

A METHOD FOR MANNHEIM OFFSETS OF RULED SURFACES

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Abstract. In this paper, using Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ of ruled surface $\varphi(s, v)$, Mannheim offsets $\varphi^*(s, v)$ with Darboux frame $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ of $\varphi(s, v)$ are defined. The striction curve, distribution parameter and orthogonal trajectory of $\varphi^*(s, v)$ are investigated by using the Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. The distribution parameters of ruled surfaces $\varphi_{\mathbf{T}^*}^*$, $\varphi_{\mathbf{g}^*}^*$ and $\varphi_{\mathbf{n}^*}^*$ are calculated. Finally, the relation between the instantaneous Pfaffian vectors of motions H/H' and H^*/H'^* are given, where $H = \{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ be the moving space along the base curve of $\varphi(s, v)$, $H^* = \{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be the moving space along the base curve of $\varphi^*(s, v)$, H' and H'^* be fixed Euclidean spaces.

Keywords: Mannheim, Darboux frame, ruled surface.

1. INTRODUCTION

A point-line is a rigid combination of a directed line and an endpoint on the line. The trajectories of the directed line and the end point are referred to as the directrix and indicatrix, respectively. The point-line trajectory is used for many industrial applications such as welding, cutting, painting, milling, screwing. The tool of an industrial machine can be represented by a point-line and the trajectory of such a tool can be presented by a point-line trajectory. The point-line trajectory under a one-parameter motion is a patch on a ruled surface.

In mathematics, a ruled surface M in \mathbb{E}^3 is a surface that contains at least one-parameter family of straight lines. Thus a ruled surface has a parametrization $\varphi: \mathbb{R}^2 \rightarrow M$ of the form $\varphi(s, v) = \alpha(s) + v\mathbf{e}(s)$, $\|\mathbf{e}(s)\| = 1$ where (α) and (\mathbf{e}) are curves in \mathbb{E}^3 . The φ is called a ruled patch. The curve (α) is called the directrix or base curve of the ruled surface, and the curve (\mathbf{e}) is called the (spherical) indicatrix curve and \mathbf{e} is also called the (spherical) indicatrix vector of the ruled surface [1, 2]. To characterize a point line trajectory, the relation between the directrix and the indicatrix must be given.

The frame fields constitute an important role while examining the differential properties of curves and surfaces. A Darboux frame is a natural moving frame defined on a surface. It is the analog of the Frenet frame as applied to surface theory. A Darboux frame exists at any non-umbilic point of a surface [3]. From past to present, the geometers have defined some different offsets of curves for example the involute-evolute, Bertrand,

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Mannheim and Smarandache by using the frame fields. Offsets of curves generally more complicated than their progenitor curve. Mannheim curves is defined by R. Blum in 1966 [4]. The Mannheim partner curves which lying on different surfaces are studied according to the Darboux frame [5]. They called these new associated curves as Mannheim partner D-curves. The characterizations of these curves by using the Darboux frame of curves have been given. Properties of ruled surfaces and their offset surfaces have been examined in Euclidean and non-Euclidean spaces [2, 6-17].

In this paper, we interested in finding Mannheim offsets of ruled surface by using the Darboux frame. In the second section, we have compiled some basic facts about ruled surfaces and Darboux frame. In the third section, we define Mannheim offsets of ruled surface with Darboux frame. We investigate their characteristic properties as the striction curve, distribution parameter and orthogonal trajectory and the distribution parameters of ruled surfaces $\varphi_{\mathbf{T}^*}, \varphi_{\mathbf{g}^*}$ and $\varphi_{\mathbf{n}^*}$. Lastly, we obtain the relations between the instantaneous Pfaffian vectors of motions H/H' and H^* , where $H = \{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ be the moving space along the base curve of $\varphi(s, v)$, $H^* = \{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be the moving space along the base curve of $\varphi^*(s, v)$, H' and H^* be fixed Euclidean spaces.

2. PRELIMINARIES

This section is devoted to the basic notions on the ruled surfaces and Darboux frame in the Euclidean 3-space.

Definition 1. A ruled surface has a parametrization $\varphi: \mathbb{R}^2 \rightarrow M$ of the form

$$\varphi(s, v) = \alpha(s) + v\mathbf{e}(s), \|\mathbf{e}(s)\| = 1$$

where (α) and (\mathbf{e}) are curves in \mathbb{R}^3 . The curve (α) is called the directrix or base curve of the ruled surface, and the curve (\mathbf{e}) is called the spherical indicatrix curve and \mathbf{e} is also called the spherical indicatrix vector of the ruled surface. The rulings are the straight lines $v \rightarrow \alpha(s) + v\mathbf{e}(s)$ [1, 2].

Definition 2. The ruled surface is said to be a noncylindrical ruled surface provided that $\langle \mathbf{e}_s, \mathbf{e}_s \rangle \neq 0$ [1].

Definition 3. The striction point on ruled surface is the foot of the common perpendicular line of the successive rulings on the main ruling. The set of striction points of the noncylindrical ruled surface generates its striction curve [2]. It is given by

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, \mathbf{e}_s \rangle}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle} \mathbf{e}(s). \quad (1)$$

Theorem 1. If successive rulings intersect, the ruled surface is called developable [2].

Definition 4. The distribution parameter of the noncylindrical ruled surface is defined by [2]:

$$P_e = \frac{\det(\alpha_s, \mathbf{e}, \mathbf{e}_s)}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle}. \quad (2)$$

Theorem 2. The ruled surface is developable if and only if $P_e = 0$ [2].

Definition 5. A curve which intersects perpendicularly each one of rulings is called an orthogonal trajectory of the ruled surface. It is calculated by

$$\langle \mathbf{e}, d\varphi \rangle = 0. \quad (3)$$

Since the base curve $\alpha(s)$ is a space curve, there exists the moving Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along the curve. By using the unit normal vector field of the (ruled) surface \mathbf{n} , we can define $\mathbf{g} = \mathbf{n} \times \mathbf{T}$ unit vector, which satisfies $\langle \mathbf{T}, \mathbf{g} \rangle = \langle \mathbf{n}, \mathbf{g} \rangle = 0$, [3]. So, we get the Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ of the ruled surface φ . The derivative formulae of the Darboux frame can be defined by

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{g}} \\ \dot{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \quad (4)$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of the curve $\alpha(s)$. Here and subsequently, dot denotes the derivative with respect to the arc length parameter of a curve and prime denotes the derivative with respect to the arbitrary parameter of the curve.

Let $\varphi(s, v)$ be a ruled surface with Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. Using of the equation (4), we get the Pfaffian forms (connection forms) of the system $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$

$$\left. \begin{aligned} \omega_1 &= \tau_g, \\ \omega_2 &= -\kappa_n, \\ \omega_3 &= \kappa_g. \end{aligned} \right\} \quad (5)$$

For the instantaneous Pfaffian vector of motion H/H' , where $H = sp\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ be the moving space along the base curve α of $\varphi(s, v)$, H' be fixed Euclidean space, we have,

$$\boldsymbol{\omega} = \omega_1 \mathbf{T} + \omega_2 \mathbf{g} + \omega_3 \mathbf{n}. \quad (6)$$

3. A METHOD FOR MANNHEIM OFFSETS OF RULED SURFACES USING THE DARBOUX FRAME

This section is original part of our study. Section 3 is giving a method to obtain Mannheim offsets of ruled surfaces by using the Darboux frame.

3.1. CHARACTERISTIC PROPERTIES OF MANNHEIM OFFSETS

Definition 6. Let $\varphi(s, v)$ with Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\varphi^*(s, v)$ with Darboux frame $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be two ruled surfaces given by the equations:

$$\begin{aligned}\varphi(s, v) &= \alpha(s) + v\mathbf{e}(s), \|\mathbf{e}(s)\| = 1, \\ \varphi^*(s, v) &= \alpha^*(s) + v\mathbf{e}^*(s), \|\mathbf{e}^*(s)\| = 1.\end{aligned}$$

$\varphi(s, v)$ is said to be Mannheim offset of $\varphi^*(s, v)$, if there exists a one-to-one correspondence between their points such that \mathbf{g} of $\varphi(s, v)$ and \mathbf{n}^* of $\varphi^*(s, v)$ are linearly dependent.

\mathbf{n}^* is always perpendicular to the spherical indicatrix vector \mathbf{e}^* . So, we can write that

$$\mathbf{e}^* = \mathbf{T}^* \cos \phi^* + \mathbf{g}^* \sin \phi^*, \quad (7)$$

where ϕ^* is the angle between the vectors \mathbf{T}^* and \mathbf{e}^* .

The relation between the Darboux vectors of Mannheim offset φ^* of φ is given by:

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{g}^* \\ \mathbf{n}^* \end{bmatrix} = \begin{bmatrix} \cos \psi & 0 & \sin \psi \\ \sin \psi & 0 & -\cos \psi \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}, \quad (8)$$

where ψ is the angle between the tangent vectors \mathbf{T} and \mathbf{T}^* .

The equation of the offset ruled surface φ^* , in the terms of the base curve of φ and $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ Darboux frame, can be written as

$$\begin{aligned}\varphi^*(s, v) &= \alpha^*(s) + v\mathbf{e}^*(s) \\ &= \underbrace{\alpha(s) + R\mathbf{g}(s)}_{\alpha^*(s)} + v \underbrace{\left[\cos(\psi - \phi^*)\mathbf{T} + \sin(\psi - \phi^*)\mathbf{n} \right]}_{\mathbf{e}^*(s)},\end{aligned} \quad (9)$$

where R is the distance function between the corresponding points. The distance function is calculated as $R = \text{constant}$, [5].

The vectors \mathbf{e}^* and \mathbf{e}_s^* can be given in terms of Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ as:

$$\begin{aligned}\mathbf{e}^* &= \cos(\psi - \phi^*)\mathbf{T} + \sin(\psi - \phi^*)\mathbf{n}, \\ \mathbf{e}_s^* &= -\sin(\psi - \phi^*)\left[\kappa_n + (\psi - \phi^*)'\right]\mathbf{T} + \left[\kappa_g \cos(\psi - \phi^*) - \tau_g \sin(\psi - \phi^*)\right]\mathbf{g} \\ &\quad + \cos(\psi - \phi^*)\left[\kappa_n + (\psi - \phi^*)'\right]\mathbf{n}.\end{aligned}\quad (10)$$

We can give the striction curve of the noncylindrical ruled surface with Darboux Frame φ^* as follows:

$$c^*(s) = \alpha(s) + Rg(s) - \frac{\left[\kappa_n + (\psi - \phi^*)'\right]\left[-(1 - R\kappa_g)\sin(\psi - \phi^*) + R\tau_g\right]}{\left[\kappa_n + (\psi - \phi^*)'\right]^2 + \left[\kappa_g \cos(\psi - \phi^*) - \tau_g \sin(\psi - \phi^*)\right]}\mathbf{e}^*(s).\quad (11)$$

We can write the distribution parameter of the noncylindrical ruled surface with Darboux Frame φ^* as follows:

$$P_{e^*} = \frac{\det(\alpha_s^*, \mathbf{e}^*, \mathbf{e}_s^*)}{\|\mathbf{e}_s^*\|^2}$$

or

$$P_{e^*} = \frac{R\tau_g\kappa_g(1 - 2\sin^2(\psi - \phi^*)) + \sin(\psi - \phi^*)\left[\tau_g \sin(\psi - \phi^*) + \cos(\psi - \phi^*)(-R\tau_g - \kappa_g(1 - R\kappa_g))\right]}{\left[\kappa_n + (\psi - \phi^*)'\right]^2 + \left[\kappa_g \cos(\psi - \phi^*) - \tau_g \sin(\psi - \phi^*)\right]}.\quad (12)$$

Moreover, from the equation (3), we can write the orthogonal trajectories of φ^* as follows:

$$\begin{aligned}\langle \mathbf{e}^*, \alpha_s^* \rangle ds &= -dv, \\ \left[(1 - R\kappa_g)\cos(\psi - \phi^*) + R\tau_g \sin(\psi - \phi^*)\right] ds &= -dv.\end{aligned}\quad (13)$$

Theorem 3. Let (φ, φ^*) be a pair of Mannheim ruled surfaces with Darboux frame. The distribution parameters of the ruled surfaces $\varphi_{\mathbf{T}}^*$, $\varphi_{\mathbf{g}}^*$ and $\varphi_{\mathbf{n}}^*$ are given as follows:

$$\begin{aligned}
P_{T^*} &= \frac{\det(\alpha_s^*, \mathbf{T}^*, \mathbf{T}_s^*)}{\|\mathbf{T}_s^*\|^2} = 0, \\
P_{g^*} &= \frac{R\kappa_g \tau_g (2\sin^2 \psi - 1) + \cos \psi \sin \psi \left[\kappa_g (1 - R\kappa_g) + \tau_g (1 + R\tau_g) \right]}{(\kappa_n + \psi')^2 + (\kappa_g \sin \psi + \tau_g \cos \psi)^2}, \\
P_{n^*} &= \frac{\tau_g}{\kappa_g^2 + \tau_g^2}.
\end{aligned} \tag{14}$$

Proof: The main idea of the proof is to use the definition of the distribution parameters of the ruled surfaces $\varphi_{T^*}^*$, $\varphi_{g^*}^*$ and $\varphi_{n^*}^*$. The proof is straightforward.

Let us give two corollaries of this theorem.

Corollary 1. Let (φ, φ^*) be a pair of Mannheim ruled surfaces with Darboux frame. If α is a principal line, then the ruled surface $\varphi_{n^*}^*$ is developable.

Corollary 2. Let (φ, φ^*) be a pair of the oriented ($\psi = 0^\circ$) Mannheim ruled surfaces with Darboux frame. If α is a principal line or geodesic curve, then the ruled surface $\varphi_{g^*}^*$ is developable.

3.2. THE INTEGRAL INVARIANTS FOR MANNHEIM OFFSETS

Let $\varphi(s, v)$ with Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\varphi^*(s, v)$ with Darboux frame $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be a pair of Mannheim ruled surfaces. In this subsection, we have investigated the relation between the instantaneous Pfaffian vectors of motions H/H' and H^*/H'^* , where $H = sp\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ be the moving space along the base curve α of $\varphi(s, v)$, $H^* = sp\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be the moving space along the base curve α^* of $\varphi^*(s, v)$, H' and H'^* be fixed Euclidean spaces.

The derivative formulae of the Darboux frame $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ with respect to the arc length parameter s^* of the curve α^* can be given by the equation:

$$\begin{bmatrix} \dot{\mathbf{T}}^* \\ \dot{\mathbf{g}}^* \\ \dot{\mathbf{n}}^* \end{bmatrix} = \begin{bmatrix} 0 & -(\psi' + \kappa_n) s' & (\kappa_g \cos \psi - \tau_g \sin \psi) s' \\ (\psi' + \kappa_n) s' & 0 & (\kappa_g \sin \psi + \tau_g \cos \psi) s' \\ -(\kappa_g \cos \psi - \tau_g \sin \psi) s' & -(\kappa_g \sin \psi + \tau_g \cos \psi) s' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}^* \\ \mathbf{g}^* \\ \mathbf{n}^* \end{bmatrix}, \tag{15}$$

where $\frac{ds}{ds^*} = s'$.

Thus, we have

$$\left. \begin{aligned} \omega_1^* &= (\tau_g \cos \psi + \kappa_g \sin \psi) s' = (\omega_1 \cos \psi + \omega_3 \sin \psi) s' \\ \omega_2^* &= (\tau_g \sin \psi - \kappa_g \cos \psi) s' = (\omega_1 \sin \psi - \omega_3 \cos \psi) s' \\ \omega_3^* &= -(\psi' + \kappa_n) s' = (\omega_2 - \psi') s' \end{aligned} \right\} \quad (16)$$

for the Pfaffian forms of the system $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$, and we get

$$\boldsymbol{\omega}^* = [(\omega_1 \cos \psi + \omega_3 \sin \psi) \mathbf{T}^* + (\omega_1 \sin \psi - \omega_3 \cos \psi) \mathbf{g}^* + (\omega_2 - \psi') \mathbf{n}^*] s', \quad (17)$$

for the instantaneous Pfaffian vector of motion H^* / H^* . From the equation (8), we obtain

$$\boldsymbol{\omega}^* = (-\boldsymbol{\omega} + \psi' \mathbf{g}^*) s', \quad (18)$$

for the relation between the instantaneous Pfaffian vectors of motions H / H' and H^* / H^* .

Theorem 4. Let α and α^* be two curves in ruled surface with Darboux frame φ . α and α^* are the Mannheim offsets if and only if α is a principal line.

Proof: Let α and α^* be Mannheim offsets in a ruled surface φ . Therefore, we can write $\alpha^*(s) = \alpha(s) + R\mathbf{g}(s)$. Since α^* is a curve in φ , then the normal of the ruled surface $\mathbf{n}(s)$ must be orthogonal to $\alpha^*(s)$, $\langle \alpha^*(s), \mathbf{n}(s) \rangle = \langle (1 - R\kappa_g)\mathbf{T}(s) + R\tau_g\mathbf{n}(s), \mathbf{n}(s) \rangle = R\tau_g = 0$. If the distance function R is equal to zero, then α and α^* are the same curves. For this reason, τ_g must be equal to zero and so α is a principal line.

Conversely, let α be a principal line. In this case, we can write $\dot{\mathbf{g}} = -\kappa_g \mathbf{T}$. The following equation must be available to α and α^* be Mannheim offsets $\alpha^*(s) = \alpha(s) + R\mathbf{g}(s)$. We can find the vector $\alpha^*(s)$ from this equation as $\alpha^*(s) = (1 - R\kappa_g)\mathbf{T}(s)$. The normal vector $\mathbf{n}(s)$ is orthogonal to the vector $\alpha^*(s)$. So, we can say that α and α^* are Mannheim offsets.

Example 1. We can write the hyperboloid of one sheet as:

$$\varphi(s, v) = \left(3 \cos \frac{s}{3}, 3 \sin \frac{s}{3}, 0 \right) + v \frac{1}{5} \left(-3 \sin \frac{s}{3}, 3 \cos \frac{s}{3}, 4 \right). \quad (19)$$

If we take $\phi^* = 45^\circ$ and $\psi = 45^\circ$ in the equation (19), namely $\phi^* - \psi = 0^\circ$, then the Mannheim offsets of this ruled surface are:

$$\varphi^*(s, v) = \left(3 \cos \frac{s}{3}, 3 \sin \frac{s}{3}, R \right) + v \left(-\sin \frac{s}{3}, \cos \frac{s}{3}, 0 \right). \quad (20)$$

We can see these surfaces in Fig. 1, here we take $R = 3$ in the equation (20).

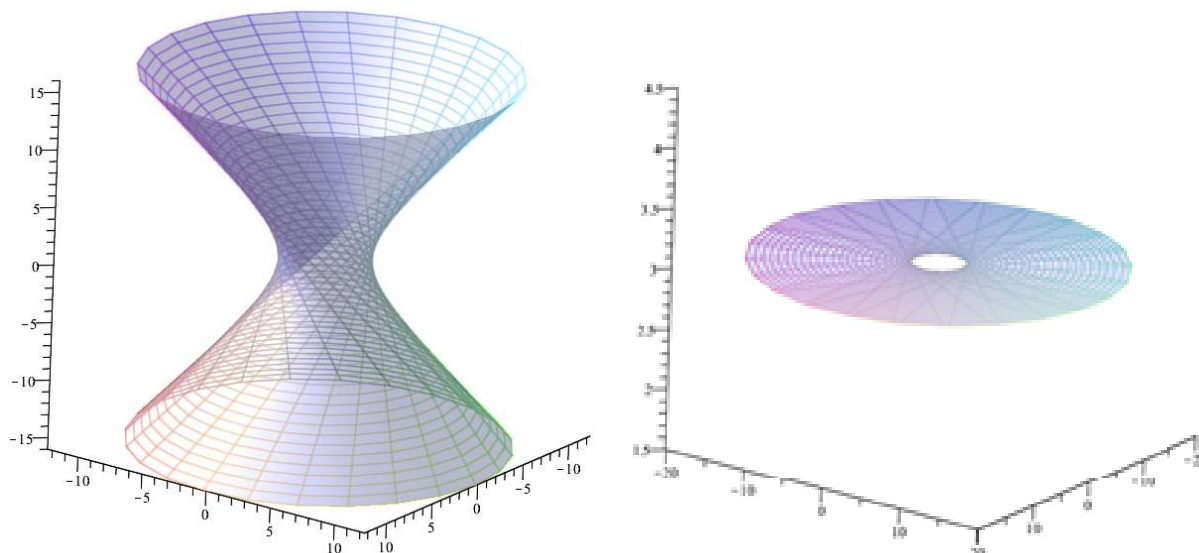


Figure 1. The hyperboloid of one sheet and its Mannheim ruled surface for $\phi^* = 45^\circ$, $\psi = 45^\circ$ and $R = 3$.

Example 2. We can write the helicoid as:

$$\varphi(s, v) = (0, 0, s) + v(\cos s, \sin s, 0). \quad (21)$$

If we take $\phi^* = 135^\circ$ and $\psi = 45^\circ$ in the equation (21), namely $\phi^* - \psi = 90^\circ$, then Mannheim offsets of the helicoid are

$$\varphi^*(s, v) = (R \sin s, R \cos s, s) + v(-\sin s, \cos s, 0) \quad (22)$$

We can see these surfaces in Fig. 2, here we take $R = 50$ in the equation (22).

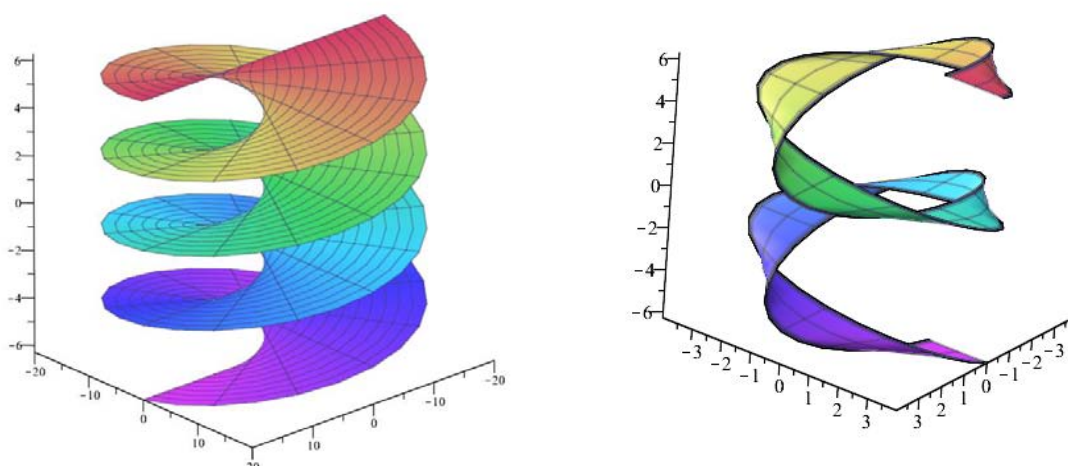


Figure 2. The helicoid and its Mannheim ruled surface for $\phi^* = 135^\circ$, $\psi = 45^\circ$ and $R = 50$

4. CONCLUSION

We defined Mannheim offsets of ruled surface by using the Darboux frame and developed the theory of the offsets of ruled surfaces in the Euclidean 3-space. Moreover, we investigated the characteristic properties of Mannheim ruled surfaces according to the Darboux frame.

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