ORIGINAL PAPER

ON THE NATURAL LIFT CURVE AND THE INVOLUTE CURVE

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Abstract. *In this study, the relation between Frenet vectors of the natural lift curve $\overline{\alpha}$ of the curve α and Frenet vectors of a β the involute of α is given in IR^3 and IR_1^3.*

Keywords: *Natural Lift Curve, Involute-Evolute, Frenet Vectors.*

1. INTRODUCTION

The concepts of the natural lift curve and geodesic sprays have first been given by Thorpe in [1]. Thorpe provied the natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M. Çalışkan at al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [2]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$. Some properties of $\overline{M}$ -vector field Z defined on a hypersurface M of $\overline{M}$ were studied by Agashe in [3]. $\overline{M}$ -integral curve of Z and $\overline{M}$ -geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\overline{\alpha}$ of the curve α (in $\overline{M}$) is an $\overline{M}$ -integral curve of the geodesic spray Z iff α is an $\overline{M}$ -geodesic in [4]. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [5]. The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [5]. Çalışkan and Ergün defined $\overline{M}$ -vector field Z, $\overline{M}$ -geodesic spray, $\overline{M}$ -integral curve of Z, $\overline{M}$ -geodesic in [6].The anologue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [6]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [7]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [8].

One of the most significant curve is an involute of a given curve. The concept of involute was first studied by Huygens when he was considering clocks without pendula for use on ships at sea. An involute of a given curve is some other curve that always remains perpendicular to the tangent lines to that given curve. This can also be thought as the process of winding or unwinding a string tautly around a curve. The original curve is called an evolute. In addition to this, involute-evolute curve couple is a well known concept in the

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classical differential geometry, see [9-11]. The basic local theory of space curve are mainly developed by the Frenet-Serret theorem which expresses the derivative of a geometrically chosen basis of E^3 by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated, see for the details [12].

Let α be a regular curve with $\alpha''(s) \neq 0$. For every point of $\alpha(s)$, the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where

$$T(s) = \alpha'(s),\ N(s) = \frac{\alpha''}{\|\alpha''\|},\ B(s) = T(s) \times N(s)$$

are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N.$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [12].

Let M be a hypersurface in IR^3 and let $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s))\ (\text{for all } t \in I)$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

For any parametrized curve $\alpha : I \to M$, $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)|_{\alpha(s)}\right) = D_{\alpha'(s)}\alpha'(s)$$

where D is the Levi-Civita connection on IR^3 [1].

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector W which satisfies $T' = W \times T$, $N' = W \times N$ and $B' = W \times B$. The vector W is called the Darboux vector. In terms of Frenet vectors T, N and B, Darboux vector is given by $W = \tau T + \kappa B$. Also, we have $\kappa = \|W\|\cos\varphi$, $\tau = \|W\|\sin\varphi$, where φ is the angle between the Darboux vector of α and binormal vector $B(s)$.

Let $\alpha(s)$, $L_1 \le s \le L_2$, be an arc length curve and $\overline{\alpha}(s)$, $L_1 \le s \le L_2$, be natural lift of α. Then we have

$$\begin{bmatrix} \overline{T}(s) \\ \overline{N}(s) \\ \overline{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where $\{T(s), N(s), B(s)\}$ and $\{\overline{T}(s), \overline{N}(s), \overline{B}(s)\}$ are Frenet frames of the curves α and $\overline{\alpha}$, respectively, and φ is the angle between the Darboux vector and binormal vector .

Definition 1: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be two regular curves in IR^3. $\beta(s^*)$ is called the involute of $\alpha(s)$ ($\alpha(s)$ is called the evolute of $\beta(s^*)$) if

$$\langle T(s), T^*(s) \rangle = 0 .$$

In this situIation, (α, β) is called the involute-evolute curve couple.

Lemma 1: Let β be a curve the involute of α. The relations between the Frenet vectors of the (α, β) is as follow,

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Minkowski 3-space IR_1^3 is the vector space IR^3 equipped with the Lorentzian inner product g given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathrm{IR}^3$. A vector $X = (x_1, x_2, x_3) \in \mathrm{IR}^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ or $X = 0$ and lightlike (or null) if $g(X, X) = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in IR_1^3 can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $s \in I \subset \mathrm{IR}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp . $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp . $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X, X)|}$ [13].

The vectors $X=(x_1,x_2,x_3)$, $Y=(y_1,y_2,y_3)\in \mathrm{IR}_1^3$ are orthogonal if and only if $g(X,X)=0$ [14].

Now let X and Y be two vectors in IR_1^3 , then the Lorentzian cross product is given by [15]

$$X\times Y=\begin{vmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$
$$=(x_2y_3-x_3y_2,\ x_1y_3-x_3y_1,\ x_2y_1-x_1y_2).$$

We denote by $\{T(s),N(s),B(s)\}$ the moving Frenet frame along the curve α . Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α , respectively.

Let α be a unit speed timelike curve with curvature κ and torsion τ . So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

$$T\times N=-B,\quad N\times B=T,\quad B\times T=-N,$$

where $\times$ is the Lorentzian cross product in IR_1^3 [7] . The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s),N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T,N,B\}$ has the same orientation as IR_1^3 . Then, Frenet formulas are given by [7]

$$T'=\kappa N,\ N'=\kappa T+\tau B,\ B'=-\tau N.$$

Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T\times N=-B,\quad N\times B=-T,\quad B\times T=N.$$

The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s),N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T,N,B\}$ has the same orientation as IR_1^3 . Then, Frenet formulas are given by [7]

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The binormal vector field $B(s)$ is the unique timelike unit vector field perpendicular to the spacelike plane $\{T(s),N(s)\}$ at every point $\alpha(s)$ of α , such that $\{T,N,B\}$ has the same orientation as IR_1^3 .

Then, Frenet formulas are given by [7]

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = \tau N.$$

Lemma 2: Let X and Y be nonzero Lorentz orthogonal vectors in IR_1^3 . If X is timelike, then Y is spacelike [14].

Lemma 3: Let X and Y be positive (negative) timelike vectors in IR_1^3 . Then $g(X,Y) \leq \|X\|\|Y\|$ whit equality if and only if X and Y are linearly dependent [14].

Lemma 4:

i) Let X and Y be positive (negative) timelike vectors in IR_1^3 . By Lemma 2, there is a unique nonnegative real number $\phi(X,Y)$ such that

$$g(X,Y) = \|X\|\|Y\|\cosh\phi(X,Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$ [14].

ii) Let X and Y be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then we have

$$|g(X,Y)| \leq \|X\|\|Y\|.$$

Hence, there is a unique real number $\phi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\|Y\|\cos\phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian spacelike angle between X and Y [14].

iii) Let X and Y be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then, we have

$$g(X,Y) > \|X\|\|Y\|.$$

Hence, there is a unique positive real number $\phi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\|\|Y\|\cosh\phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [14].

iv) Let X be a spacelike vector and Y be a positive timelike vector in IR_1^3. Then there is a unique nonnegative real number $\phi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\|Y\| \sinh \phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [14].

Let P be a surface in IR_1^3 and let $\alpha : I \to P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \ (\text{for all } t \in I),$$

where X is a smooth tangent vector field on P [13]. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where $T_p P$ is the tangent space of P at p and $\chi(P)$ is the space of tangent vector fields on P.

For any parametrized curve $\alpha : I \to P$, $\overline{\alpha} : I \to TP$ is given by

$$\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TP [5].

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length timelike curve. Then, the natural lift $\overline{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\overline{T}(s), \overline{N}(s), \overline{B}(s)\}$ of $\overline{\alpha}$.

a) Let the natural lift $\overline{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$\begin{bmatrix} \overline{T}(s) \\ \overline{N}(s) \\ \overline{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If W is a timelike vector, then we have

$$\begin{bmatrix} \overline{T}(s) \\ \overline{N}(s) \\ \overline{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

b) Let the natural lift $\overline{\alpha}$ of α is a spacelike curve with spacelike binormal.

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Let $\alpha(s),\ L_1 \le s \le L_2,$ be an arc length spacelike curve with spacelike binormal. Then, the natural lift $\overline{\alpha}$ of α is a timelike curve . We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\overline{T}(s), \overline{N}(s), \overline{B}(s)\}$ of $\overline{\alpha}$.

Let the natural lift $\overline{\alpha}$ of α is a timelike curve.

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Definition 2: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ be a regular timelike curve and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be a regular spacelike curve in IR_1^3. $\beta(s^*)$ is called the involute of $\alpha(s)$ ($\alpha(s)$ is called the evolute of $\beta(s^*)$) if

$$\langle T(s), T^*(s) \rangle = 0 .$$

In this situation, (α, β) is called a timelike-spacelike involute-evolute curve couple.

Let $\alpha(s)$, $L_1 \le s \le L_2$, be an arc length timelike curve. Then, β is a spacelike curve with timelike or spacelike binormal.

a) Let β is a spacelike curve with spacelike binormal.

i) If the Darboux vector W of the curve α is a timelike vector, then we have

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\varphi & 1 & \cosh\varphi \\ -\cosh\varphi & 0 & -\sinh\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If the Darboux vector W of the curve α is a spacelike vector, then we have

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$$\langle T(s), T^*(s) \rangle = 0 .$$

In this situIation, (α, β) is called a spacelike-timelike involute-evolute curve couple.

Let $\alpha(s)$, $L_1 \le s \le L_2$, be an arc length spacelike curve with timelike or spacelike binormal. Then, β is a spacelike curve.

a) Let α be an arc length spacelike curve with spacelike binormal. Then, β is a spacelike curve.

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

b) Let α be an arc length spacelike curve timelike binormal. Then, β is a spacelike curve.

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Definition 4: Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ be a regular spacelike curve and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be a regular spacelike curve in IR_1^3. $\beta(s^*)$ is called the invoüte of $\alpha(s)$,($\alpha(s)$ is called the evolute of $\beta(s^*)$) , if

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a) Let α be an arc length spacelike curve with spacelike binormal. Then, β is a spacelike curve.

i) If β is a spacelike curve with spacelike binormal, then we have

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ -\cos\varphi & 0 & -\sin\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

ii) If β is a spacelike curve with timelike binormal, then we have

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ \cos\varphi & 0 & \sin\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

b) Let α be an arc length spacelike curve with timelike binormal. Then, β is a spacelike curve.

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2) If the Darboux vector W of the curve α is a timelike vector, then we have

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ii) Let β be a spacelike curve with timelike binormal,

1) If the Darboux vector W of the curve α is a spacelike vector, then we have

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2. ON THE NATURAL LIFT CURVE AND THE INVOLUTE CURVE

Let $\overline{\alpha}$ be the natural ift of α and β be the involute of α in IR^3.

$$\begin{aligned} T^* &= \overline{T} \\ N^* &= -\cos\varphi T + \sin\varphi B = -\cos\varphi\left(-\cos\varphi\overline{N} + \sin\varphi\overline{B}\right) + \sin\varphi\left(\sin\varphi\overline{N} + \cos\varphi\overline{B}\right) \\ &= \overline{N} \\ B^* &= \sin\varphi T + \cos\varphi B \\ &= \sin\varphi\left(-\cos\varphi\overline{N} + \sin\varphi\overline{B}\right) + \cos\varphi\left(\sin\varphi\overline{N} + \cos\varphi\overline{B}\right) \\ &= B \end{aligned}$$

where $\{T, N, B\}$, $\{\overline{T}, \overline{N}, \overline{B}\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of the curves α , $\overline{\alpha}$ and β , respectively.

So from the above calculations we can give the following propositions.

Proposition 1: If $\overline{\alpha}$ is natural lift curve of α and β is the involute of α in IR^3, then

$$sp\left\{\overline{T}, \overline{N}, \overline{B}\right\} = sp\left\{T^*, N^*, B^*\right\} .$$

Proposition 2: Let $\overline{\alpha}$ be the natural ift of α and β be the involute of α in IR^3.

Then

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Let $\overline{\alpha}$ be the natural ift of α and β be the involute of α in IR_1^3.

Proposition 4: Let α be an arc length timelike curve. Then, β is a spacelike curve with spacelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = -\overline{N}$$
$$B^* = \overline{B}$$

ii) If the Darboux vector W of the curve α is a timelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = -\overline{B}$$
$$B^* = \overline{N}$$

Proposition 5: Let α be an arc length timelike curve. Then, β is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{B}$$
$$B^* = \overline{N}$$

ii) If the Darboux vector W of the curve α is a timelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Proposition 6: Let α be an arc length a spacelike curve with spacelike binormal.
Then, β is a timelike curve.

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Proposition 7: Let α be an arc length a spacelike curve with timelike binormal. Then, β is a timelike curve.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

ii) If the Darboux vector W of the curve α is a timelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Proposition 8: Let α be an arc length a spacelike curve with spacelike binormal. Then, β is a spacelike curve with timelike binormal.

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Proposition 9: Let α be an arc length a spacelike curve with spacelike binormal. Then, β is a spacelike curve with spacelike binormal.

$$T^* = \overline{T}$$
$$N^* = -\overline{N}$$
$$B^* = \overline{B}$$

Proposition 10: Let α be an arc length a spacelike curve with timelike binormal. Then, β is a spacelike curve with spacelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = -\overline{N}$$
$$B^* = \overline{B}$$

ii) If the Darboux vector W of the curve α is a timelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = -\overline{N}$$
$$B^* = \overline{B}$$

Proposition 11: Let α be an arc length a spacelike curve with timelike binormal. Then, β is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

ii) If the Darboux vector W of the curve α is a timelike vector, then we have

$$T^* = \overline{T}$$
$$N^* = \overline{N}$$
$$B^* = \overline{B}$$

Proposition 3: If $\overline{\alpha}$ is natural lift curve of α and β is the involute of α in IR_1^3. then $sp\{\overline{T}, \overline{N}, \overline{B}\} = sp\{T^*, N^*, B^*\}$.

3. EXAMPLES

Example 1: Given the arclength curve

$$\alpha(s) = \left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}\right)$$

its Frenet apparatus are

$$T(s) = \left(-\cos\left(\frac{\sqrt{2}}{2}s\right), -\sin\left(\frac{\sqrt{2}}{2}s\right), 0\right),$$
$$N(s) = \left(\sin\left(\frac{\sqrt{2}}{2}s\right), -\cos\left(\frac{\sqrt{2}}{2}s\right), 0\right),$$
$$B(s) = (0,0,1),$$

The natural lift of the curve α is

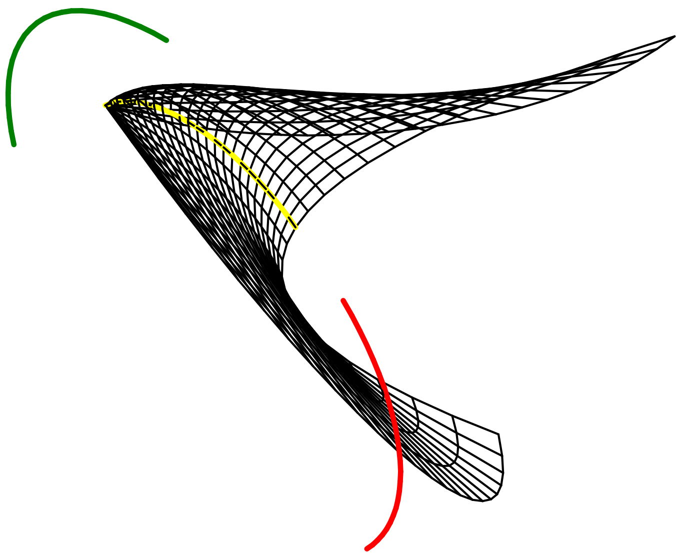
$$\overline{\alpha}(s)=\left(-\cos\left(\frac{\sqrt{2}}{2}s\right),-\sin\left(\frac{\sqrt{2}}{2}s\right),0\right)$$

and the involute of α for can be given by the equation

$$\begin{aligned}\beta(s)&=\left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right)+\lambda\left(-\cos\left(\frac{\sqrt{2}}{2}s\right),-\sin\left(\frac{\sqrt{2}}{2}s\right),0\right)\\&=\left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right)-\lambda\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right)-\lambda\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right),\lambda\in\mathrm{IR},\end{aligned}$$

for $\lambda=-\frac{\sqrt{2}}{2}$, we have

$$\begin{aligned}\beta(s)&=\left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right)+\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right)+\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right)\\&=\left(-\frac{\sqrt{2}}{2}\left(\sin\left(\frac{\sqrt{2}}{2}s\right)-\cos\left(\frac{\sqrt{2}}{2}s\right)\right),\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\sqrt{2}}{2}s\right)+\sin\left(\frac{\sqrt{2}}{2}s\right)\right),\frac{\sqrt{2}}{2}\right).\end{aligned}$$



Example 2: Given the arclength curve

$$\alpha(s)=\left(\frac{\sqrt{3}}{2}\sin s,\frac{1}{2}s,\frac{\sqrt{3}}{2}\cos s\right)$$

its Frenet apparatus are

$$T(s)=\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2}-\frac{\sqrt{3}}{2}\sin s\right),$$

$$N(s)=(-\sin s,0,-\cos s),$$

$$B(s)=\left(-\frac{1}{2}\cos s,\frac{\sqrt{3}}{2},\frac{1}{2}\sin s\right),$$

The natural lift of the curve α is

$$\overline{\alpha}(s)=\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2}-\frac{\sqrt{3}}{2}\sin s\right)$$

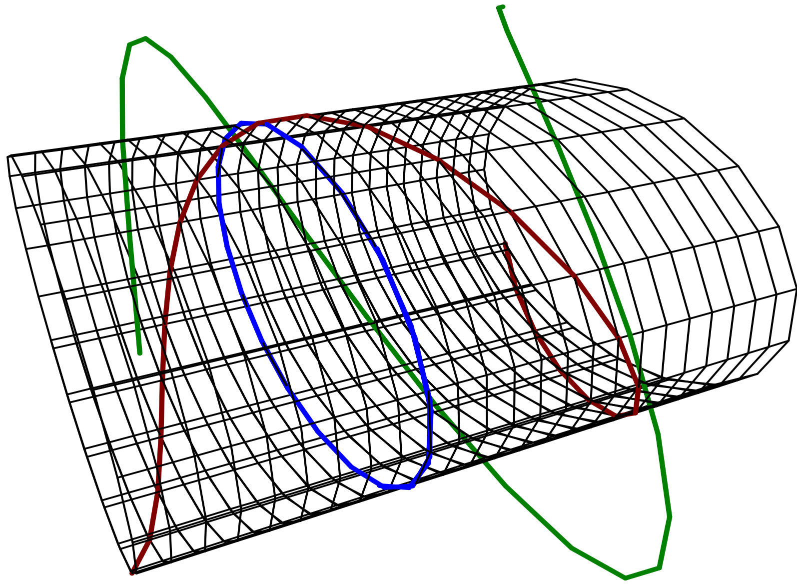
and the involute of α for can be given by the equation

$$\beta(s)=\left(\frac{\sqrt{3}}{2}\sin s,\frac{1}{2}s,\frac{\sqrt{3}}{2}\cos s\right)+\lambda\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2}-\frac{\sqrt{3}}{2}\sin s\right),\lambda\in \mathrm{IR},$$

for $\lambda=-\frac{\sqrt{2}}{2}$, we have

$$\beta(s)=\left(\frac{\sqrt{3}}{2}\sin s,\frac{1}{2}s,\frac{\sqrt{3}}{2}\cos s\right)-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2}-\frac{\sqrt{3}}{2}\sin s\right)$$

$$=\left(\frac{\sqrt{3}}{2}\sin s-\frac{\sqrt{6}}{4}\cos s,\frac{1}{2}s-\frac{\sqrt{2}}{4},\frac{\sqrt{3}}{2}\cos s+\frac{\sqrt{6}}{4}\sin s\right)$$



Example 3: Let

$$\alpha(s) = (\cosh s, 0, \sinh s)$$

be a timelike curve. It is easy to show that

$$T(s) = (\sinh s, 0, \cosh s),$$
$$N(s) = (\cosh s, 0, \sinh s),$$
$$B(s) = (0, 1, 0),$$

In this situation, the natural lift of α and the involute of α for can be given by the equations

$$\overline{\alpha}(s) = (\sinh s, 0, \cosh s)$$

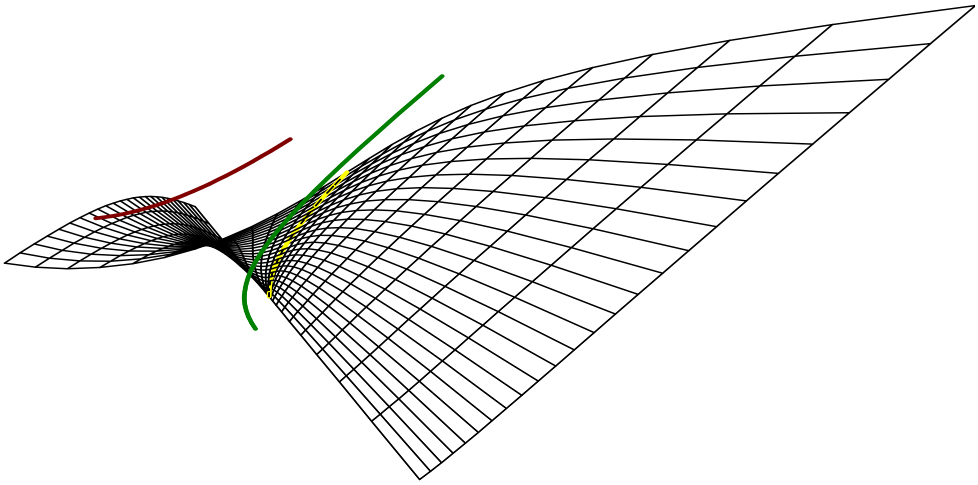
and

$$\beta(s) = (\cosh s, 0, \sinh s) + \lambda(\sinh s, 0, \cosh s), \lambda \in \mathrm{IR},$$

for $\lambda = \frac{\sqrt{7}-1}{3}$, we have

$$\beta(s) = (\cosh s, 0, \sinh s) + \tfrac{\sqrt{7}-1}{3}(\sinh s, 0, \cosh s)$$

$$\beta(s) = \left(\cosh s + \tfrac{\sqrt{7}-1}{3}\sinh s, 0, \sinh s + \tfrac{\sqrt{7}-1}{3}\cosh s\right)$$



Example 4: Given the arclength timelike curve

$$\alpha(s)=\left(\frac{5}{3}s,\frac{4}{9}\cos 3s,\frac{4}{9}\sin 3s\right)$$

its Frenet apparatus are

$$T(s)=\left(\frac{5}{3},-\frac{4}{3}\sin 3s,\frac{4}{3}\cos 3s\right),$$
$$N(s)=(0,-\cos 3s,-\sin 3s),$$
$$B(s)=\left(-\frac{4}{3},\frac{5}{3}\sin 3s,-\frac{5}{3}\cos 3s\right),$$

The natural lift of the curve α is

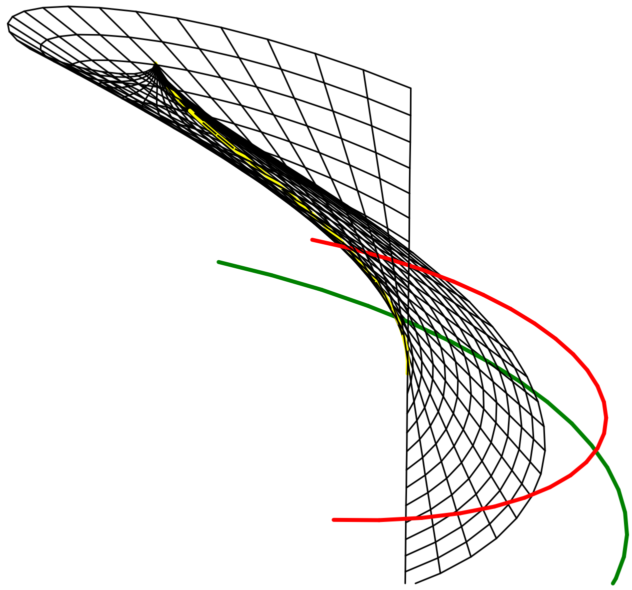
$$\bar{\alpha}(s)=\left(\frac{5}{3},-\frac{4}{3}\sin 3s,\frac{4}{3}\cos 3s\right)$$

and the involute of α for can be given by the equation

$$\beta(s)=\left(\frac{5}{3}s,\frac{4}{9}\cos 3s,\frac{4}{9}\sin 3s\right)+\lambda\left(\frac{5}{3},-\frac{4}{3}\sin 3s,\frac{4}{3}\cos 3s\right),\lambda\in \mathrm{IR},$$

for $\lambda=\frac{5}{9}$, we have

$$\beta(s)=\left(\frac{5}{3}s,\frac{4}{9}\cos 3s,\frac{4}{9}\sin 3s\right)+\frac{5}{3}\left(\frac{5}{3},-\frac{4}{3}\sin 3s,\frac{4}{3}\cos 3s\right),$$
$$=\left(\frac{5}{3}s+\frac{25}{9},\frac{4}{9}\cos 3s-\frac{20}{9}\sin 3s,\frac{4}{9}\sin 3s+\frac{20}{9}\cos 3s\right)$$



Example 5: Let

$$\alpha(s)=\left(\frac{1}{3}\sinh\left(\sqrt{3}s\right),\frac{2}{3}\sqrt{3}s,\frac{1}{3}\cosh\left(\sqrt{3}s\right)\right)$$

be a unit speed spacelike hyperbolic helix with

$$T=\frac{\sqrt{3}}{3}\left(\cosh\left(\sqrt{3}s\right),2,\sinh\left(\sqrt{3}s\right)\right)$$
$$N=\left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right),\qquad \kappa=1\ and\ \tau=2$$
$$B=\frac{\sqrt{3}}{3}\left(2\cosh\left(\sqrt{3}s\right),1,2\sinh\left(\sqrt{3}s\right)\right)$$

In this situation, the natural lift of α and the involute of α for can be given by the equations

$$\overline{\alpha}(s)=\left(\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3},\frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right)$$

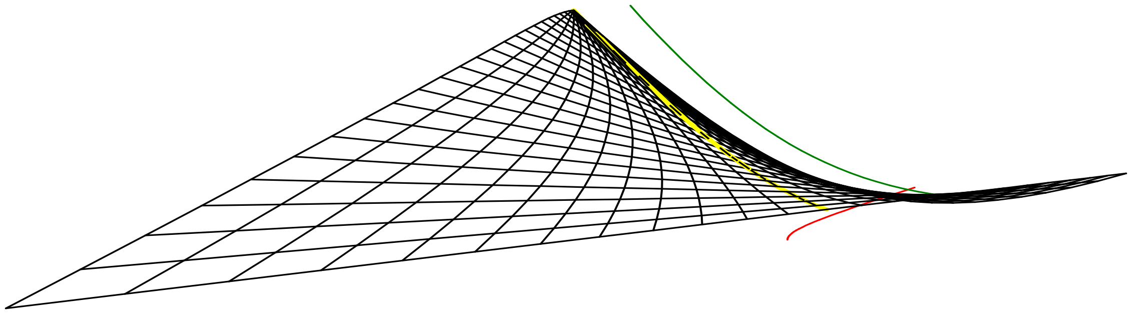
and

$$\beta(s)=\left(\frac{1}{3}\sinh\left(\sqrt{3}s\right),\frac{2}{3}\sqrt{3}s,\frac{1}{3}\cosh\left(\sqrt{3}s\right)\right)+\lambda\frac{\sqrt{3}}{3}\left(\cosh\left(\sqrt{3}s\right),2,\sinh\left(\sqrt{3}s\right)\right),\lambda\in\mathrm{IR},$$

for $\lambda=\frac{\sqrt{7}-1}{3}$, we have

$$\beta(s)=\left(\frac{1}{3}\sinh\left(\sqrt{3}s\right),\frac{2}{3}\sqrt{3}s,\frac{1}{3}\cosh\left(\sqrt{3}s\right)\right)+\frac{\sqrt{7}-1}{3}\left(\frac{\sqrt{3}}{3}\left(\cosh\left(\sqrt{3}s\right),2,\sinh\left(\sqrt{3}s\right)\right)\right),$$

$$=\left(\frac{1}{3}\sinh\left(\sqrt{3}s\right)+\frac{\sqrt{7}-1}{3}\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{2}{3}\sqrt{3}s+\frac{2\sqrt{7}-2}{3},\frac{1}{3}\cosh\left(\sqrt{3}s\right)+\frac{\sqrt{7}-1}{3}\frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right)$$



Example 6: Let

$$\alpha(s)=(\sinh s,0,\cosh s)$$

be a timelike curve. It is easy to show that

$$T(s)=(\cosh s,0,\sinh s),$$
$$N(s)=(\sinh s,0,\cosh s),$$
$$B(s)=(0,-1,0),$$
$$\kappa(s)=1,\ \tau(s)=0.$$

In this situation, the natural lift of α and the involute of α for can be given by the equations

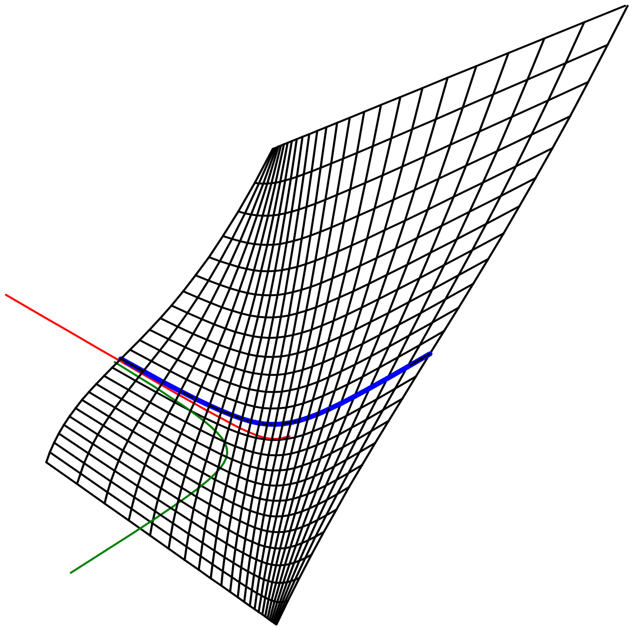
$$\overline{\alpha}(s)=(\cosh s,0,\sinh s)$$

and

$$\beta(s)=(\sinh s,0,\cosh s)+\lambda(\cosh s,0,\sinh s),\lambda\in\mathrm{IR},$$

for $\lambda=\frac{\sqrt{7}}{3}$, we have

$$\beta(s) = (\sinh s, 0, \cosh s) + \frac{\sqrt{7}}{3}(\cosh s, 0, \sinh s)$$
$$= \left(\sinh s + \frac{\sqrt{7}}{3}\cosh s, 0, \cosh s + \frac{\sqrt{7}}{3}\sinh s \right)$$



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