_3 & p_2 \end{pmatrix}$$

are positive innerwise.

The following results [8,17] give the rate of convergence for solutions of a system of difference equations.

Let us consider the system of difference equations

$$X_{n+1} = \left(A + B_n \right) X_n, \ n \in \mathrm{N}_0, \tag{2.5}$$

where X_n is an m-dimensional vector $A \in C^{m\times m}$ is a constant matrix, and $B : Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B_n\| \to 0 \tag{2.6}$$

as $n \to \infty$.

Theorem 2.4. (Perron's First Theorem) *Suppose that condition* (2.6) *holds. If* X_n *is a solution of* (2.5), *then either* $X_n = 0$ *for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{2.7}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Thereom 2.5 (Perron's Second Theorem) *Suppose that condition* (2.6) *holds. If* X_n *is a solution of* (2.5), *then either* $X_n = 0$ *for all large n or*

$$\rho = \lim_{n\to\infty} \left(\|X_n\|\right)^{1/n} \tag{2.8}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

3. STABILITY OF THE SYSTEM

In this section, we investigate the local and the global stability of the equilibrium points of system (1.2). It is easy to see that $(\overline{x}_1, \overline{y}_1) = (0,0)$ is always an equilibrium point of system (1.2). When $\alpha > 1$ and $\beta > 1$, the unique positive equilibrium point of system is $(\overline{x}_2, \overline{y}_2) = \left((\alpha-1)^{1/r(k+1)}, (\beta-1)^{1/r(k+1)}\right)$. In addition, if $\alpha = 1$ and $\beta \neq 1$, then $(\overline{x}_3, \overline{y}_3) = (c_1, 0)$ and if $\alpha \neq 1$ and $\beta = 1$, then $(\overline{x}_4, \overline{y}_4) = (0, c_2)$, where c_1 and c_2 are arbitrary real numbers.

Theorem 3.1. *The following statements hold:*

i) *If* $\alpha < 1$ *and* $\beta < 1$, *then the equilibrium point* $(\overline{x}_1, \overline{y}_1) = (0,0)$ *of system* (1.2) *is locally asymptotically stable.*

ii) *If* $\alpha > 1$ *or* $\beta > 1$, *then the equilibrium point* $(\overline{x}_1, \overline{y}_1) = (0,0)$ *of system* (1.2) *is unstable.*

iii) *If* $\alpha > 1$ *and* $\beta > 1$, *then the equilibrium point* $(\overline{x}_2, \overline{y}_2) = \left((\alpha-1)^{1/r(k+1)}, (\beta-1)^{1/r(k+1)}\right)$ *of system* (1.2) *is unstable*

Proof: First, we write system (1.2) in the form of system (2.2) such that $X_n = \left(x_n, \ldots, x_{n-k}, y_n, \ldots, y_{n-k}\right)^T$, the map F is

$$F\left(\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_k \\ t_0 \\ t_1 \\ \vdots \\ t_k \end{pmatrix}\right) = \begin{pmatrix} \alpha z_k / (1+\prod_{i=0}^{k} t_i^r) \\ z_0 \\ \vdots \\ z_{k-1} \\ \alpha t_k / (1+\prod_{i=0}^{k} z_i^r) \\ t_0 \\ \vdots \\ t_{k-1} \end{pmatrix}.$$

i) The linearized system of (1.2) about the equilibrium point $\overline{X}_0 = (0,\ldots,0)^T$ is given by

$$X_{n+1} = J_F(\overline{X}_0) X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ \vdots \\ x_{n-k} \\ y_n \\ \vdots \\ y_{n-k} \end{pmatrix} \text{ and } J_F(\overline{X}_0) = \begin{pmatrix} 0 & \cdots & \alpha & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \beta \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}_{(2k+2)\times(2k+2)}.$$

The characteristic polynomial of $J_F(\overline{X}_0)$ is given by

$$\left(\lambda^{k+1} - \alpha\right)\left(\lambda^{k+1} - \beta\right) = 0. \tag{3.1}$$

It is easy to see that if $\alpha < 1$ and $\beta < 1$, then all the roots of the characteristic equation (3.1) lie in the open unit disk $|\lambda| < 1$. So, the unique equilibrium point $(\overline{x}_1, \overline{y}_1) = (0,0)$ of (1.2) is locally asymptotically stable.

ii) It is clearly seen that if $\alpha > 1$ or $\beta > 1$, then at least one root of (3.1) has a modulus greater than one. In this case, the equilibrium point $(\overline{x}_1, \overline{y}_1) = (0,0)$ of (1.2) is unstable.

iii) The linearized system of (1.2) about the positive equilibrium point

$$\overline{X}_{\alpha,\beta} = \left((\alpha-1)^{1/r(k+1)},\ldots,(\alpha-1)^{1/r(k+1)},(\beta-1)^{1/r(k+1)},\ldots,(\beta-1)^{1/r(k+1)}\right)^T$$

is given by

$$X_{n+1} = J_F(\overline{X}_{\alpha,\beta}) X_n,$$

where

$$X_n=\begin{pmatrix} x_n\\ \vdots\\ x_{n-k}\\ y_n\\ \vdots\\ y_{n-k}\end{pmatrix},\ J_F(\overline{X}_{\alpha,\beta})=\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 & A & \cdots & A & A\\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
B & B & B & \cdots & B & 0 & 0 & \cdots & 0 & 1\\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}_{(2k+2)\times(2k+2)},$$

$$A=\frac{r(\alpha-1)(\beta-1)^{1/r(k+1)}}{\alpha(\alpha-1)^{1/r(k+1)}} \text{ and } B=\frac{r(\beta-1)(\alpha-1)^{1/r(k+1)}}{\beta(\beta-1)^{1/r(k+1)}}.$$

The characteristic polynomial of $J_F(\overline{X}_{\alpha,\beta})$ is given by

$$P(\lambda)=\lambda^{2k+2}-AB\lambda^{2k}-2AB\lambda^{2k-1}-\ldots-(kAB+2)\lambda^{k+1}-(k+1)AB\lambda^{k}-kAB\lambda^{k-1}-\ldots-2AB\lambda-AB+1$$

or

$$P(\lambda)=\left(\lambda^{k+1}-1\right)^2-\frac{r^2(\alpha-1)(\beta-1)}{\alpha\beta}\left(\lambda^k\sum_{i=0}^{k}(i+1)\lambda^{k-i}+\sum_{i=0}^{k-1}(k-i)\lambda^{k-i-1}\right). \tag{3.2}$$

We get $P(1)=\frac{-r^2(k+1)^2(\alpha-1)(\beta-1)}{\alpha\beta}<0$ from characteristic polynomial (3.2) if $\alpha>1$ and $\beta>1$. So, from Theorem 2.2 and Theorem 2.3, we can say that if $\alpha>1$ and $\beta>1$, then the positive equilibrium point $\left((\alpha-1)^{1/r(k+1)},(\beta-1)^{1/r(k+1)}\right)$ of system (1.2) is unstable.■

Theorem 3.2. *If $\alpha<1$ and $\beta<1$, then the equilibrium point $(\overline{x}_1,\overline{y}_1)=(0,0)$ of system (1.2) is globally asymptotically stable.*

Proof: From Theorem 3.1, we know that if $\alpha<1$ and $\beta<1$, then the equilibrium point $(\overline{x}_1,\overline{y}_1)=(0,0)$ of system (1.2) is locally asymptotically stable. Hence we only show that

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=0. \tag{3.3}$$

From system (1.2), we have that

$$0<x_{n+1}=\frac{\alpha x_{n-k}}{1+\prod_{i=0}^{k}y_{n-i}^r}<\alpha x_{n-k},\ 0<y_{n+1}=\frac{\beta y_{n-k}}{1+\prod_{i=0}^{k}x_{n-i}^r}<\beta y_{n-k}, \tag{3.4}$$

for $n\in\mathbb{N}_0$. From (3.4), we have the inequalities

$$0 < x_n < \alpha x_{n-k},\ 0 < y_{n+1} < \beta y_{n-k}$$

from which it follows that

$$0 < x_{nk-i} < \alpha^n x_{-i},\ 0 < y_{nk-i} < \beta^n y_{-i}, \tag{3.5}$$

where $x_{-i}, y_{-i}, i = 0,1,\ldots,k,$ are the initial conditions. Consequently, by taking limits of inequalities in (3.5) when $\alpha < 1$ and $\beta < 1$, ~~then~~ we have the limits in (3.3) which completes the proof.■

Theorem 3.3. *Let* $\alpha > 1$ *and* $\beta > 1$. *If* $x_{-k}x_{-k+1}\cdots x_0 = (\beta-1)^{1/r}$ *and* $y_{-k}y_{-k+1}\cdots y_0 = (\alpha-1)^{1/r}$, *then System* (1.2) *has the periodic solutions with period* $k+1$.

Proof: First, we note that if $x_{-k}x_{-k+1}\cdots x_0 = (\beta-1)^{1/r}$ and $y_{-k}y_{-k+1}\cdots y_0 = (\alpha-1)^{1/r}$, then $x_{-i}y_{-i} > 0\ (i = 0,1,\ldots,k)$ and so $x_n y_n > 0$ for $n \in \mathbb{N}_0$. Thus, we can multiply both sides of the first equation of (1.2) by $\prod_{i=0}^{k-1} x_{n-i}$ and both sides of the second equation of (1.2) by $\prod_{i=0}^{k-1} y_{n-i}$ as follows:

$$\prod_{i=0}^{k} x_{n-i+1} = \frac{\alpha \prod_{i=0}^{k} x_{n-i}}{1+\prod_{i=0}^{k} y_{n-i}^r},\ \prod_{i=0}^{k} y_{n-i+1} = \frac{\beta \prod_{i=0}^{k} y_{n-i}}{1+\prod_{i=0}^{k} x_{n-i}^r},\ n \in \mathbb{N}_0. \tag{3.6}$$

By applying the change of variables

$$\prod_{i=0}^{k} x_{n-i} = t_n,\ \prod_{i=0}^{k} y_{n-i} = z_n \tag{3.7}$$

to (3.6), we get the system

$$t_{n+1} = \frac{\alpha t_n}{1+z_n^r},\ z_{n+1} = \frac{\beta z_n}{1+t_n^r},\ n \in \mathbb{N}_0. \tag{3.8}$$

On the other hand, the changes of variables (3.7) yields the equations

$$x_n = \frac{t_n}{t_{n-1}} x_{n-k+1},\ y_n = \frac{z_n}{z_{n-1}} y_{n-k+1},\ n \in \mathbb{N}_0. \tag{3.9}$$

Obviously, the nonzero equilibrium solutions of (3.8) satisfies (3.9). If $\alpha > 1$ and $\beta > 1$, then (3.8) has the positive equilibrium point $(\overline{t}_2, \overline{z}_2) = \left((\beta-1)^{1/r}, (\alpha-1)^{1/r}\right)$. The equilibrium solution $(t_n, z_n) = \left((\beta-1)^{1/r}, (\alpha-1)^{1/r}\right)$ satisfies Eq. (3.9). In this case, from the assumptions $x_{-k}x_{-k+1}\cdots x_0 = (\beta-1)^{1/r}$, $y_{-k}y_{-k+1}\cdots y_0 = (\alpha-1)^{1/r}$ and system (1.2), we get that

$$x_{n+1} = x_{n-k},\ y_{n+1} = y_{n-k},\ n \in \mathbb{N}_0,$$

which implies that system (1.2) has the periodic solutions with period $k+1$. So, the proof is completed. ■

4. OSCILLATION BEHAVIOR AND EXISTENCE OF UNBOUNDED SOLUTIONS

In the following result, we are concerned with the oscillation of positive solutions of system (1.2) about the equilibrium point $(\overline{x}_2, \overline{y}_2) = \left((\alpha-1)^{1/r(k+1)}, (\beta-1)^{1/r(k+1)}\right)$.

Theorem 4.1. *Assume that* $\alpha > 1, \beta > 1$ *and* $(x_n, y_n)_{n=-k}^{\infty}$ *be a positive solution of system* (1.2) *such that:*

i) $x_{-k}, x_{-k+1}, \ldots, x_0 \geq \overline{x}_2, \ y_{-k}, y_{-k+1}, \ldots, y_0 < \overline{y}_2$; *or*

ii) $x_{-k}, x_{-k+1}, \ldots, x_0 < \overline{x}_2, \ y_{-k}, y_{-k+1}, \ldots, y_0 \geq \overline{y}_2$.

Then $\{(x_n, y_n)\}_{n=-k}^{\infty}$ *non-oscillates about the equilibrium point* $(\overline{x}_2, \overline{y}_2)$.

Proof: Assume that (i) holds, the case (ii) is similar and will be omitted. From (1.2), we have

$$x_1 = \frac{\alpha x_{-k}}{1+\prod_{i=0}^{k} y_{-i}^{r}} \geq \frac{\alpha \overline{x}_2}{1+\overline{y}_2^{\,r(k+1)}} = \overline{x}_2$$

$$y_1 = \frac{\beta y_{-k}}{1+\prod_{i=0}^{k} x_{-i}^{r}} < \frac{\beta \overline{y}_2}{1+\overline{x}_2^{\,r(k+1)}} = \overline{y}_2.$$

Then the result follows by induction. ■

In the following theorem, we show the existence of unbounded solutions for system (1.2).

Theorem 4.2. *Assume that* $\alpha > 1, \beta > 1$ *and* $(x_n, y_n)_{n=-k}^{\infty}$ *be a positive solution of system* (1.2). *Then the following statements are true:*

i) *If* $x_{-k}, x_{-k+1}, \ldots, x_0 \in (0, \overline{x}_2), \ y_{-k}, y_{-k+1}, \ldots, y_0 \in (\overline{y}_2, \infty)$. *Then* $\lim_{n\to\infty} x_n = 0$ *and* (y_n) *is unbounded.*

ii) *If* $x_{-k}, x_{-k+1}, \ldots, x_0 \in (\overline{x}_2, \infty), \ y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, \overline{y}_2)$. *Then* $\lim_{n\to\infty} y_n = 0$ *and* (x_n) *is unbounded.*

Proof: In the proof we will handle only the condition (i) since another can be shown similarly.

i) From above theorem, we can assume without loss of generality that the positive solution $(x_n, y_n)_{n=-k}^{\infty}$ of system (1.2) is such that

$$x_n < \overline{x}_2, \ \overline{y}_2 < y_n, \ n = 0, 1, \ldots.$$

Then

$$x_{(k+1)(n+1)+i}=\frac{\alpha x_{(k+1)n+i}}{1+\prod_{i=0}^{k} y^{r}_{(k+1)(n+1)-1-i}}<\frac{\alpha x_{(k+1)n+i}}{1+\overline{y}_2^{r(k+1)}}=x_{(k+1)n+i},$$

$$y_{(k+1)(n+1)+i}=\frac{\alpha y_{(k+1)n+i}}{1+\prod_{i=0}^{k} x^{r}_{(k+1)(n+1)-1-i}}>\frac{\alpha y_{(k+1)n+i}}{1+\overline{x}_2^{r(k+1)}}=y_{(k+1)n+i},$$

from which it follows that

$$\lim_{n\to\infty} x_n=0,\ \lim_{n\to\infty} y_n=\infty,$$

which completes the proof.■

5. RATE OF CONVERGENCE

In this section, we estimate the rate of convergence of a solution which converges to the equilibrium point $\left(\overline{x}_1,\overline{y}_1\right)=\left(0,0\right)$ of system (1.2) when $\alpha<1$ and $\beta<1$. The result of this section is the following theorem.

Theorem 5.1. *Let* $\alpha<1$ *and* $\beta<1$ *and the sequence* $\left(x_n,y_n\right)_{n\geq 0}$ *be a positive solution of system* (1.2). *Then, the error vector* $E_n=\left[e_n^1,\dots,e_{n-k}^1,e_n^2,\dots,e_{n-k}^2\right]^T$ *of every solution of system* (1.2) *satisfies both of the asymptotic relations*

$$\rho=\lim_{n\to\infty}\frac{\left\|E_{n+1}\right\|}{\left\|E_n\right\|},\ \rho=\lim_{n\to\infty}\left(\left\|E_n\right\|\right)^{1/n}, \tag{5.1}$$

where $e_{n-i}^1=x_{n-i}-\overline{x}_1$, $e_{n-i}^2=y_{n-i}-\overline{y}_1$ $\left(i=0,1,\dots,k\right)$ *and* ρ *is equal to the modulus of one of the eigenvalues of matrix* $J_F(\overline{X}_0)$.

Proof: First, we will find a system which satisfies the error terms. The error terms are given by

$$e_{n+1}^1=\sum_{i=0}^{k}P_n^i e_{n-i}^1+\sum_{i=0}^{k}Q_n^i e_{n-i}^2,\ e_{n+1}^2=\sum_{i=0}^{k}R_n^i e_{n-i}^1+\sum_{i=0}^{k}S_n^i e_{n-i}^2$$

where $P_n^j=0$, $P_n^k=\frac{\alpha}{1+\prod_{i=0}^{k}y_{n-i}^r}$, $Q_n^s=\frac{-\alpha r x_{n-k}y_n^r\cdots y_{n-s-1}^r y_{n-s}^{r-1}y_{n-s+1}^r\cdots y_{n-k}^r}{\left(1+\prod_{i=0}^{k}y_{n-i}^r\right)^2}$, $R_n^s=\frac{-\beta r y_k x_n^r\cdots x_{n-s-1}^r x_{n-s}^{r-1}x_{n-s+1}^r\cdots x_{n-k}^r}{\left(1+\prod_{i=0}^{k}x_{n-i}^r\right)^2}$, $S_n^j=0$, $S_n^k=\frac{\beta}{1+\prod_{i=0}^{k}x_{n-i}^r}$, $0\leq j\leq k-1$, $0\leq s\leq k$, $n\in\mathbb{N}_0$. It is easily seen that

$$\lim_{n\to\infty} P_n^j = 0,\ \lim_{n\to\infty} P_n^k = \frac{\alpha}{1+\overline{y}_1^{r(k+1)}} = \alpha,\ \lim_{n\to\infty} Q_n^s = \frac{-\alpha r \overline{x}_1 \overline{y}_1^{r(k+1)-1}}{\left(1+\overline{y}_1^{r(k+1)}\right)^2} = 0,$$

$$\lim_{n\to\infty} R_n^s = \frac{-\beta r \overline{y}_1 \overline{x}_1^{r(k+1)-1}}{\left(1+\overline{x}_1^{r(k+1)}\right)^2} = 0,\ \lim_{n\to\infty} S_n^j = 0,\ \lim_{n\to\infty} S_n^k = \frac{\beta}{1+\overline{x}_1^{r(k+1)}} = \beta.$$

Now, we have the limiting system of the error terms at $(\overline{x}_1, \overline{y}_1) = (0,0)$ as follows:

$$E_{n+1} = ME_n,$$

where $M = \begin{pmatrix} 0 & \cdots & \alpha & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \beta \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}_{(2k+2)\times(2k+2)}$, which equals to the matrix $J_F(\overline{X}_0)$. From Theorem 2.4 and Theorem 2.5, we have the results in (5.1).■

6. NUMERICAL EXAMPLES

Example 6.1. *Let* $k=3,\ r=4,\ \alpha=0.7,\ \beta=0.8$. *In this case, the system is as follows:*

$$x_{n+1} = \frac{0.7x_{n-3}}{1+y_n^4 y_{n-1}^4 y_{n-2}^4 y_{n-3}^4},\ y_{n+1} = \frac{0.8y_{n-3}}{1+x_n^4 x_{n-1}^4 x_{n-2}^4 x_{n-3}^4},\ n \in \mathbb{N}_0, \tag{6.1}$$

with the unique equilibrium point $(\overline{x}_1, \overline{y}_1) = (0,0)$. *We illusrate the solution which corresponds the initial conditions* $x_{-3}=2.2,\ x_{-2}=31,\ x_{-1}=0.13,\ x_0=1,\ y_{-3}=0.12,\ y_{-2}=21,$ $y_{-1}=3,\ y_0=2$ *of* (6.1) *in Fig. 1.*

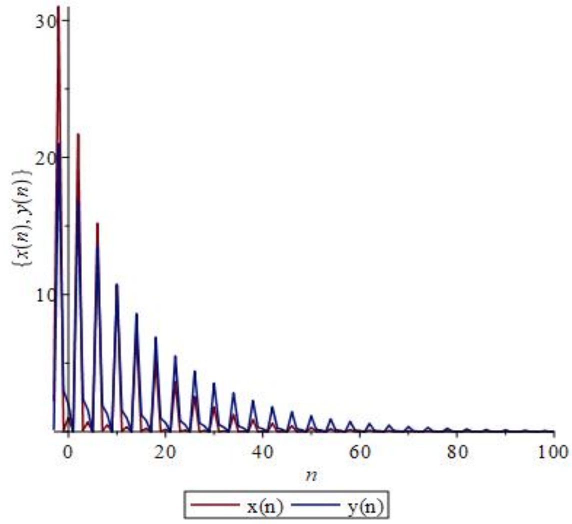

Figure 1. Plot for solution of Eq. (6.1).

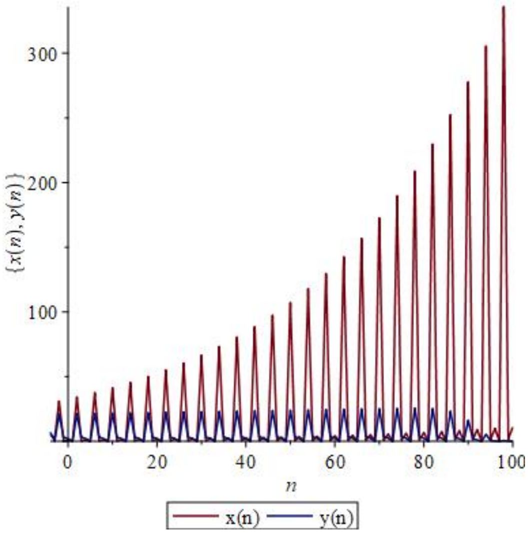

Figure 2. Plot for solution of Eq. (6.2).

Example 6.2: *Let* $k=3,\ r=4,\ \alpha=1.1,\ \beta=1.01$. *In this case, the system is as follows:*

$$x_{n+1}=\frac{1.1x_{n-3}}{1+y_n^4y_{n-1}^4y_{n-2}^4y_{n-3}^4},\ y_{n+1}=\frac{1.01y_{n-3}}{1+x_n^4x_{n-1}^4x_{n-2}^4x_{n-3}^4},\ n\in\mathbb{N}_0. \tag{6.2}$$

We illusrate the solution which corresponds the initial conditions $x_{-3}=2.2,\ x_{-2}=31,\ x_{-1}=0.13,\ x_0=1,\ y_{-3}=0.12,\ y_{-2}=21,\ y_{-1}=3,\ y_0=2$ *of* (6.2) *in Figure 2.*

Example 6.3. *Let* $k=3,\ r=4,\ \alpha=2,\ \beta=17$. *In this case, the system is as follow:*

$$x_{n+1}=\frac{2x_{n-3}}{1+y_n^4y_{n-1}^4y_{n-2}^4y_{n-3}^4},\ y_{n+1}=\frac{17y_{n-3}}{1+x_n^4x_{n-1}^4x_{n-2}^4x_{n-3}^4},\ n\in\mathbb{N}_0. \tag{6.3}$$

We illusrate the solution which corresponds the initial conditions $x_{-3}=0.2,\ x_{-2}=4,\ x_{-1}=10,\ x_0=0.25,\ y_{-3}=3,\ y_{-2}=0.33,\ y_{-1}=2,\ y_0=0.5$ *of* (6.3) *in Figure 3.*

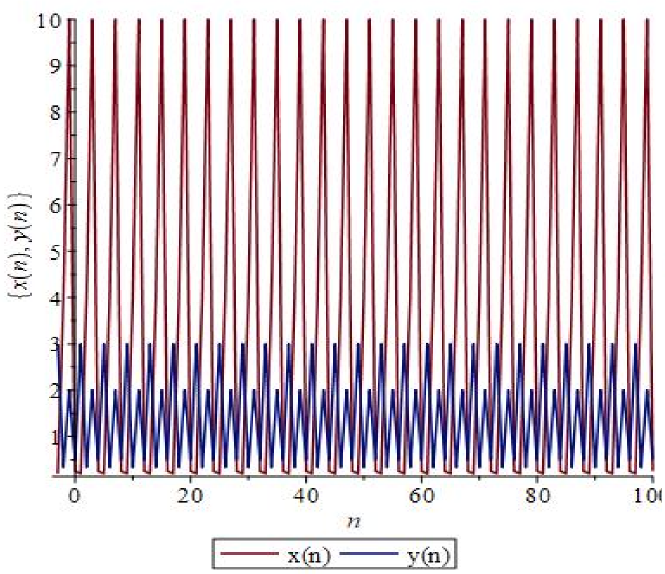

Figure 3: Plot for solution of Eq. (6.3).

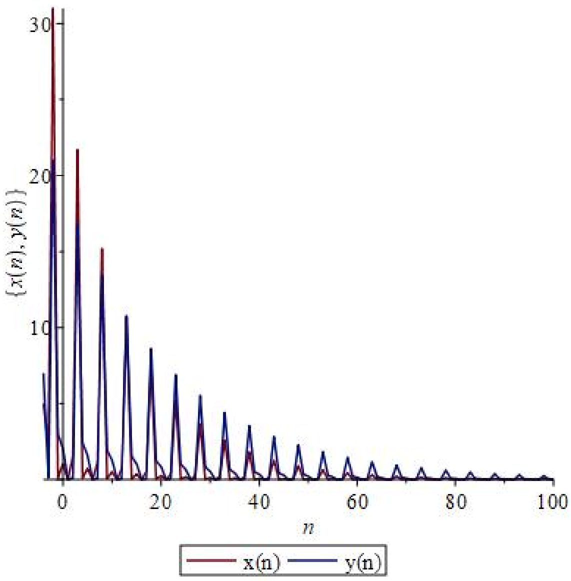

Figure 4. Plot for solution of Eq. (6.4).

Example 6.4. *Let* $k=4,\ r=3,\ \alpha=0.7,\ \beta=0.8$. *In this case, the system is as follows:*

$$x_{n+1}=\frac{0.7x_{n-4}}{1+y_n^3y_{n-1}^3y_{n-2}^3y_{n-3}^3y_{n-4}^3},\ y_{n+1}=\frac{0.8y_{n-4}}{1+x_n^3x_{n-1}^3x_{n-2}^3x_{n-3}^3x_{n-4}^3},\ n\in\mathbb{N}_0, \tag{6.4}$$

We illusrate the solution which corresponds the initial conditions $x_{-3}=2.2,\ x_{-2}=31,\ x_{-1}=0.13,\ x_0=1,\ y_{-3}=0.12,\ y_{-2}=21,\ y_{-1}=3,\ y_0=2$ *of* (6.4) *in Figure 4.*

Example 6.5. *Let* $k=4,\ r=3,\ \alpha=1.1,\ \beta=1.01$. *In this case, the system is as follows:*

$$x_{n+1}=\frac{1.1x_{n-4}}{1+y_n^3y_{n-1}^3y_{n-2}^3y_{n-3}^3y_{n-4}^3},\ y_{n+1}=\frac{1.01y_{n-4}}{1+x_n^3x_{n-1}^3x_{n-2}^3x_{n-3}^3x_{n-4}^3},\ n\in\mathbb{N}_0, \tag{6.5}$$

We illusrate the solution which corresponds the initial conditions $x_{-3} = 2.2,\ x_{-2} = 31,\ x_{-1} = 0.13,\ x_0 = 1,\ y_{-3} = 0.12,\ y_{-2} = 21,\ y_{-1} = 3,\ y_0 = 2$ *of* (6.5) *in Figure 5.*

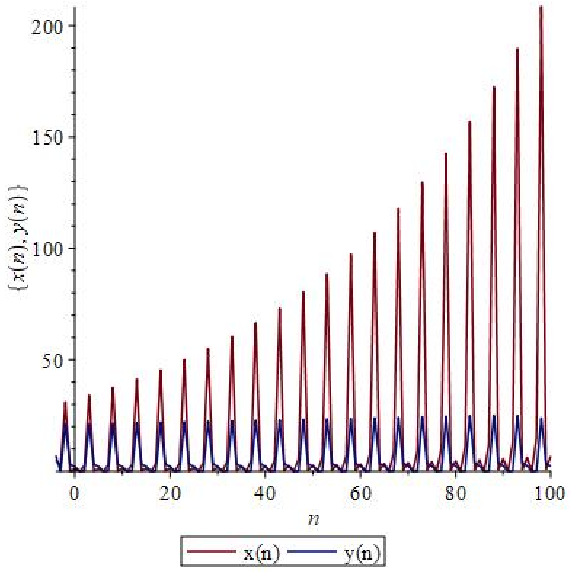

Figure 5. Plot for solution of Eq. (6.5).

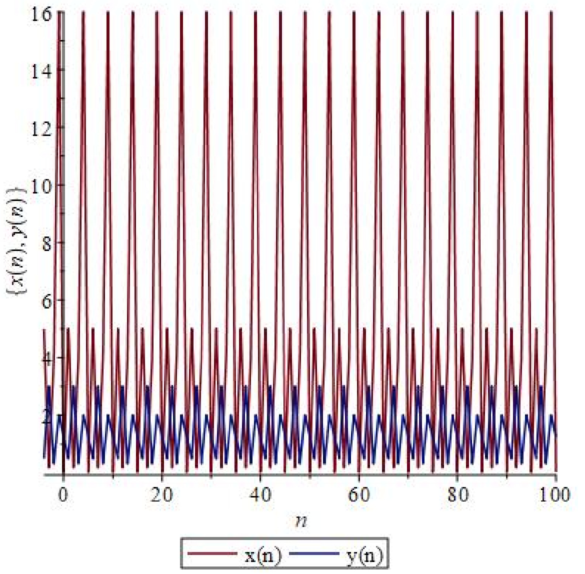

Figure 6. Plot for solution of Eq. (6.6).

Example 6.6.: *Let* $k = 4,\ r = 3,\ \alpha = 2,\ \beta = 17$. *In this case, the system is as follow:*

$$x_{n+1} = \frac{2x_{n-4}}{1 + y_n^3 y_{n-1}^3 y_{n-2}^3 y_{n-3}^3 y_{n-4}^3},\ y_{n+1} = \frac{17 y_{n-4}}{1 + x_n^3 x_{n-1}^3 x_{n-2}^3 x_{n-3}^3 x_{n-4}^3},\ n \in \mathbb{N}_0, \tag{6.6}$$

We illusrate the solution which corresponds the initial conditions $x_{-3} = 0.2,\ x_{-2} = 4,\ x_{-1} = 10,\ x_0 = 0.25,\ y_{-3} = 3,\ y_{-2} = 0.33,\ y_{-1} = 2,\ y_0 = 0.5$ *of* (6.6) *in Figure 6.*

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