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LUCAS COLLOCATION METHOD FOR SYSTEM OF HIGH-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. *In this paper, a numerical collocation method based on Lucas polynomials is presented to solve the system of linear functional differential equations with variable coefficients under the mixed conditions. This method transforms the functional system along with conditions into a matrix equation by means of collocation points and the truncated Lucas series. Furthermore, by use of an error analysis technique based on residual function, we improve effectiveness of the method. Our results are illustrated and corroborated with some numerical experiments.*

Keywords: *System of functional differential equations; Lucas polynomials and series; Collocation method; Residual error analysis.*

1. INTRODUCTION

The systems of differential, difference, differential-difference and delay differential equations have been confronted in many scientific and technological problems such as engineering, astrophysics, biology, chemical reactions, and mechanics. Most of the systems have no analytic solutions, so numerical techniques have been required such as Adomian decomposition method [1, 2], Differential transformation method [3], Spline approximation [4-9], Runge-Kutta method [10], Variational iteration method [11], Homotopy perturbation method [12], Homotopy analysis method [13], Legendre pseudospectral method [14], Taylor collocation method [15-17], Laguerre collocation method [18-19], Lucas-Taylor collocation method [20], Variable multistep method [21], Linear multistep method [22].

In this study, we introduce a novel collocation method based on Lucas polynomials for solving the system of linear functional differential equations in the form

$$L\left[y_i(t)\right]=\sum_{n=0}^{m}\sum_{j=1}^{k}\left[p_{i,j}^{n}(t)y_j^{(n)}(\lambda t+\mu)+q_{i,j}^{n}(t)y_j^{(n)}(t)\right]=g_i(t),\;\; i=1,2,...,k,\quad 0\le a\le t\le b \qquad (1)$$

under the mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j}y_j^{(n)}(a)+b_{i,n}^{j}y_j^{(n)}(b)\right]=c_{j,i},\quad i=0,1,...,m-1,\quad j=1,2,...,k \qquad (2)$$

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where $y_j^{(0)}(t)=y_j(t)$, $j=1,2,...,k$ are unknown functions; $p_{i,j}^n(t)$, $q_{i,j}^n(t)$ and $g_i(t)$ are continuous functions on $[a,b]$, and $a_{i,n}^j$, $b_{i,n}^j$, $c_{j,i}$, λ and μ are real constant coefficients.

We assume that the system (1) under the mixed conditions (2) has approximate solutions in the truncated Lucas series form

$$y_j(t)\cong y_{j,N}(t)=\sum_{n=0}^{N}a_{j,n}L_n(t),\quad j=1,2,...,k \tag{3}$$

where $a_{j,n}$, $(n=0,1,2,...,N)$ are unknown Lucas coefficients and $L_n(t), n=0,1,...,N$ are the Lucas polynomials defined by

$$L_0(t)=2\ ;\ L_n(t)=\sum_{k=0}^{[n/2]}\frac{n}{n-k}\binom{n-k}{k}t^{n-2k},\ (n\geq 1)\quad [n/2]=\begin{cases} n/2 & ,\ n \text{ even}\\ (n-1)/2 & ,\ n \text{ odd}\end{cases}\quad [23, 24].$$

In addition, in order to find the approximate solutions of the problem (1)- (2), we can use the collocation points defined by

$$t_k=a+\frac{b-a}{N}k,\quad k=0,1,...,N\ . \tag{4}$$

2. MATERIALS AND METHODS

2.1. MATERIALS

In this section, we convert the equations (1)-(3) to the matrix forms. In order to achieve our goal, firstly, we can write the Lucas polynomials $L_n(t)$ in the matrix form

$$\mathbf{L}(t)=\mathbf{T}(t)\mathbf{D}^T \tag{5}$$

where

$$\mathbf{L}(t)=\begin{bmatrix}L_0(t) & L_1(t) & L_2(t) & \cdots & L_N(t)\end{bmatrix},\ \mathbf{T}(t)=\begin{bmatrix}1 & t & t^2 & \cdots & t^N\end{bmatrix}$$

and if N is odd,

$$\mathbf{D}=\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0\\
\frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0\\
0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0\\
\frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{n-1}{(n-1)/2}\binom{(n-1)/2}{(n-1)/2} & 0 & \frac{n-1}{(n+1)/2}\binom{(n+1)/2}{(n-3)/2} & 0 & \frac{n-1}{(n+3)/2}\binom{(n+3)/2}{(n-5)/2} & & 0\\
0 & \frac{n}{(n+1)/2}\binom{(n+1)/2}{(n-1)/2} & 0 & \frac{n}{(n+3)/2}\binom{(n+3)/2}{(n-3)/2} & 0 & \cdots & \frac{n}{n}\binom{n}{0}
\end{bmatrix}.$$

If N is even,

$$\mathbf{D}=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0 \\ \frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{n-1}{n/2}\binom{n/2}{(n-2)/2} & 0 & \frac{n-1}{(n+2)/2}\binom{(n+2)/2}{(n-4/2} & 0 & \cdots & 0 \\ \frac{n}{n/2}\binom{n/2}{n/2} & 0 & \frac{n}{(n+2)/2}\binom{(n+2)/2}{(n-2)/2} & 0 & \frac{n}{(n+4)/2}\binom{(n+4)/2}{(n-4)/2} & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}.$$

$y_{j,N}(t)$ approximate solutions in Eq. (3) can be expressed as

$$y_{j,N}(t)=\mathbf{L}(t)\mathbf{A}_j, \quad j=1,2,...,k \tag{6}$$

where

$$\mathbf{A}_j=\begin{bmatrix} a_{j,0} & a_{j,1} & a_{j,2} & \cdots & a_{j,N} \end{bmatrix}^T .$$

By using (5) and (6), we obtain the relation

$$y_{j,N}(t)=\mathbf{T}(t)\mathbf{D}^T\mathbf{A}_j . \tag{7}$$

Besides, it is obvious that the relation between $\mathbf{T}(t)$ and its derivatives $\mathbf{T}^{(k)}(t)$ can be written as

$$\mathbf{T}^{(k)}(t)=\mathbf{T}(t)\mathbf{B}^k \tag{8}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, from the relations (7) and (8), we obtain the matrix relations

$$y_{j,N}^{(i)}(t)=\mathbf{T}(t)\mathbf{B}^i\mathbf{D}^T\mathbf{A}_j , \quad i=0,1,...,m \text{ and } j=1,2,...,k . \tag{9}$$

By putting $t\to\lambda t+\mu$ into the relation (9), we have

$$y_{j,N}^{(i)}(\lambda t+\mu)=\mathbf{T}(\lambda t+\mu)\mathbf{B}^i\mathbf{D}^T\mathbf{A}_j , \quad i=0,1,...,m \text{ and } j=1,2,...,k . \tag{10}$$

On the other hand, it is well-known that the relation between $\mathbf{T}(\lambda t+\mu)$ and $\mathbf{T}(t)$ is

$$\mathbf{T}(\lambda t+\mu)=\mathbf{T}(t)\mathbf{M}(\lambda,\mu) \tag{11}$$

where for $\lambda \neq 0$ and $\mu \neq 0$ [29]

$$\mathbf{M}(\lambda,\mu)=\begin{bmatrix} \binom{0}{0}\lambda^0\mu^0 & \binom{1}{0}\lambda^0\mu^1 & \binom{2}{0}\lambda^0\mu^2 & \cdots & \binom{N}{0}\lambda^0\mu^N \\ 0 & \binom{1}{1}\lambda^1\mu^0 & \binom{2}{1}\lambda^1\mu^1 & \cdots & \binom{N}{1}\lambda^1\mu^{N-1} \\ 0 & 0 & \binom{2}{2}\lambda^2\mu^0 & \cdots & \binom{N}{2}\lambda^2\mu^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}\lambda^N\mu^0 \end{bmatrix}$$

and $\lambda \neq 0$ and $\mu = 0$

$$\mathbf{M}(\lambda,0)=\begin{bmatrix} \lambda^0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda^1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^N \end{bmatrix}.$$

Substituting (11) into (10 yields

$$y_{j,N}^{(i)}(\lambda t+\mu)=\mathbf{T}(t)\mathbf{M}(\lambda,\mu)\mathbf{B}^i\mathbf{D}^T\mathbf{A}_j\,,\quad i=0,1,\ldots,m \text{ and } j=1,2,\ldots,k \tag{12}$$

In this case, the matrix relations (9) and (12) can be expressed as

$$\mathbf{Y}^{(i)}(t)=\overline{\mathbf{T}}(t)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\,,\quad i=0,1,\ldots,m \tag{13}$$

and

$$\mathbf{Y}^{(i)}(\lambda t+\mu)=\overline{\mathbf{T}}(t)\overline{\mathbf{M}}(\lambda,\mu)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\,,\quad i=0,1,\ldots,m \tag{14}$$

where

$$\mathbf{Y}^{(i)}(t)=\begin{bmatrix} y_{1,N}^{(i)}(t) \\ y_{2,N}^{(i)}(t) \\ \vdots \\ y_{k,N}^{(i)}(t) \end{bmatrix},\ \mathbf{Y}^{(i)}(\lambda t+\mu)=\begin{bmatrix} y_{1,N}^{(i)}(\lambda t+\mu) \\ y_{2,N}^{(i)}(\lambda t+\mu) \\ \vdots \\ y_{k,N}^{(i)}(\lambda t+\mu) \end{bmatrix},\ \overline{\mathbf{T}}(t)=\begin{bmatrix} \mathbf{T}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(t) \end{bmatrix},\ \mathbf{A}=\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}$$

$$\overline{\mathbf{M}}(\lambda,\mu)=\begin{bmatrix} \mathbf{M}(\lambda,\mu) & 0 & \cdots & 0 \\ 0 & \mathbf{M}(\lambda,\mu) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}(\lambda,\mu) \end{bmatrix},\overline{\mathbf{B}}=\begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix},\overline{\mathbf{D}}=\begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix}.$$

2.2. METHODS

Firstly, the system (1) can be written in the matrix form

$$\sum_{i=0}^{m}\left[\mathbf{P}_i(t)\mathbf{Y}^{(i)}(\lambda t+\mu)+\mathbf{Q}_i(t)\mathbf{Y}^{(i)}(t)\right]=\mathbf{G}(t) \tag{15}$$

where

$$\mathbf{P}_i(t)=\begin{bmatrix} p_{1,1}^i(t) & p_{1,2}^i(t) & \cdots & p_{1,k}^i(t) \\ p_{2,1}^i(t) & p_{2,2}^i(t) & \cdots & p_{2,k}^i(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}^i(t) & p_{k,2}^i(t) & \cdots & p_{k,k}^i(t) \end{bmatrix},\ \mathbf{Q}_i(t)=\begin{bmatrix} q_{1,1}^i(t) & q_{1,2}^i(t) & \cdots & q_{1,k}^i(t) \\ q_{2,1}^i(t) & q_{2,2}^i(t) & \cdots & q_{2,k}^i(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{k,1}^i(t) & q_{k,2}^i(t) & \cdots & q_{k,k}^i(t) \end{bmatrix}$$

and

$$\mathbf{G}(t)=\begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_k(t) \end{bmatrix}.$$

By using the collocation points (4), into Eq.(15) we obtain the system of the matrix equations

$$\sum_{i=0}^{m}\left[\mathbf{P}_i(t_k)\mathbf{Y}^{(i)}(\lambda t_k+\mu)+\mathbf{Q}_i(t_k)\mathbf{Y}^{(i)}(t_k)\right]=\mathbf{G}(t_k),\quad k=0,1,...,N\,. \tag{16}$$

or the compact form

$$\sum_{i=0}^{m}\left[\mathbf{P}_i\overline{\mathbf{Y}}^{(i)}+\mathbf{Q}_i\mathbf{Y}^{(i)}\right]=\mathbf{G} \tag{17}$$

where

$$\mathbf{P}_i=\begin{bmatrix} \mathbf{P}_i(t_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_i(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_i(t_N) \end{bmatrix},\ \overline{\mathbf{Y}}^{(i)}=\begin{bmatrix} \mathbf{Y}^{(i)}(\lambda t_0+\mu) \\ \mathbf{Y}^{(i)}(\lambda t_1+\mu) \\ \vdots \\ \mathbf{Y}^{(i)}(\lambda t_N+\mu) \end{bmatrix},$$

$$\mathbf{Q}_i=\begin{bmatrix} \mathbf{Q}_i(t_0) & 0 & \cdots & 0 \\ 0 & \mathbf{Q}_i(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Q}_i(t_N) \end{bmatrix},\ \mathbf{Y}^{(i)}=\begin{bmatrix} \mathbf{Y}^{(i)}(t_0) \\ \mathbf{Y}^{(i)}(t_1) \\ \vdots \\ \mathbf{Y}^{(i)}(t_N) \end{bmatrix}\text{ and }\mathbf{G}=\begin{bmatrix} \mathbf{G}(t_0) \\ \mathbf{G}(t_1) \\ \vdots \\ \mathbf{G}(t_N) \end{bmatrix}.$$

From the relations (13) and (14) along with the collocation points (4), we gain

$$\mathbf{Y}^{(i)}(t_k)=\overline{\mathbf{T}}(t_k)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\text{ and }\mathbf{Y}^{(i)}(\lambda t_k+\mu)=\overline{\mathbf{T}}(t_k)\overline{\mathbf{M}}(\lambda,\mu)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\ ,\ k=0,1,2,...,N$$

or briefly,

$$\mathbf{Y}^{(i)}=\mathbf{T}\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\text{ and }\overline{\mathbf{Y}}^{(i)}=\mathbf{T}\overline{\mathbf{M}}(\lambda,\mu)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}\ ,\ k=0,1,2,...,N \tag{18}$$

where

$$\mathbf{T}=\begin{bmatrix}\overline{\mathbf{T}}(t_0)\\ \overline{\mathbf{T}}(t_1)\\ \vdots\\ \overline{\mathbf{T}}(t_N)\end{bmatrix},\quad \overline{\mathbf{T}}(t_k)=\begin{bmatrix}\mathbf{T}(t_k) & 0 & \cdots & 0\\ 0 & \mathbf{T}(t_k) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{T}(t_k)\end{bmatrix}$$

By substituting (18) into Eq.(17), we obtain the fundamental matrix equation as

$$\left\{\sum_{i=0}^{m}\left[\mathbf{P}_i\mathbf{T}\overline{\mathbf{M}}(\lambda,\mu)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}+\mathbf{Q}_i\mathbf{T}\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\right]\right\}\mathbf{A}=\mathbf{G}\,. \tag{19}$$

Eq. (19) equivalent to Eq.(1) can be shown by

$$\mathbf{W}\,\mathbf{A}=\mathbf{G}\ \text{ or } [\mathbf{W};\mathbf{G}]. \tag{20}$$

where we have a system of $k(N+1)$ linear algebraic equations with $k(N+1)$ unknown Lucas coefficients:

$$\mathbf{W}=\sum_{i=0}^{m}\left[\mathbf{P}_i\mathbf{T}\overline{\mathbf{M}}(\lambda,\mu)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}+\mathbf{Q}_i\mathbf{T}\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\right]=\left[w_{p,q}\right],\quad p,q=1,2,...,k(N+1)\,.$$

By using Eq. (2) and the relation (13), we have the matrix form for the conditions as

$$\sum_{j=0}^{m-1}\left[\mathbf{a_j}\overline{\mathbf{T}}(a)+\mathbf{b_j}\overline{\mathbf{T}}(b)\right]\left(\overline{\mathbf{B}}\right)^j\overline{\mathbf{D}}\mathbf{A}=\mathbf{C} \tag{21}$$

where

$$\mathbf{a_j}=\begin{bmatrix}\mathbf{a_j^1} & 0 & \cdots & 0\\ 0 & \mathbf{a_j^2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{a_j^k}\end{bmatrix},\quad \mathbf{b_j}=\begin{bmatrix}\mathbf{b_j^1} & 0 & \cdots & 0\\ 0 & \mathbf{b_j^2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{b_j^k}\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}\mathbf{c_1}\\ \mathbf{c_2}\\ \vdots\\ \mathbf{c_k}\end{bmatrix}$$

and

$$\mathbf{a_j^i}=\begin{bmatrix}a_{0,j}^i\\ a_{1,j}^i\\ \vdots\\ a_{m-1,j}^i\end{bmatrix},\quad \mathbf{b_j^i}=\begin{bmatrix}b_{0,j}^i\\ b_{1,j}^i\\ \vdots\\ b_{m-1,j}^i\end{bmatrix},\quad \mathbf{c_i}=\begin{bmatrix}c_{i,0}\\ c_{i,1}\\ \vdots\\ c_{i,m-1}\end{bmatrix},\quad i=0,1,...,k\,.$$

Thus, the fundamental matrix equation for conditions is

$$\mathbf{U}\mathbf{A}=\mathbf{C}\ \text{ or } [\mathbf{U};\mathbf{C}] \tag{22}$$

where

$$\mathbf{U}=\sum_{j=0}^{m-1}\left[\mathbf{a_j}\overline{\mathbf{T}}(a)+\mathbf{b_j}\overline{\mathbf{T}}(b)\right]\left(\overline{\mathbf{B}}\right)^j\overline{\mathbf{D}}\,.$$

Therefore, the rows of the matrix (22) are replaced by last rows of the matrix (20), we obtain the new augmented matrix

$$\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] \tag{23}$$

If we have a singular matrix $\widetilde{\mathbf{W}}$, rows of the matrix (22) can be replaced by any rows of the matrix (20). If $\operatorname{rank}\widetilde{\mathbf{W}} = \operatorname{rank}\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] = k\left(N+1\right)$, then we have

$$\mathbf{A} = \left(\widetilde{\mathbf{W}}\right)^{-1}\widetilde{\mathbf{G}}\,. \tag{24}$$

Accordingly, the unknown Lucas coefficients matrix $\mathbf{A}$ is determined. So we can find the Lucas polynomial solutions

$$y_{j,N}(t) = \sum_{n=0}^{N} a_{j,n} L_n(t), \quad j = 1,2,...,k\,.$$

3. RESIDUAL ERROR ANALYSIS

In this section, we develop an error estimation technique for the Lucas polynomial approximations of the problem (1)-(2) by means of the residual correction method [25-28] and then, by using this technique we improve the approximate solution.

Firstly, residual function of the method can be defined as

$$R_{i,N}(t) = L\left[y_{i,N}(t)\right] - g_i(t), \quad i = 1,2,...,k \tag{25}$$

where $L\left[y_{i,N}(x)\right] \cong g_i\left(x\right)$ and $y_{i,N}(t)$, $i = 1,2,...,k$ are the Lucas polynomial solutions (3) of the problems (1)-(2). Then $y_{j,N}(t)$ correspond the problem

$$\begin{cases} \sum_{n=0}^{m}\sum_{j=1}^{k}\left[p_{i,j}^{n}(t)y_{j,N}^{(n)}(\lambda t+\mu) + q_{i,j}^{n}(t)y_{j,N}^{(n)}(t)\right] = g_i(t) + R_{i,N}(t), \quad i = 1,2,...,k \\ \sum_{n=0}^{m-1}\left[a_{i,n}^{j}y_{j,N}^{(n)}(a) + b_{i,n}^{j}y_{j,N}^{(n)}(b)\right] = c_{j,i}, \quad i = 0,1,..,m-1, \quad j = 1,2,...,k. \end{cases}$$

Further, the error function $e_{j,N}(t)$ can be determined as

$$e_{j,N}(t) = y_j(t) - y_{j,N}(t) \tag{26}$$

where $y_j(t)$, $j = 1,2,...,k$ are the exact solutions of the problem (1)-(2). From Eqs.(1), (2), (25) and (26), we obtain the system of the error differential equations

$$L\left[e_{i,N}(t)\right] = L\left[y_i(t)\right] - L\left[y_{i,N}(t)\right] = -R_{i,N}(t)$$

with the homogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j}e_{j,N}^{(n)}(a)+b_{i,n}^{j}e_{j,N}^{(n)}(b)\right]=0,\quad i=0,1,\ldots,m-1,\quad j=1,2,\ldots,k$$

or openly, the error problem

$$\begin{cases}\sum_{n=0}^{m}\sum_{j=1}^{k}\left[p_{i,j}^{n}(t)e_{j,N}^{(n)}(\lambda t+\mu)+q_{i,j}^{n}(t)e_{j,N}^{(n)}(t)\right]=-R_{i,N}(t),\quad i=1,2,\ldots,k\\ \sum_{n=0}^{m-1}\left[a_{i,n}^{j}e_{j,N}^{(n)}(a)+b_{i,n}^{j}e_{j,N}^{(n)}(b)\right]=0,\quad i=0,1,..,m-1,\quad j=1,2,\ldots,k.\end{cases} \tag{27}$$

Here, note that the nonhomogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j}y_{j}^{(n)}(a)+b_{i,n}^{j}y_{j}^{(n)}(b)\right]=c_{j,i},\quad i=0,1,\ldots,m-1,\quad j=1,2,\ldots,k$$

and

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j}y_{j,N}^{(n)}(a)+b_{i,n}^{j}y_{j,N}^{(n)}(b)\right]=c_{j,i},\quad i=0,1,\ldots,m-1,\quad j=1,2,\ldots,k$$

are diminished to the homogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j}e_{j,N}^{(n)}(a)+b_{i,n}^{j}e_{j,N}^{(n)}(b)\right]=0,\quad i=0,1,\ldots,m-1,\quad j=1,2,\ldots,k\ .$$

The error problem (27) can be settled by using the presented method in Section 2. So, we obtain the approximation $e_{j,N,M}(t)$ to $e_{j,N}(t)$ as follows:

$$e_{j,N,M}(t)=\sum_{n=0}^{M}a_{j,n}^{*}L_{n}(t),\quad M>N,\quad j=1,2,\ldots,k\ .$$

Consequently, the corrected Lucas polynomial solution $y_{j,N,M}(t)=y_{j,N}(t)+e_{j,N,M}(t)$ is obtained by means of the polynomials $y_{j,N}(t)$ and $e_{j,N,M}(t)$. In addition, we construct the error function $e_{j,N}(t)=y_{j}(t)-y_{j,N}(t)$, the estimated error function $e_{j,N,M}(t)$ and the corrected error function $E_{j,N,M}(t)=e_{j,N}(t)-e_{j,N,M}(t)=y_{j}(t)-y_{j,N,M}(t)$ [20, 29].

4. RESULTS AND DISCUSSION

4.1. RESULTS

In this section, some numerical examples are introduced to show the applicability and reliability of the present method. The examples are computed by *Maple* and *Matlab*. In examples, Lucas polynomial solutions $y_{j,N}(t)$, corrected Lucas polynomial solutions $y_{j,N,M}(t)=y_{j,N}(t)+e_{j,N,M}(t)$ and corrected absolute error functions $\left|E_{j,N,M}(t)\right|$ are calculated.

Example 4.1 [5]. First, we consider the system of linear differential-difference equations

$$\begin{cases} y_1^{(2)}(t) - y_1(t) + y_2(t) - y_1(t-0.2) = -e^{t-0.2} + e^{-t} \\ y_2^{(2)}(t) + y_1(t) - y_2(t) - y_2(t-0.2) = -e^{-t+0.2} + e^{t} \end{cases}, \quad 0 \le t \le 1 \tag{28}$$

with the initial conditions

$$y_1(0) = 1,\ y_1^{(1)}(0) = 1,\ y_2(0) = 1,\ y_2^{(1)}(0) = -1$$

which has the exact solution $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. Also, we get $k = 2$, $m = 2$, $p_{1,1}^2 = 0$, $p_{1,1}^1 = 0$, $p_{1,1}^0 = -1$, $p_{1,2}^2 = 0$, $p_{1,2}^1 = 0$, $p_{1,2}^0 = 0$, $p_{2,1}^2 = 0$, $p_{2,1}^1 = 0$, $p_{2,1}^0 = 0$, $p_{2,2}^2 = 0$, $p_{2,2}^1 = 0$, $p_{2,2}^0 = -1$, $q_{1,1}^2 = 1$, $q_{1,1}^1 = 0$, $q_{1,1}^0 = -1$, $q_{1,2}^2 = 0$, $q_{1,2}^1 = 0$, $q_{1,2}^0 = 1$, , $q_{2,1}^2 = 0$, $q_{2,1}^1 = 0$, $q_{2,1}^0 = 1$, $q_{2,2}^2 = 1$, $q_{2,2}^1 = 0$, $q_{2,2}^0 = -1$, $g_1(t) = -e^{t-0.2} + e^{-t}$, $g_2(t) = -e^{-t+0.2} + e^{t}$.

The approximate solutions $y_{1,3}(t)$ and $y_{2,3}(t)$ for $N = 3$ is given by

$$y_{i,3}(t) = \sum_{n=0}^{3} a_{i,n} L_n(t), \quad (i = 1, 2).$$

For $a = 0$, $b = 1$ and $N = 3$, we have

$$\left\{ t_0 = 0,\ t_1 = \frac{1}{3},\ t_2 = \frac{2}{3},\ t_3 = 1 \right\}.$$

From Eq. (19), the fundamental matrix equation of the problem (28) becomes

$$\left\{ \mathbf{P}_0 \mathbf{T} \overline{\mathbf{M}}(1, -0.2) \overline{\mathbf{D}} + \mathbf{Q}_0 \mathbf{T} \overline{\mathbf{D}} + \mathbf{Q}_2 \mathbf{T} \left(\overline{\mathbf{B}} \right)^2 \overline{\mathbf{D}} \right\} \mathbf{A} = \mathbf{G}.$$

By the present method, the approximate solutions of the problem (28) for $N = 3$ are obtained as

$$y_{1,3}(t) = 1 + t + 0.4998390890t^2 + 0.1980790563t^3$$
$$y_{2,3}(t) = 1 - t + 0.4998625020t^2 - 0.1416577389t^3$$

Now, we consider the error problem

$$\begin{cases} e_{1,3}^{(2)}(t) - e_{1,3}(t) + e_{2,3}(t) - e_{1,3}(t-0.2) = -R_{1,3}(t) \\ e_{2,3}^{(2)}(t) + e_{1,3}(t) - e_{2,3}(t) - e_{2,3}(t-0.2) = -R_{2,3}(t) \end{cases} \tag{29}$$

with the contitions $e_{1,3}(0) = 0,\ e_{2,3}(0) = 0$; where the residual functions are

$$\begin{cases} R_{1,3}(t) = y_{1,3}^{(2)}(t) - y_{1,3}(t) + y_{2,3}(t) - y_{1,3}(t-0.2) + e^{t-0.2} - e^{-t} \\ R_{2,3}(t) = y_{2,3}^{(2)}(t) + y_{1,3}(t) - y_{2,3}(t) - y_{2,3}(t-0.2) + e^{-t+0.2} - e^{t} \end{cases}$$

By solving the error problem (29) for $M = 4,6,9$ we find the following results: for $M = 4$,

$$e_{1,3,4}(t) = -(0.2081668171e-15) - (0.1387778781e-16)t + (0.1895940517e-3)t^2 - (0.3569216741e-1)t^3 + (0.5379181378e-1)t^4$$

$$e_{2,3,4}(t) = -(0.1387778781e-16) + (0.4163336342e-16)t + (0.1166475490e-3)t^2 - (0.2205659316e-1)t^3 + (0.3261028178e-1)t^4;$$

$$y_{1,3,4}(t) = 1 + t + 0.5000286831t^2 + 0.1623868889t^3 + (0.5379181378e-1)t^4$$

$$y_{2,3,4}(t) = 1 - t + 0.4999791495t^2 - 0.1637143321t^3 + (0.3261028178e-1)t^4;$$

for $M = 6$,

$$y_{1,3,6}(t) = 1 + t + 0.5000004261t^2 + 0.1666307377t^3 + (0.4188079941e-1)t^4 + (0.7812426246e-2)t^5 + (0.1948989067e-2)t^6$$

$$y_{2,3,6}(t) = 1 - t + 0.4999997458t^2 - 0.1666459902t^3 + (0.4154049178e-1)t^4 - (0.8009909817e-2)t^5 + (0.9988166218e-3)t^6;$$

for $M = 9$,

$$y_{1,3,9}(t) = 1 + t + 0.4999999999t^2 + 0.1666666721t^3 + (0.4166660258e-1)t^4 + (0.1387760219e-2)t^6 + (0.2005132956e-3)t^7 + (0.2251659443e-4)t^8 + (0.4070657195e-5)t^9 + (0.8333692357e-2)t^5$$

$$y_{2,3,9}(t) = 1 - t + 0.4999999996t^2 - 0.1666666445t^3 + (0.4166645863e-1)t^4 + (0.1386479131e-2)t^6 - (0.1948093965e-3)t^7 + (0.2171775472e-4)t^8 - (0.1368089205e-5)t^9 - (0.8332390671e-2)t^5.$$

which are the corrected Lucas polynomial solutions for several M values.

Table 1. Comparison of $y_1(t)$ exact and numerical solutions for $N = 3$ and $M = 4,6,9$ of the problem (28).

	Exact Solution	**Lucas Polynomial Solution**	**Corrected Lucas Polynomial Solutions**		
t_i	$y_1(t_i)$	$y_{1,3}(t_i)$	$y_{1,3,4}(t_i)$	$y_{1,3,6}(t_i)$	$y_{1,3,9}(t_i)$
0	**1**	**1**	**1**	**1**	**1**
0.1	1.105170918076	1.105196469946	1.105168052901	1.105170903152	1.105170918076
0.3	1.349858807576	1.350333652530	1.349822741171	1.349858707751	1.349858807578
0.5	1.648721270700	1.649719654288	1.648667520249	1.648721089975	1.648721270704
0.8	2.225540928492	2.221313493786	2.225193571225	2.225540477467	2.225540928492
1.0	2.718281828459	2.697918145300	2.716207385780	2.718273378523	2.718281827703

Table 2. Comparison of $y_2(t)$ exact and numerical solutions for $N=3$ and $M=4,6,9$ of the problem (28).

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solutions		
t_i	$y_2(t_i)$	$y_{2,3}(t_i)$	$y_{2,3,4}(t_i)$	$y_{2,3,6}(t_i)$	$y_{2,3,9}(t_i)$
0	1	1	1	1	1
0.1	0.904837418036	0.904856967281	0.904839338191	0.904837426417	0.904837418041
0.3	0.740818220682	0.741162866230	0.740841979771	0.740818277426	0.740818220705
0.5	0.606530659713	0.607258408138	0.606568638474	0.606530765239	0.606530659744
0.8	0.449328964117	0.447383238963	0.449522089062	0.449329222298	0.449328964146
1.0	0.367879441171	0.358204763100	0.368875099180	0.367883154185	0.367879442459

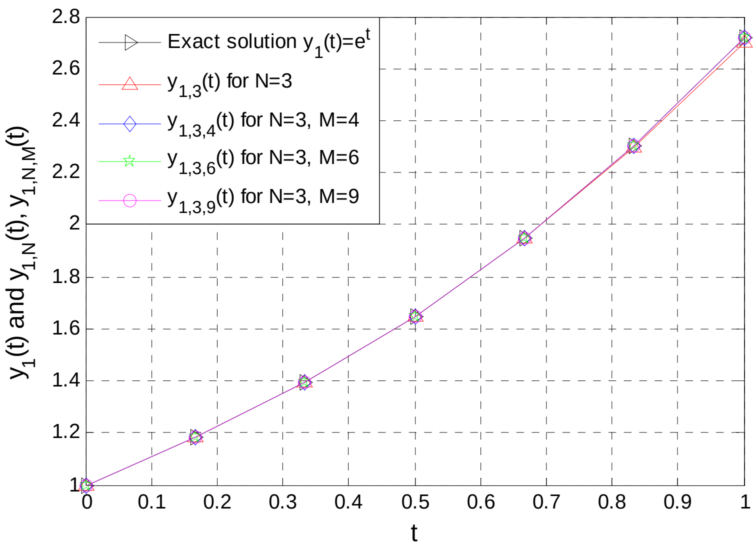

Figure 1. Comparision of the exact solution $y_1(t)$ and the approximate solutions $y_{1,N}(t)$, $y_{1,N,M}(t)$ for the problem (28).

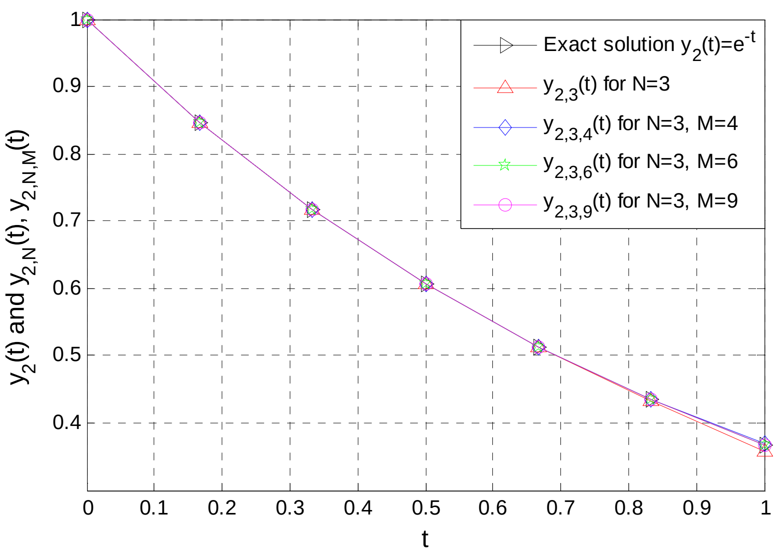

Figure 2. Comparision of the exact solution $y_2(t)$ and the approximate solutions $y_{2,N}(t)$, $y_{2,N,M}(t)$ for the problem (28).

It is seen from Table 1, Table 2, Fig.1 and Fig. 2 that the corrected approximate solutions are very close to the exact solutions when the values of N and M are increased; namely the obtained solutions improve whenever N and M are increased.

Table 3. Comparison of $\left|E_{1,N,M}(t)\right|$ corrected absolute errors for $N=3$ and $M=4,6,9$ of the problem (28).

Corrected Absolute Errors $\left\|E_{1,N,M}(t)\right\| = \left\|y_1(t) - y_{1,N,M}(t)\right\|$			
t_i	$\left\|E_{1,3,4}(t_i)\right\|$	$\left\|E_{1,3,6}(t_i)\right\|$	$\left\|E_{1,3,9}(t_i)\right\|$
0	0	0	0
0.1	0.2865e-5	0.1492e-7	0.6747e-12
0.3	0.3607e-4	0.9983e-7	0.1916e-11
0.5	0.5375e-4	0.1807e-6	0.4171e-11
0.8	0.3474e-3	0.4510e-6	0.1754e-12
1.0	0.2074e-2	0.8450e-5	0.7558e-9

Table 4. Comparison of $\left|E_{2,N,M}(t)\right|$ corrected absolute errors for $N=3$ and $M=4,6,9$ of the problem (28).

Corrected Absolute Errors $\left\|E_{2,N,M}(t)\right\| = \left\|y_2(t) - y_{2,N,M}(t)\right\|$			
t_i	$\left\|E_{2,3,4}(t_i)\right\|$	$\left\|E_{2,3,6}(t_i)\right\|$	$\left\|E_{2,3,9}(t_i)\right\|$
0	0	0	0
0.1	0.1920e-5	0.8381e-8	0.4711e-11
0.3	0.2376e-4	0.5674e-7	0.2280e-10
0.5	0.3798e-4	0.1055e-6	0.3171e-10
0.8	0.1931e-3	0.2582e-6	0.2912e-10
1.0	0.9957e-3	0.3713e-5	0.1288e-8

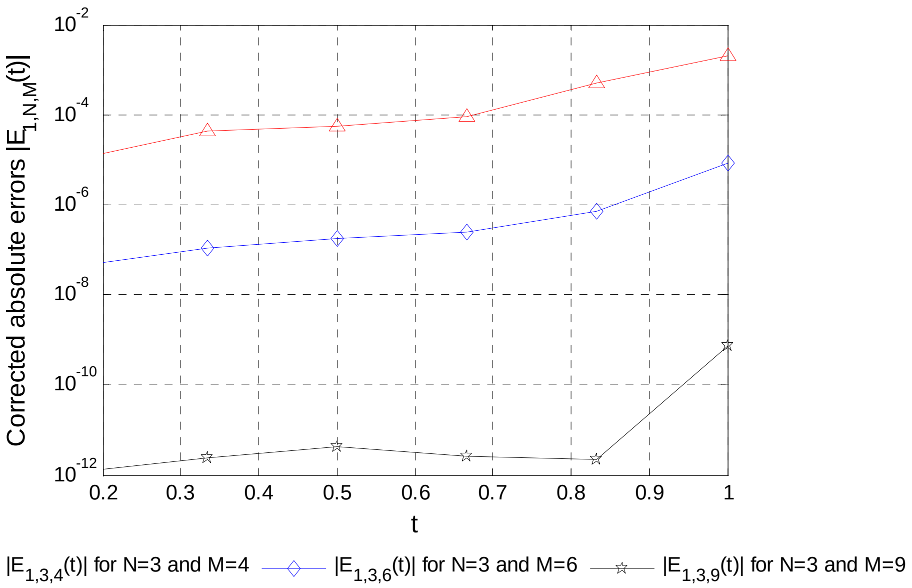

Figure 3. Comparision of the corrected absolute error functions $\left|E_{1,N,M}(t)\right|$ for the problem (28).

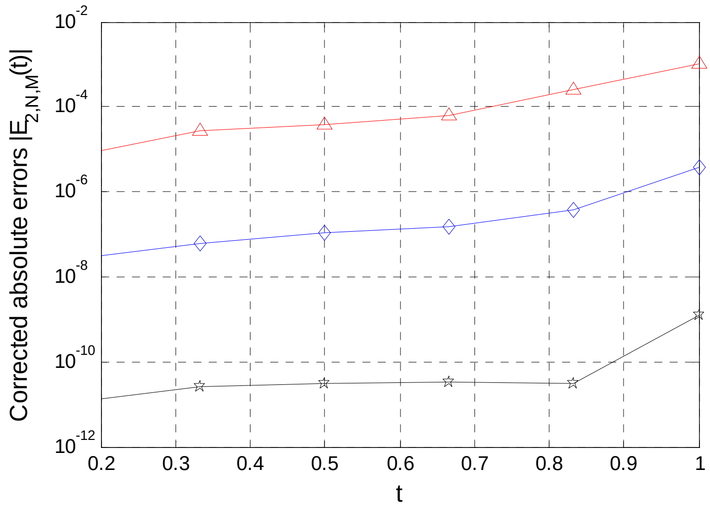

Figure 4. Comparision of the corrected absolute error functions $\left|E_{2,N,M}(t)\right|$ for the problem (28).

Tables 3-4 and Figs. 3-4 show that the corrected absolute errors are very close to zero when the value of M is increased. Thus, we can say that Lucas collocation method is very effective for the system of differential-difference equations (28).

Table 5. Comparison of the errors of the polynomial spline function method and the present method for the problem (28).

	Spline Function Approximation (Third Degree) in [5]	**Present Method for $N=3, M=9$**
x_i	Absolute error for $y_1(x_i)$	Corrected absolute error $\left|E_{1,3,9}(x_i)\right|$
0.1	8.5e-8	0.6747e-12
0.2	2.3e-6	0.1263e-11
0.3	1.7e-5	0.1916e-11
0.4	5.7e-5	0.3328e-11
0.5	1.8e-4	0.4271e-11
	Spline Function Approximation (Third Degree) in [5]	**Present Method for $N=3, M=9$**
x_i	Absolute error for $y_2(x_i)$	Corrected absolute error $\left|E_{2,3,9}(x_i)\right|$
0.1	8.2e-8	0.4711e-11
0.2	2.2e-6	0.1481e-10
0.3	1.7e-5	0.2280e-10
0.4	5.6e-5	0.2993e-10
0.5	1.6e-4	0.3171e-10

In Table 5, we show the comparison between the absolute errors obtained by the present method and the spline function method [5]. This represents that the present method is more effective than the spline function approximation since the corrected absolute errors approximate to zero.

Example 4.2 Let us consider the system of linear differential-difference equations

$$\begin{cases} y_1^{(1)}(t) - 2y_2(t+1) + ty_3(t+1) = \cos(t) - 2\cos(t+1) + te^{t+1} \\ y_2^{(1)}(t) - ty_3(t) + y_1(t+1) = -\sin(t) - te^t + \sin(t+1) \\ y_3^{(1)}(t) + 2ty_2(t+1) = e^t + 2t\cos(t+1) \end{cases}, \quad 0 \le t \le 1 \qquad (30)$$

under the initial conditions

$$y_1(0) = 0,\ y_2(0) = 1,\ y_3(0) = 1$$

which has the exact solution $y_1(t) = \sin(t)$, $y_2(t) = \cos(t)$ and $y_3(t) = e^t$. In this problem $k = 3$, $m = 1$, $p_{1,1}^0 = 0$, $p_{1,2}^0 = -2$, $p_{1,3}^0 = t$, $p_{2,1}^0 = 1$, $p_{2,2}^0 = 0$, $p_{2,3}^0 = 0$, $p_{3,1}^0 = 0$, $p_{3,2}^0 = 2t$, $p_{3,3}^0 = 0$, $q_{1,1}^0 = 0$, $q_{1,2}^0 = 0$, $q_{1,3}^0 = 0$, $q_{2,1}^0 = 0$, $q_{2,2}^0 = 0$, $q_{2,3}^0 = -t$, $q_{3,1}^0 = 0$, $q_{3,2}^0 = 0$, $q_{3,3}^0 = 0$, $q_{1,1}^1 = 1$, $q_{1,2}^1 = 0$, $q_{1,3}^1 = 0$, $q_{2,1}^1 = 0$, $q_{2,2}^1 = 1$, $q_{2,3}^1 = 0$, $q_{3,1}^1 = 0$, $q_{3,2}^1 = 0$, $q_{3,3}^1 = 1$.

The approximate solutions $y_{1,2}(t)$, $y_{2,2}(t)$ and $y_{3,2}(t)$ for $N = 2$ is given by

$$y_{i,2}(t) = \sum_{n=0}^{2} a_{i,n} L_n(t), \quad i = 1,2,3.$$

The set of the collocation points for $a = 0$, $b = 1$ and $N = 2$: $\left\{ t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 = 1 \right\}$.

The fundamental matrix equation of the problem (30):

$$\left\{ \mathbf{Q}_1 \mathbf{T}\overline{\mathbf{B}}\overline{\mathbf{D}} + \mathbf{Q}_0 \mathbf{T}\overline{\mathbf{D}} + \mathbf{P}_0 \mathbf{T}\overline{\mathbf{M}}(1,1)\overline{\mathbf{D}} \right\} \mathbf{A} = \mathbf{G}.$$

The approximate solutions of the problem (30) for $N = 2$:

$$y_{1,2}(t) = -0.251239644865421t^2 + 1.03204905836365t$$
$$y_{2,2}(t) = 1 - 0.504334736259701t^2 + (0.60661571309666e - 1t$$
$$y_{3,2}(t) = 1 + 0.763219271987656t^2 + t$$

The error problem the with conditions $e_{1,3}(0) = 0, e_{2,3}(0) = 0$:

$$\begin{cases} e_{1,2}^{(1)}(t) - 2e_{2,2}(t+1) + te_{3,2}(t+1) = -R_{1,2}(t) \\ e_{2,2}^{(1)}(t) - te_{3,2}(t) + e_{1,2}(t+1) = -R_{2,2}(t) \\ e_{3,2}^{(1)}(t) + 2te_{2,2}(t+1) = -R_{3,2}(t) \end{cases} \qquad (31)$$

such that the residual functions

$$\begin{cases} R_{1,2}(t) = y_{1,2}^{(1)}(t) - 2y_{2,2}(t+1) + ty_{3,2}(t+1) - \cos(t) + 2\cos(t+1) - te^{t+1} \\ R_{2,2}(t) = y_{2,2}^{(1)}(t) - ty_{3,2}(t) + y_{1,2}(t+1) + \sin(t) + te^t - \sin(t+1) \\ R_{3,2}(t) = y_{3,2}^{(1)}(t) + 2ty_{2,2}(t+1) - e^t - 2t\cos(t+1) \end{cases}$$

By solving the error problem (31) for $M=3$, the estimated Lucas error functions $e_{1,2,3}(t)$, $e_{2,2,3}(t)$ and $e_{3,2,3}(t)$ are obtained as

$$e_{1,2,3}(t)=-(0.306354322370542e-1)t+0.226934157274993t^2-0.132044522913125t^3$$
$$e_{2,2,3}(t)=-(0.642542018248143e-1)t-(0.337192752493939e-1)t^2$$
$$+(0.826557614556811e-1)t^3$$
$$e_{3,2,3}(t)=(0.222044604925031e-15)t-0.308088718648511t^2+0.277863061548920t^3$$

So, we get corrected Lucas polynomial solutions $y_{1,2,3}(t)$, $y_{2,2,3}(t)$ and $y_{3,2,3}(t)$ as

$$y_{1,2,3}(t)=-(0.243054876e-1)t^2+1.001413626t-0.132044522913125t^3$$
$$y_{2,2,3}(t)=1-0.5380540116t^2-(0.359263051e-2)t+(0.826557614556811e-1)t^3$$
$$y_{3,2,3}(t)=1+0.4551305534t^2+t+0.277863061548920t^3$$

Then we find the corrected Lucas polynomial solutions for different N and M values. For $N=2$ and $M=5$

$$y_{1,2,5}(t)=(0.55705857e-2)t^2+0.9974917492t-(0.888178419700125e-15)$$
$$-(0.875212928058344e-2)t^4+(0.907209538656195e-2)t^5-0.158123753483435t^3$$
$$y_{2,2,5}(t)=1-0.5019386015t^2-(0.378756276e-2)t+(0.172672246031444e-2)t^3$$
$$+(0.493722756612796e-1)t^4-(0.632465274877506e-2)t^5$$
$$y_{3,2,5}(t)=1+0.5009550711t^2+t+0.165528916238553t^3+(0.328715570166613e-1)t^4$$
$$+(0.166736506943974e-1)t^5$$

and for $N=2$ and $M=8$

$$y_{1,2,8}(t)=(0.6043e-6)t^2+1.000202858t-0.166913195573201t^3-(0.582645043323282e-12)$$
$$-(0.933219411791697e-4)t^4+(0.837334913035903e-2)t^5$$
$$-(0.303996026850939e-3)t^7+(0.261124712466199e-4)t^8$$
$$+(0.155819536111323e-3)t^6$$
$$y_{2,2,8}(t)=1-0.4998356983t^2+(0.2275445e-4)t+(0.479235722288252e-4)t^3$$
$$+(0.415353270238095e-1)t^4-(0.573084566504178e-4)t^5$$
$$+(0.434322557651967e-5)t^7+(0.180467342965063e-4)t^8$$
$$-(0.133165258256263e-2)t^6$$
$$y_{3,2,8}(t)=1+0.4998984805t^2+t+0.166663870141448t^3+(0.418021426376214e-1)t^4$$
$$+(0.838293154864544e-2)t^5+(0.132142315784733e-2)t^6$$
$$+(0.470459287207348e-4)t^8+(0.176601771137541e-3)t^7$$

Table 6. Numerical results of the exact solution $y_1(t)$ and the approximate solutions $\{y_{1,N}(t), y_{1,N,M}(t)\}$ for $N=2$ and $M=3,5,8$ of the Problem (30)

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solutions		
t_i	$y_1(t_i)$	$y_{1,2}(t_i)$	$y_{1,2,3}(t_i)$	$y_{1,2,5}(t_i)$	$y_{1,2,8}(t_i)$
0	0	0	0	-0.8882e-15	-0.5826e-12
0.1	0.099833417	0.100692509	0.099766263	0.099645973	0.099853453
0.3	0.295520207	0.287003149	0.294671392	0.295430689	0.295573896
0.5	0.479425539	0.453214618	0.478124876	0.480109547	0.479493428
0.8	0.717356091	0.664845874	0.717968593	0.719987084	0.717390147
1.0	0.841470985	0.780809413	0.845063615	0.845258548	0.841448230

Table 7. Numerical results of the exact solution $y_2(t)$ and the approximate solutions $\{y_{2,N}(t), y_{2,N,M}(t)\}$ for $N=2$ and $M=3,5,8$ of the Problem (30)

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solutions		
t_i	$y_2(t_i)$	$y_{2,2}(t_i)$	$y_{2,2,3}(t_i)$	$y_{2,2,5}(t_i)$	$y_{2,2,8}(t_i)$
0	1	1	1	1	1
0.1	0.995004165	1.001022810	0.994342853	0.994608458	0.995008118
0.3	0.955336489	0.972808345	0.952729055	0.954120425	0.955358236
0.5	0.877582562	0.904247102	0.874022152	0.875725530	0.877631908
0.8	0.696706709	0.725755026	0.695091078	0.694763749	0.696796838
1.0	0.540302306	0.556326835	0.541009119	0.539048181	0.540403736

Table 8. Numerical results of the exact solution $y_3(t)$ and the approximate solutions $\{y_{3,N}(t), y_{3,N,M}(t)\}$ for $N=2$ and $M=3,5,8$ of the Problem (30)

	Exact Solution	Lucas Polynomial Solution	Corrected Lucas Polynomial Solutions		
t_i	$y_3(t_i)$	$y_{3,2}(t_i)$	$y_{3,2,3}(t_i)$	$y_{3,2,5}(t_i)$	$y_{3,2,8}(t_i)$
0	1	1	1	1	1
0.1	1.105170918	1.107632193	1.104829169	1.105178534	1.105169914
0.3	1.349858808	1.368689734	1.348464052	1.349862014	1.349850761
0.5	1.648721271	1.690804818	1.648515521	1.648505406	1.648704415
0.8	2.225540928	2.288460334	2.233549442	2.224289862	2.225527338
1.0	2.718281828	2.763219272	2.732993615	2.716029195	2.718292496

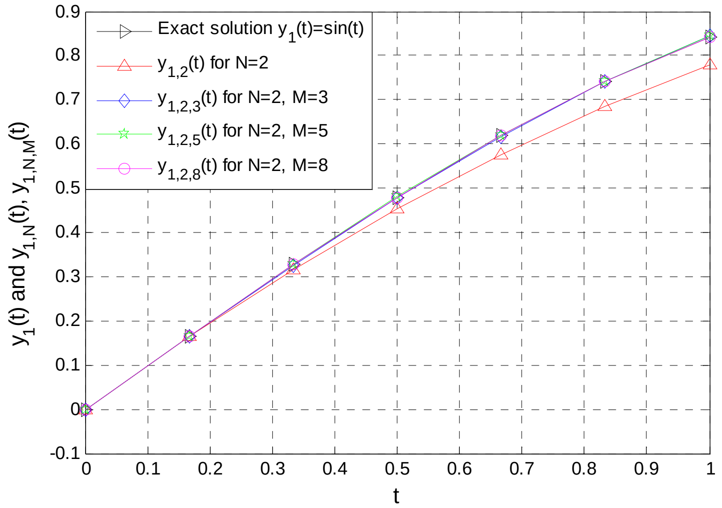

Figure 5. Comparision of the exact solution $y_1(t)$ and the approximate solutions $y_{1,N}(t)$, $y_{1,N,M}(t)$ for the problem for the problem (30).

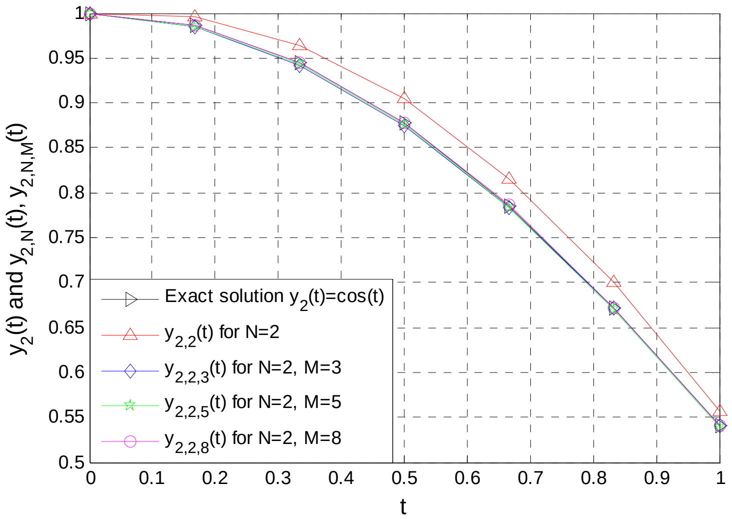

Figure 6. Comparision of the exact solution $y_2(t)$ and the approximate solutions $y_{2,N}(t)$, $y_{2,N,M}(t)$ for the problem for the problem (30).

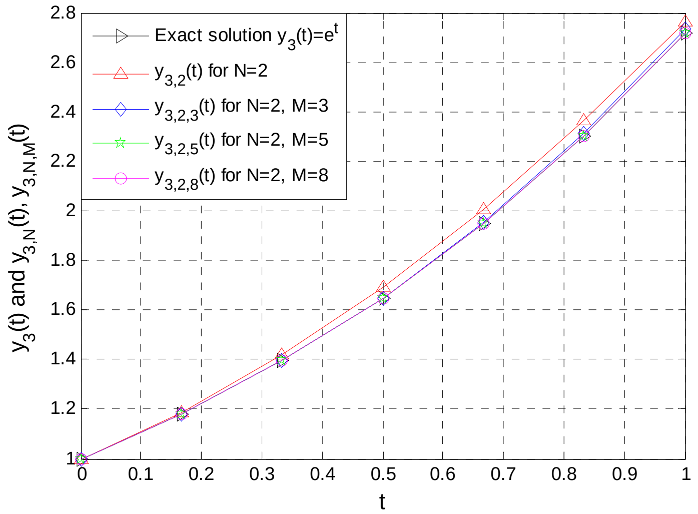

Figure 7. Comparision of the exact solution $y_3(t)$ and the approximate solutions $y_{3,N}(t)$, $y_{3,N,M}(t)$ for the problem for the problem (30).

It is seen from Tables 6-8 and Figs. 5-7 that the approximate solutions are quite close to the exact solutions when the value of M is increased. Namely the accuracy of the solution increase as the values of M increase.

Table 9. Comparison of the corrected absolute errors $|E_{1,N,M}(t)|$ for $N=2$ and $M=3,5,8$ of the Problem (30)

Corrected Absolute Errors $\lvert E_{1,N,M}(t)\rvert = \lvert y_1(t)-y_{1,N,M}(t)\rvert$			
t_i	$\lvert E_{1,2,3}(t_i)\rvert$	$\lvert E_{1,2,5}(t_i)\rvert$	$\lvert E_{1,2,8}(t_i)\rvert$
0	0	0.8882e-15	0.5826e-12
0.1	0.6715e-4	0.1874e-3	0.2004e-4
0.3	0.8488e-3	0.8952e-4	0.5369e-4
0.5	0.1301e-2	0.6840e-3	0.6789e-4
0.8	0.6125e-3	0.2631e-2	0.3406e-4
1.0	0.3593e-2	0.3786e-2	0.2275e-4

Table 10. Comparison of the corrected absolute errors $|E_{2,N,M}(t)|$ for $N=2$ and $M=3,5,8$ of the Problem (30)

Corrected Absolute Errors $\lvert E_{2,N,M}(t)\rvert = \lvert y_2(t)-y_{2,N,M}(t)\rvert$			
t_i	$\lvert E_{2,2,3}(t_i)\rvert$	$\lvert E_{2,2,5}(t_i)\rvert$	$\lvert E_{2,2,8}(t_i)\rvert$
0	0	0	0
0.1	0.6613e-3	0.3957e-3	0.3953e-5
0.3	0.2607e-2	0.1216e-2	0.2175e-4
0.5	0.3564e-2	0.1857e-2	0.4935e-4
0.8	0.1616e-2	0.1943e-2	0.9013e-4
1.0	0.7068e-3	0.1254e-2	0.1014e-3

Table 11. Comparison of the corrected absolute errors $|E_{3,N,M}(t)|$ for $N=2$ and $M=3,5,8$ of the Problem (30)

Corrected Absolute Errors $\lvert E_{3,N,M}(t)\rvert = \lvert y_3(t)-y_{3,N,M}(t)\rvert$			
t_i	$\lvert E_{3,2,3}(t_i)\rvert$	$\lvert E_{3,2,5}(t_i)\rvert$	$\lvert E_{3,2,8}(t_i)\rvert$
0	0	0	0
0.1	0.3417e-3	0.7615e-5	0.1004e-5
0.3	0.1395e-2	0.3206e-5	0.8047e-5
0.5	0.2057e-3	0.2159e-3	0.1686e-4
0.8	0.8009e-2	0.1251e-2	0.1359e-4
1.0	0.1471e-1	0.2253e-2	0.1067e-4

Tables 9-11 show that corrected absolute errors are very close to zero when the value of M is increased. Thus, we say that Lucas collocation method is very effective for the problem (30).

5. CONCLUSIONS

Generally, it is analytically difficult to solve the functional systems such as the high-order linear differential-difference equations system. Hereby, approximate solutions are required. In this study, a new collocation method based on the Lucas polynomials have been introduced with the aid of the residual error technique for solving system of linear functional differential equations. As it has been seen from the results, the developed method is very useful and prevalent. In addition, accuracy of the solutions increase whenever and are increased which can be seen from tables and figures. Also, one of the important advantages of the method is that the approximate solutions can be obtained in a short time by using computer programmes such as Maple and Matlab [29-34].

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