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APPROXIMATION BY THE SUMMATION INTEGRAL TYPE OPERATORS BASED ON LUPAS-SZASZ BASIS FUNCTIONS

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Abstract. *In this paper, the summation integral type operators based on Lupas and Szász basis functions are introduced. The degree of approximation of these operators is examined in terms of Ditzian-Totik modulus of smoothness and corresponding K-functional. The rate of convergence by means of the Lipschitz class and the Lipschitz type maximal function is investigated. Furtermore, the properties of weigthed approximation and Voronoskaja type theorem in weighted spaces are obtained. The advantages of these operators are shown by some graphics and numerical calculations.*

Keywords: *Durrmeyer-type operators, rate of convergence, moduli of countunity, Petree-K funtional.*

1. INTRODUCTION

In [1] Agratini investigated the approximation properties the linear positive operators introduced by Lupas [12], and studied on some quantitative estimates for the degree of approximation. For $f \in C[0,\infty)$,these operators are defined by

$$L_n(f;x) = \sum_{k=0}^{\infty} \nu_{n,k} f\left(\frac{k}{n}\right), x \geq 0, \tag{1}$$

where $\nu_{n,k}(x) = \frac{(nx)_k}{2^k k!} 2^{-nx}$is Lupas basis function and $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ is the Pochammer symbol. In [1] the author obtained an asymptotic formula for these operators and gave the order of approximation in terms of modulus of continuity and also introduced integral modifications of the operators (1).

In [2], Kantorovich variant of the operators (1) was considered and the smoothness properties in terms of modulus of continuity were studied. Also, by using probabilistic methods, rate of convergence of the Kantorovich variant of the operators (1) for functions of bounded variation was obtained.

In [5], a generalization of the operators L_n given by (1) was introduced as follows

$$L_n^*(f;x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f\left(\frac{k}{b_n}\right), x \geq 0$$

where (an), (bn) are increasing and unbounded sequences of positive numbers such that

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$$\frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right), \lim_{n\to\infty} \frac{1}{b_n} = 0$$

and studied the weighted approximation properties. In [6] the authors obtained some estimates on the degree of approximation for Kantorovich type of the operators L_n^*.

Recently, the approximation properties of Lupas type operators on based Pòlya distribution were studied in [3, 4, 8, 9, 13]. Lupas type operator on based Pòlya distribution defined by

$$P_n^{\left(\frac{1}{n}\right)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{\left(\frac{1}{n}\right)}(x),$$

where $f \in C(I)$, with $I = [0,1]$, $p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}$, is a generalization of Bernstein polynomials. The L_n operators given by (1) is similar form with the Szász operators and, in a sense, it is an extension of the $P_n^{(1/n)}$ to positive real axis.

Motivated by the above works, in this paper we define a generalization of the operators L_n given by (1) and introduction its summation-integral type operators in (4). Let $C[0,\infty)$ denotes the space of all real valued continuous functions on $[0,\infty)$. For $f \in C[0,\infty)$ and $x \in [0,\infty)$, we define a generalization of the operators (1) as follows

$$L_{a_n,b_n}(f;x) = 2^{-\frac{a_n}{b_n}x} \sum_{k=0}^{\infty} \frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!} f\left(\frac{k}{a_n} b_n\right) \tag{2}$$

where $(a_n), (b_n)$ are unbounded and increasing sequences of positive real numbers such that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0 \text{ and } \frac{b_n}{a_n} \leq 1. \tag{3}$$

Now we introduce a mixed summation-integral type operator of the operators (2), having generalized Lupas and Szász basis functions in summation and integration respectively:

$$D_{a_n,b_n}(f;x) = \frac{a_n}{b_n} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^{\infty} P_{n,k}(u) f(u) du \tag{4}$$

where $P_{n,k}(x) = e^{-\frac{a_n}{b_n}x} \frac{\left(\frac{a_n}{b_n}x\right)^k}{k!}$, $l_{n,k}(x) = \frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!} 2^{-\frac{a_n}{b_n}x}$ and $(a_n), (b_n)$ are the same with the definition of (2).

The aim of these paper is to study some local and global results and the degree of approximation, using the second order Ditzian-Totik modulus of smoothness and the corresponding K-functional and give an estimate for functions in a Lipschitz type space by means of the operators $D_{a_n,b_n}(f;x)$. We also investigate the properties of the weighted approximation and give a Voronoskaja type asymptotic result by the operators $D_{a_n,b_n}(f;x)$ defined by (4) in weighted spaces of continuous functions with growth polynomial.

2. BASIC RESULTS

Lemma 1: Let $e_i(x) = x^i, i = 0,1,2,3,4$ and (a_n), (b_n) are unbounded and increasing sequences of positive real numbers satisfying the condition (3). For each $x \in [0,\infty)$ we have

$$L_{a_n,b_n}(e_0;x) = 1$$

$$L_{a_n,b_n}(e_1;x) = x$$

$$L_{a_n,b_n}(e_2;x) = x^2 + 2\frac{b_n}{a_n}x$$

$$L_{a_n,b_n}(e_3;x) = x^3 + 6\frac{b_n}{a_n}x^2 + 6\left(\frac{b_n}{a_n}\right)^2 x$$

$$L_{a_n,b_n}(e_4;x) = x^4 + 12\frac{b_n}{a_n}x^3 + 36\left(\frac{b_n}{a_n}\right)^2 x^2 + 26\left(\frac{b_n}{a_n}\right)^3 x.$$

Proof: Taking into account the equality

$$2^{\frac{a_n}{b_n}x} = \sum_{k=0}^{\infty} \frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!}, x \in 0,\infty),$$

and by simple calculations the desired results are obtained.

Remark 1: It is clear that the operators defined with (2) satisfy Korovkin-type approximation theorem on any compact set of $[0,\infty)$.

Lemma 2: Let $e_i(x) = x^i, i = 0,1,2,3,4$ and (a_n), (b_n) are unbounded and increasing sequences of positive real numbers satisfying the condition (3). For each $x \in [0,\infty)$, the operators D_{a_n,b_n} satisfy the following equalities:

$$D_{a_n,b_n}(e_0;x) = 1 \quad (5)$$

$$D_{a_n,b_n}(e_1;x) = x + \frac{b_n}{a_n} \quad (6)$$

$$D_{a_n,b_n}(e_2;x) = x^2 + 5\frac{b_n}{a_n}x + 2\left(\frac{b_n}{a_n}\right)^2 \quad (7)$$

$$D_{a_n,b_n}(e_3;x) = x^3 + 12\frac{b_n}{a_n}x^2 + 29\left(\frac{b_n}{a_n}\right)^2 x + 6\left(\frac{b_n}{a_n}\right)^3 \quad (8)$$

$$D_{a_n,b_n}(e_4;x) = x^4 + 22\frac{b_n}{a_n}x^3 + 131\left(\frac{b_n}{a_n}\right)^2 x^2 + 206\left(\frac{b_n}{a_n}\right)^3 x + 24\left(\frac{b_n}{a_n}\right)^4. \quad (9)$$

Proof: Considering the definition of the operator (4) and the properties of Pochammer symbol, we get

$$\begin{aligned} D_{a_n,b_n}(e_0;x) &= \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)e_0(u)du \\ &= \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} e^{-\frac{a_n}{b_n}u}\frac{\left(\frac{a_n}{b_n}u\right)^k}{k!}du \\ &= \sum_{k=0}^{\infty}\frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!}\ 2^{-\frac{a_n}{b_n}x}\Gamma(k+1) \\ &= L_{a_n,b_n}(e_0;x) = 1 \end{aligned}$$

and for e_1, e_2, e_3, e_4, similarly we have the results.

Remark 2: As a simple result of Lemma 2, we have

$$\mu_{n,1}(x) := D_{a_n,b_n}((t-x);x) = \frac{b_n}{a_n} \tag{10}$$

and

$$\mu_{n,2}(x) := D_{a_n,b_n}((t-x)^2;x) = 3\frac{b_n}{a_n}x + 2\left(\frac{b_n}{a_n}\right)^2, \tag{11}$$

for each $0 \leq x < \infty$, taking into account the condition (3), we get

$$D_{a_n,b_n}((t-x)^2;x) = O\left(\frac{b_n}{n}\right)(x+1) \text{ as } n \to \infty.$$

Further, using the condition (3)

$$\begin{aligned}\mu_{n,4}(x): &= D_{a_n,b_n}((t-x)^4;x) \\ &= 27\left(\frac{b_n}{a_n}\right)^2 x^2 + 182\left(\frac{b_n}{a_n}\right)^3 x + 24\left(\frac{b_n}{a_n}\right)^4 \\ &= O\left(\frac{b_n}{a_n}\right)(x^2+x+1) \text{ as } n \to \infty.\end{aligned} \tag{12}$$

Taking into account the equality (11), we have the following lemma.

Lemma 3: For all $n \in \mathbb{N}$ and $x \in [0,\infty)$, we have

$$\mu_{n,2}(x) := D_{a_n,b_n}((t-x)^2;x) \leq \frac{3b_n}{a_n}\delta^2_{a_n,b_n}(x) \tag{13}$$

where $\delta^2_{a_n,b_n}(x) = \phi^2(x) + b_n/a_n, \phi(x) = \sqrt{x}$.

Theorem 1: Let $\{D_{a_n,b_n}\}$ be the sequence of linear positive operators given by (4). Then, for all $f \in C[0,\infty)$,

$$\lim_{n\to\infty} D_{a_n,b_n}(f;x) = f(x)$$

uniformly with respect to $x \in [0,a] \subset [0,\infty), a > 0$.

Proof: From Lemma 2, we have

$$\lim_{n\to\infty} D_{a_n,b_n}(e_k;x) = e_k(x) = x^k, k = 0,1,2, \tag{14}$$

uniformly with respect to $x \in [0,a], a > 0$. Then, by the well-known Korovkin theorem, the operators $D_{a_n,b_n}(f;x)$ uniformly converges to f on any compact subset of $[0,\infty)$ as $n \to \infty$.

3. DIRECT RESULTS

In this section, we would like to give an estimate the degree of approximation for the operators $D_{a_n,b_n}(f;x)$ in terms of the modulus of continuity, moduli of smoothness and the Petree's K-functional.

We begin by recalling some definitions and notations. By $C_B[0,\infty)$, we denote the class on real valued continuous and bounded functions f defined on the interval $[0,\infty)$ with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$. For $f \in C_B[0,\infty), \delta > 0$, the m th order modulus of continuity is defined as

$$\omega_m(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |\Delta_h^m f(x)|$$

with Δ is the forward difference.

The Petree's K-functional is defined by

$$\mathcal{K}_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)}\{||f-g|| + \delta\|g''\|\}\ ,(\delta > 0),$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$ and $||.||$ is the uniform norm on $C_B[0,\infty)$. By ([7], p.10), we have the following inequality

$$\mathcal{K}_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta}) \tag{15}$$

where M a positive constant and ω_2 is the second order modulus of smoothness for $f \in C_B[0,\infty)$ is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\delta} \sup_{x,x+2h\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Now, we can give the following result:

Theorem 2: Let $\{D_{a_n,b_n}\}$ be a sequence of linear positive operators defined by (4). Then, for all $f \in C_B[0,\infty)$ and for each $x \in [0,\infty)$, the following inequality

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq 2\omega_1\left(f, \delta_{a_n,b_n}^*(x)\right) \tag{16}$$

holds. Moreover, if f has a continuous derivative on $[0,\infty)$, then we get

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq \delta_{a_n,b_n}^*(x)\left[2\omega_1\left(f', \delta_{a_n,b_n}^*(x)\right) + \|f'\|\right]$$

where $\delta_{a_n,b_n}^*(x)$ also is the same in both inequality and $\delta_{a_n,b_n}^*(x) := \left(3x\frac{b_n}{a_n} + 2\left(\frac{b_n}{a_n}\right)^2\right)^{\frac{1}{2}}$, $n \in \mathbb{N}$ and ω_1 is usual first modulus of continuity.

Proof: For every $u,t \in [0,\infty)$ and $\delta > 0$, considering the definition of modulus of continuity we can write

$$|f(u)-f(x)| \leq (1+\delta^{-1}|u-x|)\omega_1(f;\delta).$$

Using the definition of (4) we have

$$\left|D_{a_n,b_n}(f;x)-f(x)\right| = \left|\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)f(u)du - f(x)\right|$$

$$\leq \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)(1+\delta^{-1}|u-x|)\omega_1(f;\delta)du$$

$$= \omega_1(f;\delta)D_{a_n,b_n}(e_0;x)+\delta^{-1}\omega_1(f;\delta)D_{a_n,b_n}(|u-x|;x).$$

Applying Cauchy-Schwarz inequality and by (11), we have

$$\left|D_{a_n,b_n}(f;x)-f(x)\right| \leq \omega_1(f;\delta)+\delta^{-1}\omega_1(f;\delta)\left(3x\frac{b_n}{a_n}+2\left(\frac{b_n}{a_n}\right)^2\right)^{\frac{1}{2}}.$$

Choosing $\delta=\delta^*_{a_n,b_n}(x)=\left(3x\frac{b_n}{a_n}+2\left(\frac{b_n}{a_n}\right)^2\right)^{\frac{1}{2}}=\sqrt{\mu_{n,2}(x)}$ we obtain (16), for each $x\in[0,\infty)$.

Since f has a continuous derivative on $(0,\infty)$ we can write

$$f(t)-f(x)=[f'(\xi)-f'(x)](t-x)+f'(x)(t-x) \tag{17}$$

for every $x,t\in[0,\infty)$ and for any $\xi\in(x,t)$.

Applying the operators D_{a_n,b_n} to (17) and taking into account the definition of modulus of continuity for f', we have

$$\left|D_{a_n,b_n}(f;x)-f(x)\right| \leq \mu_{n,2}^{1/2}(x)\left[2\omega_1\left(f';\delta^*_{a_n,b_n}(x)\right)+\|f'\|\right]$$

which implies the desired result.

Now, we give the rate of convergence by Petree's K-functional.

Theorem 3: For each $x\in[0,\infty)$ $f\in C_B[0,\infty)$, the following inequality

$$\left|D_{a_n,b_n}(f;x)-f(x)\right| \leq 4\mathcal{K}_2\left(f,\delta^2_{a_n,b_n}(x)\right)+\omega_1\left(f;\frac{b_n}{a_n}\right) \tag{18}$$

$$\leq M\omega_2\left(f;\left(\frac{b_n}{a_n}\right)^{-\frac{1}{2}}\delta_{a_n,b_n}(x)\right)+\omega_1\left(f;\frac{b_n}{a_n}\right)$$

holds where $(\delta_{a_n,b_n}(x))^2=\phi^2(x)+\frac{b_n}{a_n}, \phi(x)=\sqrt{x}$ and M is a constant independently of n,x.

Proof: We define the auxiliary operators as follows:

$$\overline{D}_{a_n,b_n}(f;x)=D_{a_n,b_n}(f;x)+f(x)-f\left(\frac{a_nx+b_n}{a_n}\right). \tag{19}$$

Then by Lemma 2, we get

$$\overline{D}_{a_n,b_n}(e_0;x) = D_{a_n,b_n}(e_0;x) = 1$$

and

$$\overline{D}_{a_n,b_n}(e_1;x) = D_{a_n,b_n}(e_1;x) + x - \frac{a_n x + b_n}{a_n} = x$$

and $\overline{D}_{a_n,b_n}\big((t-x);x\big) = 0$. Let $g \in C_B^2[0,\infty)$ and $t \in [0,\infty)$. Using the Taylor formula, we get

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the $\overline{D}_{a_n,b_n}$ operator to both sides of this equality, we have

$$\overline{D}_{a_n,b_n}(g;x) - g(x) = D_{a_n,b_n}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{\frac{a_n x+b_n}{a_n}} \left(\frac{a_n x + b_n}{a_n} - u\right) g''(u)du.$$

Hence we obtain

$$\begin{aligned}
&\left|\overline{D}_{a_n,b_n}(g;x) - g(x)\right| \\
&\quad\le D_{a_n,b_n}\left(\int_x^t |t-u||g''(u)|du;x\right) + \int_x^{\frac{a_n x+b_n}{a_n}} \left|\frac{a_n x+b_n}{a_n} - u\right| |g''(u)|du \\
&\quad\le \|g''\| \left(D_{a_n,b_n}((t-x)^2;x) + \left(\frac{a_n x+b_n}{a_n} - x\right)^2\right). \qquad (20)
\end{aligned}$$

From Lemma 3, by (11) and by using the condition (3), we find

$$\begin{aligned}
D_{a_n,b_n}((t-x)^2;x) + \left(\frac{a_n x+b_n}{a_n} - x\right)^2 &= \frac{3b_n}{a_n}x + 2\left(\frac{b_n}{a_n}\right)^2 + \left(\frac{b_n}{a_n}\right)^2 \qquad (21)\\
&= \frac{3b_n}{a_n}x + 3\left(\frac{b_n}{a_n}\right)^2 \\
&= \frac{3b_n}{a_n}(\delta_{a_n,b_n}(x))^2.
\end{aligned}$$

Using the inequalities (20) and (21) we get

$$\left|\overline{D}_{a_n,b_n}(g;x) - g(x)\right| \le \frac{3b_n}{a_n}(\delta_{a_n,b_n}(x))^2\|g''\| \qquad (22)$$

For $f \in C_B[0,\infty)$ we can write

$$\begin{aligned}
\left|D_{a_n,b_n}(f;x)\right| &\le \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} p_{n,k}(u)|f(u)|du \\
&\le \|f\|\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} p_{n,k}(u)du \\
&= \|f\|D_{a_n,b_n}(e_0;x) = \|f\|.
\end{aligned}$$

Hence for all $f \in C_B[0,\infty)$, we find

$$\left|\overline{D}_{a_n,b_n}(f;x)\right| \leq \left|D_{a_n,b_n}(f;x)\right| + |f(x)| + \left|f\left(\frac{a_nx+b_n}{a_n}\right)\right| \leq 3\|f\| \tag{23}$$

Combining (22) and (23), for $f \in C_B[0,\infty)$ and $g \in C_B^{(2)}[0,\infty)$, we have

$$\begin{aligned}&\left|D_{a_n,b_n}(f;x) - f(x)\right| \\ &\leq \left|\overline{D}_{a_n,b_n}((f-g);x)\right| + |(f-g)(x)| + \left|\overline{D}_{a_n,b_n}(g;x) - g(x)\right| + \left|f\left(\frac{a_nx+b_n}{a_n}\right) - f(x)\right| \\ &\leq 4\|f-g\| + \frac{3b_n}{a_n}(\delta_{a_n,b_n}(x))^2 + \omega_1\left(f,\frac{b_n}{a_n}\right).\end{aligned}$$

Taking the infimum over all $g \in C_B^2[0,\infty)$,we reach the result (18) and by using the inequality (15) we find

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq M\omega_2\left(f;(b_n/a_n)^{\frac{1}{2}}\right) + \omega_1\left(f;\frac{b_n}{a_n}\right)$$

which implies the proof.

Let $f \in C_B[0,\infty)$ and $\phi(x) = \sqrt{x}, x \in [0,\infty)$. The second order Ditzian-Totik modulus of smoothness and corresponding K- functional are given by, respectively,

$$\begin{aligned}\omega_2^{\phi}(f,\sqrt{\delta}) &= \sup_{0<h\leq\sqrt{\delta}} \sup_{x\pm h\phi(x)\in[0,\infty)} |f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|, \\ \widetilde{K}_{2,\phi(x)}(f,\delta) &= \inf\{||f-g|| + \delta||\phi^2 g''|| + \delta^2||g''||: g \in C^2(\phi)\}, (\delta > 0),\end{aligned}$$

where $C^2(\phi) = \{g \in C[0,\infty): g' \in AC_{loc}[0,\infty), \phi^2 g'' \in C[0,\infty)\}$ and $g' \in AC_{loc}[0,\infty)$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,\infty)$. It is known that there exists a positive constant $M > 0$, such that

$$\widetilde{\mathcal{K}}_{2,\phi(x)}(f,\delta) \leq M\omega_2^{\phi}(f,\sqrt{\delta}) \tag{24}$$

(see [7], p.68).

Theorem 4: Let $f \in C_B[0,\infty)$. For $x \in [0,\infty)$,

$$||D_{a_n,b_n}(f) - f|| \leq 4\widetilde{\mathcal{K}}_{2,\phi(x)}\left(f,\left(\frac{b_n}{a_n}\right)^{1/2}\right),$$

where $\phi(x) = \sqrt{x}$.

Proof: By the means of the auxiliary operators defined by

$$\overline{D}_{a_n,b_n}(f;x) = D_{a_n,b_n}(f;x) + f(x) - f\left(\frac{a_nx+b_n}{a_n}\right)$$

we obtain

$$\overline{D}_{a_n,b_n}(e_0;x) = 1, \overline{D}_{a_n,b_n}(e_1;x) = x.$$

For $g \in C^2(\phi)$, by using Taylor's formula of g, on proceeding as in the proof of Theorem 3, we get

$$\left|\overline{D}_{a_n,b_n}(g;x) - g(x)\right| \le D_{a_n,b_n}\left(\left|\int_x^t |t-u||g''(u)|du\right|;x\right) + \int_x^{\frac{a_nx+b_n}{a_n}} \left|\frac{a_nx+b_n}{a_n} - u\right| |g''(u)|du. \quad (25)$$

Because the function $\delta^2_{a_n,b_n}(x) = \phi^2(x) + b_n/a_n$ is concave on $x \in [0,\infty)$, for $u = \tau x + (1-\tau)t, \tau \in (0,1)$, we get

$$\frac{|t-u|}{\delta^2_{a_n,b_n}(u)} = \frac{\tau|t-x|}{\delta^2_{a_n,b_n}(\tau x+(1-\tau)t)} \le \frac{\tau|t-x|}{\delta^2_{a_n,b_n}(x)\tau+\delta^2_{a_n,b_n}(t)(1-\tau)} \le \frac{|t-x|}{\delta^2_{a_n,b_n}(x)}. \quad (26)$$

From the inequalities (25) and (26), we obtain

$$\begin{aligned}
|\overline{D}_{a_n,b_n}(g;x) - g(x)| &\le \left\|\delta^2_{a_n,b_n}g''\right\| D_{a_n,b_n}\left(\int_x^t \frac{|t-u|}{\delta^2_{a_n,b_n}(x)}du;x\right) \\
&+ \left\|\delta^2_{a_n,b_n}g''\right\|\left(\int_x^{\frac{a_nx+b_n}{a_n}} \frac{|\frac{a_nx+b_n}{a_n}-u|}{\delta^2_{a_n,b_n}(x)}du\right) \\
&\le \frac{1}{\delta^2_{a_n,b_n}(x)}\left\|\delta^2_{a_n,b_n}g''\right\|\left[\mu_{n,2}(x) + \left(\frac{a_nx+b_n}{a_n} - x\right)^2\right] \\
&\le \frac{1}{\delta^2_{a_n,b_n}(x)}\left\|\delta^2_{a_n,b_n}g''\right\|\left[\mu_{n,2}(x) + \left(\frac{b_n}{a_n}\right)^2\right].
\end{aligned}$$

As in the proof of the Theorem 2, we can write

$$|\overline{D}_{a_n,b_n}(g;x) - g(x)| \le \frac{3b_n}{a_n}\left\|(\delta_{a_n,b_n})^2 g''\right\|. \quad (27)$$

Using (23) and (27) we have for $f \in C_B[0,\infty)$

$$\begin{aligned}
\left|\overline{D}_{a_n,b_n}(f;x) - f(x)\right| &\le \left|\overline{D}_{a_n,b_n}((f-g);x)\right| + \left|\overline{D}_{a_n,b_n}(g;x) - g(x)\right| + |g(x) - f(x)| \\
&\le 4\|f-g\| + \frac{3b_n}{a_n}\left\|\phi^2 g''\right\| + 3\left(\frac{b_n}{a_n}\right)^2\left\|g''\right\|
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C^2(\phi)$ we obtain

$$\left|\overline{D}_{a_n,b_n}(f;x) - f(x)\right| \le 4\widetilde{K}_{2,\phi(x)}\left(f, \frac{b_n}{a_n}\right).$$

Now we consider following [14] the Lipschitz-type space for this operators;

$$Lip^*_M(\eta) = \left\{f \in C_B[0,\infty): |f(t) - f(x)| \le M\frac{|t-x|^\eta}{(t+x)^{\frac{\eta}{2}}}; x,t \in (0,\infty)\right\},$$

where M is a positive constant and $\eta \in (0,1]$.

Theorem 5: Let $f \in Lip_M^*(\eta)$. Then, for all $x \in (0,\infty)$, we get

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq M\left(\frac{\mu_{n,2}(x)}{x}\right)^{\eta/2} \tag{28}$$

where $\mu_{n,2}(x)$ is the same with Remark 2.

Proof: First, we start with the case $\eta = 1$. We can write

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)|f(u)-f(x)|du$$
$$\leq M\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)\frac{|u-x|}{\sqrt{u+x}}du$$

Using the fact that $\frac{1}{\sqrt{u+x}} \leq \frac{1}{\sqrt{x}}$ and before the Hölder's inequality and after the Cauchy-Schwarz inequality, we may write

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq M\frac{1}{\sqrt{x}}\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)|u-x|du$$
$$= M\frac{1}{\sqrt{x}}D_{a_n,b_n}(|t-x|;x)$$
$$\leq M\left(\frac{\mu_{n,2}(x)}{x}\right)^{1/2}.$$

The desired result is obtained for $\eta = 1$. Now, we prove the case $0 < \eta < 1$. Applying the Hölder's inequality with $p = \frac{1}{\eta}$ and $q = \frac{1}{1-\eta}$, we have

$$\left|D_{a_n,b_n}(f;x) - f(x)\right| \leq \frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)|f(u)-f(x)|du$$
$$\leq \left\{\sum_{k=0}^{\infty} l_{n,k}(x)\left(\frac{a_n}{b_n}\int_0^{\infty} P_{n,k}(u)|f(u)-f(x)|du\right)^{\frac{1}{\eta}}\right\}^{\eta}$$
$$\leq \left\{\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)|f(u)-f(x)|^{\frac{1}{\eta}}du\right\}^{\eta}$$
$$\leq M\left\{\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)\frac{|u-x|}{\sqrt{u+x}}du\right\}^{\eta}$$
$$\leq \frac{M}{x^{\frac{\eta}{2}}}\left\{\frac{a_n}{b_n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} P_{n,k}(u)|u-x|du\right\}^{\eta}$$
$$\leq \frac{M}{x^{\frac{\eta}{2}}}\left(D_{a_n,b_n}(|t-x|;x)\right)^{\eta} \leq M\left(\frac{\mu_{n,2}(x)}{x}\right)^{\eta/2},$$

so, we obtain (28).

It is obvious that, $Lip_M^*(\eta) \subset Lip_M(\eta)$ where $Lip_M(\eta) = \{f: |f(t) - f(x)| \leq$ $Mt-x\eta,x,t\in 0,\infty$. Therefore, Teorem 5 shows that the degree of approximation is better in $Lip_M^*(\eta)$.

4. LOCAL APPROXIMATION

In this section, we obtain some pointwise estimates of rate of convergence of the operators (4). First, we give a connection between the local smoothness on f and local approximation.

Let E be any subset of $[0,1)$ and $\eta \in (0,1]$. Then $Lip_{M_f}(E,\eta)$ denotes the space of all functions $f \in C_B[0,\infty)$ satisfying the condition

$$|f(t)-f(x)| \leq M_f|t-x|^{\eta}, \forall t \in \overline{E} \text{ and } x \in [0,\infty),$$

where M_f is a constant depending on f and $\overline{E}$ denotes the closure of E in $[0,\infty)$.

Theorem 6: Let $f \in C_B[0,\infty) \cap Lip_{M_f}(E,\eta)$, $\eta \in (0,1]$ and E is a any bounded subset of $[0,\infty)$. Then, for each $x \in [0,\infty)$, we have

$$\left|D_{a_n,b_n}(f;x)-f(x)\right| \leq M_f\left\{\left(\frac{3b_n}{a_n}\delta^2_{a_n,b_n}(x)\right)^{\eta/2}+\left(d(x,E)\right)^{\eta}\right\}$$

where M_f is a constant depending on f and $d(x,E)$ is a distance between point x and E that is

$$d(x,E)=\inf\{|t-x|: t \in E\}.$$

Proof: If $\overline{E}$ is a closure of subset E in $[0,\infty)$, there exists at least a point $x_0 \in \overline{E}$ such that

$$d(x,E)=|x_0-x|.$$

Using the triangle inequality, we get

$$|f(t)-f(x)| \leq |f(t)-f(x_0)|+|f(x)-f(x_0)| \tag{29}$$

and operating D_{a_n,b_n} on both sides of (29) and using the definition of $Lip_{M_f}(E,\eta)$, we get

$$\begin{aligned}\left|D_{n,b_n}(f;x)-f(x)\right| &\leq D_{a_n,b_n}(|f(t)-f(x_0)|;x)+D_{a_n,b_n}(|f(x)-f(x_0)|;x)\\ &\leq M_f\{D_{a_n,b_n}(|t-x_0|^{\eta};x)+|x-x_0|^{\eta}\}\end{aligned}$$

Applying the Hölder's inequality with $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$, using Lemma 3, it follows that

$$\begin{aligned}\left|D_{a_n,b_n}(f;x)-f(x)\right| &\leq M_f\left\{\left(D_{a_n,b_n}(|t-x|^{\eta p};x)\right)^{1/p}\left(D_{a_n,b_n}(1^q;x)\right)^{1/q}+\left(d(x,E)\right)^{\eta}\right\}\\ &\leq M_f\left\{\left(D_{a_n,b_n}((t-x)^2;x)\right)^{\frac{\eta}{2}}+\left(d(x,E)\right)^{\eta}\right\}\\ &\leq M_f\left\{\left(\frac{3b_n}{a_n}\delta^2_{a_n,b_n}(x)\right)^{\eta/2}+(d(x,E)^{\eta})\right\}.\end{aligned}$$

Hence, the proof is completed.

5. WEIGHTED APPROXIMATION

Let $B_{x^2}[0,\infty)$ denotes the space of functions defined on $[0,\infty)$ and satisfying the condition $|f(x)| \le M_f\,(1+x^2)$ where M_f is a constant depending on f and $B_{x^2}[0,\infty)$ is endowed with the weighted norm $\|f\|_{x^2} = \sup_{x\ge 0} \frac{|f(x)|}{1+x^2}$.

Let $C_{x^2}[0,\infty)$ denotes the space of all continuous functions on real semi-axis belonging to the class $B_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ be the space of functions in $C_{x^2}[0,\infty)$ such that the limit $\lim_{x\to\infty} f(x)(1+x^2)^{-1}$exists and is finite. For each $f \in C^*_{x^2}[0,\infty)$ and $\rho(x) = 1+x^2$, the weighted modulus of continuity is defined by

$$\Omega_\rho(f;\delta) = \sup_{\substack{x\ge 0\\ |h|\le\delta}} \frac{|f(x+h)-f(x)|}{\rho(x)\,\rho(h)}.$$

Notice that, the basic properties of $\Omega_\rho(f;\delta)$ is similar to properties of usual modulus of continuity (see [10]).

Theorem 7: Let $f \in C^*_{x^2}[0,\infty)$. Then

$$\left\|D_{a_n,b_n}(f) - f\right\|_{x^2} \to 0$$

as $n \to \infty$.

Proof: Considering Lemma 2, we get following equalities,

$$\left\|D_{a_n,b_n}(e_0) - e_0\right\|_\rho = \sup_{x\ge 0} \frac{|D_{a_n,b_n}(e_0)(x) - e_0(x)|}{\rho(x)} = 0$$

$$\left\|D_{a_n,b_n}(e_1) - e_1\right\|_\rho = \sup_{x\ge 0} \frac{|D_{a_n,b_n}(e_1)(x) - e_1(x)|}{\rho(x)} = \frac{b_n}{a_n}$$

and

$$\begin{aligned}\left\|D_{a_n,b_n}(e_2) - e_2\right\|_\rho &= \sup_{x\ge 0} \frac{|D_{a_n,b_n}(e_2)(x) - e_2(x)|}{\rho(x)}\\ &= \sup_{x\ge 0}\left(\frac{5b_n x}{a_n} + \frac{2(b_n)^2}{a_n^2}\right)\frac{1}{1+x^2}.\end{aligned}$$

By using the condition (3) we have $\lim_{n\to\infty}\left\|D_{a_n,b_n}(e_k) - e_k\right\|_\rho = 0$ for each $k = 0,1,2$.

From Korovkin type theorem in weighted spaces, we obtain the desired result.

Theorem 8: For $f \in C^*_{x^2}[0,\infty)$, the inequality

$$\sup_{x\ge 0} \frac{|D_{a_n,b_n}(f;x) - f(x)|}{\rho(x)^{5/2}} \le K\Omega_\rho(f;\delta'_{a_n,b_n})$$

holds for a sufficiently large n, where $\delta'_{a_n,b_n} = \sqrt{b_n/a_n}$ and K is a constant independent of n.

Proof: For every $f \in C^*_{x^2}[0,\infty)$ and $x, t \in 0,\infty)$, using the definition of Ω_ρ and by the elementary inequality

$$|f(t)-f(x)| \leq 2(1+x^2)\left(1+\left(\delta'_{a_n,b_n}\right)^2\right)\Omega_\rho(f;\delta'_{a_n,b_n})\left(1+\frac{|t-x|}{\delta'_{a_n,b_n}}\right)(1+(t-x)^2)$$

we can write

$$\begin{aligned}\left|D_{a_n,b_n}(f;x)-f(x)\right| &\leq 2(1+x^2)\left(1+\left(\delta'_{a_n,b_n}\right)^2\right)\Omega_\rho(f;\delta'_{a_n,b_n})\sum_{k=0}^{\infty} l_{n,k}(x)\\ &\times\int_0^\infty P_{n,k}(u)\left(1+\frac{1}{\delta'_{a_n,b_n}}|u-x|\right)(1+(u-x)^2)du\\ &\leq 4(1+x^2)\left(1+\left(\delta'_{a_n,b_n}\right)^2\right)\Omega_\rho(f;\delta'_{a_n,b_n})\sum_{k=0}^{\infty} l_{n,k}(x)\\ &\times\left\{1+\frac{1}{\delta'_{a_n,b_n}}\int_0^\infty P_{n,k}(u)|u-x|du+\int_0^\infty P_{n,k}(u)(u-x)^2du\right.\\ &\left.\frac{1}{\delta'_{a_n,b_n}}\int_0^\infty P_{n,k}(u)|u-x|(u-x)^2du\right\}\end{aligned}$$

Applying Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|D_{a_n,b_n}(f;x)-f(x)\right| &\leq 2(1+x^2)\left(1+\left(\delta'_{a_n,b_n}\right)^2\right)\Omega_\rho(f;\delta'_{a_n,b_n})\\ &\times\left[1+D_{a_n,b_n}((e_1-x)^2;x)+\frac{1}{\delta'_{a_n,b_n}}\sqrt{D_{a_n,b_n}((e_1-x)^2;x)}\right.\\ &\left.+\frac{1}{\delta'_{a_n,b_n}}\sqrt{D_{a_n,b_n}((e_1-x)^2;x)D_{a_n,b_n}((e_1-x)^4;x)}\right].\end{aligned}$$

By equations (11) and (12) and considering the condition (3), we get

$$\begin{aligned}\left|D_{a_n,b_n}(f;x)-f(x)\right| &\leq 2(1+x^2)\left(1+\left(\delta'_{a_n,b_n}\right)^2\right)\Omega_\rho(f;\delta'_{a_n,b_n})\\ &\times\left[1+O\left(\frac{b_n}{a_n}\right)(x+1)+\frac{1}{\delta'_{a_n,b_n}}\sqrt{O\left(\frac{b_n}{a_n}\right)(x+1)}\right.\\ &\left.+\frac{1}{\delta'_{a_n,b_n}}\sqrt{O\left(\frac{b_n}{a_n}\right)(x+1)(x^2+x+1)}\right]\end{aligned}$$

and there exist a positive constant M such that,

$$\begin{aligned}\left|D_{a_n,b_n}(f;x)-f(x)\right| &\leq 2(1+x^2)\Omega_\rho(f;\delta'_{a_n,b_n})\\ &\times\left[1+M\left(\frac{b_n}{a_n}\right)(x+1)+\frac{1}{\delta'_{a_n,b_n}}\sqrt{M\left(\frac{b_n}{a_n}\right)(x+1)}\right.\\ &\left.+\frac{1}{\delta'_{a_n,b_n}}\sqrt{M\left(\frac{b_n}{a_n}\right)(x+1)(x^2+x+1)}\right]\end{aligned}$$

Taking $K=4\left[1+M+2\sqrt{M}\right], \delta'_{a_n,b_n}=(b_n/a_n)^{1/2}$ and using the condition (3), for sufficiently large n, we obtain

$$\sup_{x\geq 0}\frac{\left|D_{a_n,b_n}(f;x)-f(x)\right|}{(1+x^2)^{5/2}}\leq K\Omega_\rho\left(f;\sqrt{\frac{b_n}{a_n}}\right).$$

We will prove a Voronoskaja type theorem for the operators $D_{a_n,b_n}(f;x)$ with $f\in C^*_{x^2}[0,\infty)$.

Theorem 9: Let $f\in C^*_{x^2}[0,\infty)$ be a function. If f is two times differentiable at a point $x\in[0,\infty)$, with f'' continuous on x we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}\left[D_{a_n,b_n}(f;x)-f(x)\right]=f'(x)+\frac{3}{2}xf''(x). \tag{30}$$

If f is two times differentiable on $[0,\infty)$, with f'' continuous on $[0,\infty)$, we get the equality (30) uniformly with respect to $x\in[0,a]\subset[0,\infty), a>0$.

Proof: Let $f\in C^*_{x^2}[0,\infty)$ and let $x\in[0,\infty)$ be a point fixed. For $t\in[0,\infty)$, by using Taylor expansion we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+r(t,x)(t-x)^2,$$

where $r(t,x)\to 0$ as $t\to x$ and $r(t,x)\in C^*_{x^2}[0,\infty)$. Applying the operator D_{a_n,b_n}, we get

$$D_{a_n,b_n}(f;x)-f(x)=D_{a_n,b_n}((t-x);x)f'(x)+\frac{1}{2}D_{a_n,b_n}((t-x)^2;x)f''(x)$$
$$+D_{a_n,b_n}(r(t,x)(t-x)^2;x).$$

Taking into account Remark 2, we reach the following result

$$\lim_{n\to\infty}\frac{a_n}{b_n}\left[D_{a_n,b_n}(f;x)-f(x)\right]=f'(x)+\lim_{n\to\infty}\frac{1}{2}\left[3x+2\frac{b_n}{a_n}\right]f''(x)$$
$$+\lim_{n\to\infty}\frac{a_n}{b_n}D_{a_n,b_n}(r(t,x)(t-x)^2;x). \tag{31}$$

From Cauchy-Schwarz inequality, we have

$$D_{a_n,b_n}(r(t,x)(t-x)^2;x)\leq\left\{D_{a_n,b_n}((r(t,x))^2;x)\right\}^{1/2}\left\{D_{a_n,b_n}((t-x)^4;x)\right\}^{1/2}.$$

The properties of the function $r(t,x)$ implies that $r^2(x,x)=0$ and $r^2(x,x)\in C^*_{x^2}[0,\infty)$. Hence from Theorem 8, we obtain

$$\lim_{n\to\infty}D_{a_n,b_n}((r(t,x))^2;x)=r^2(x,x)=0,$$

uniformly with respect to $x\in[0,a]\subset[0,\infty)$. Hence, by using (11) we get

$$\lim_{n\to\infty}\frac{a_n}{b_n}D_{a_n,b_n}(r(t,x)(t-x)^2;x)=0.$$

Then, from (31) and by using the condition (3), we reach the desired result.

6. EXAMPLES

Now, we give some numerical examples showing the degree of approximation and illustrate the convergence of the operators to a certain function by means of graphics.

Example 1. In Figs. 1-2, for $n = 30,50,150$ the convergence of D_{a_n,b_n} (yellow, green and pink in Figure 1, yellow, green and red in Figure 2) to $f(x) = x^3(x - 3/4)^2$ (blue) is illustrated with the different sequences pairs $a_n = n$, $b_n = \ln\sqrt{n + 10}$ and $a_n = n$, $b_n = \ln(n)$. It is easily seen that as the values of n increase, the convergence of $D_{a_n,b_n}(f; x)$ to $f(x)$ becomes better.

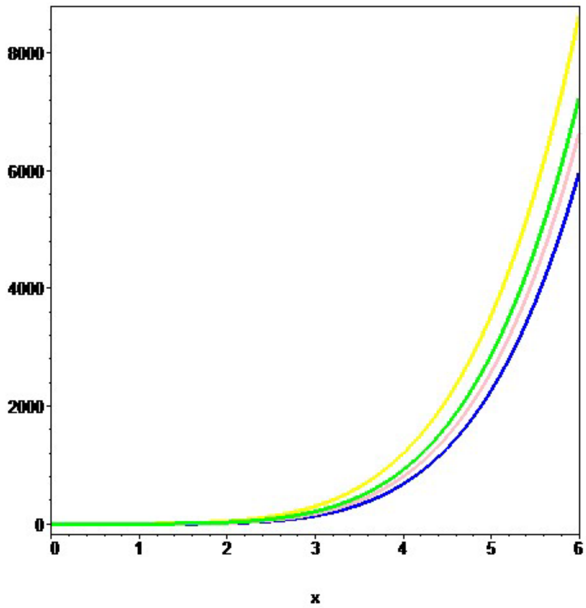

Figure 1. For $a_n = n$, $b_n = \ln\sqrt{n+10}$**, the convergence** $D_{a_n,b_n}(f;x)$ **to** $f(x)$**.**

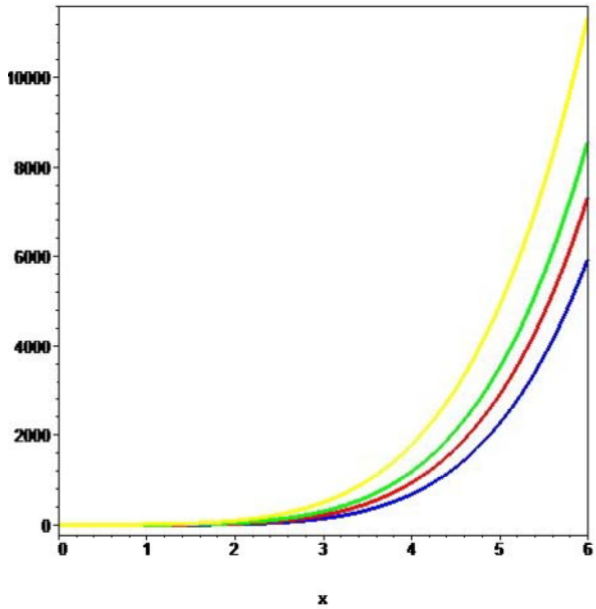

Figure 2. For $a_n = n$, $b_n = \ln n$**, the convergence** $D_{a_n,b_n}(f;x)$ **to** $f(x)$**.**

For $f(x) = x^3(x - 3/4)^2$, $x \in [0,1]$, we give the degree of $D_{a_n,b_n}(f; x)$ in term of modulus of continuity in the following table. The error bound is smaller while the values of n increase and it also changes for different (a_n) and (b_n) sequences.

Table 1. The degree of approximation of $D_{a_n,b_n}(f; x)$ in terms of modulus of continuity.

n	error bound for $a_n = n$, $b_n = \ln(n)$	error bound for $a_n = n$, $b_n = \ln\sqrt{n + 10}$
1000	.1109953430	.09326718976
1500	.1024546745	.08368864662
2000	.09583131158	.07693160462
5000	.07693160462	.05692206160
7000	.06635831382	.05043469828
100000	.02375596746	.01716194743
100000	.008631796320	.006148873274
1500000	.007181537140	.005109281812

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