

## TRAVELLING WAVE SOLUTIONS OF HIGHER ORDER NONLINEAR FRACTIONAL EVOLUTION EQUATION

KAMRAN AYUB<sup>1</sup>, NAWAB KHAN<sup>1</sup>, M YAQUB KHAN<sup>1</sup>,  
QAZI MAHMOOD UL HASSAN<sup>2</sup>

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**Abstract.** *Nonlinear mathematical problems and their solutions attain much attention in solitary waves. In soliton theory an efficient tool to attain various types of soliton solution is the exp-function technique. Under study article is devoted to find soliton wave solutions of fractional fifth order evolution equation via a reliable mathematical technique. By use of proposed technique we attain soliton wave solution of various types. The regulation of proposed algorithm is demonstrated by consequent analytical results and computational work. It is noticed that under discussion technique is user friendly with minimum computational work, also we can extend it for physical problems of different nature.*

**Keywords:** *Exp-function technique; travelling wave solution; fractional calculus; fifth order evolution equation*

### 1. INTRODUCTION

In the last few years we have observed an extraordinary progress in soliton theory. Solitons have been studied by various mathematician, physicists and engineers for their applications in physical phenomena's. First soliton waves are observed by an engineer John Scott Russell. Wide ranges of phenomena in mathematics and physics are modeled by differential equations. In nonlinear sciences it is of great importance and interest to explain physical models and attain analytical solutions. In the recent past large series of chemical, biological, physical singularities are feint by nonlinear partial differential equations. At present the prominent and valuable progress are made in the field of physical sciences. The great achievement is the development of various techniques to hunt for solitary wave solution of differential equations. In nonlinear physical sciences, an essential contribution is of exact solutions because of this we can study physical behaviors and discuss more features of the problem which give direction to more applications.

At the disruption between chaos, mathematical physics and probability, fractional calculus and differential equations are rapidly increasing branches of research. For accurate clarification of innumerable real-time models of nonlinear occurrence fractional differential equations (FDEs) of nonlinear structure have accomplished great notice. Because of its recognizable implementation in branch of sciences and engineering it turn out to be a topic of great notice for scientists. In many fields such as porous structures dynamical processes in self-similar or solute transport and fluid flow, material viscoelastic theory, bio-sciences, control theory of dynamical systems, electromagnetic theory, dynamics of earthquakes,

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<sup>1</sup> Riphah International University, Department of Mathematics, Islamabad, Pakistan.

E-mail: [kamranayub88@gmail.com](mailto:kamranayub88@gmail.com).

<sup>2</sup> University of Wah, Faculty of Basic Sciences, Department of Mathematics, Wah Cantt. Pakistan.

astrophysics, optics, signal processing, chemical physics and so on implementations of fractional models [5-8] are beneficially exerted. As a consequence, hypothesis of fractional differential equations has shown fast growth [1-8].

In the recent past, to solve a nonlinear physical problems Wu and He [9] present a well-ordered procedure called exp-function method. The technique under study has prospective to deal with the complex nonlinearity of the models with the flexibility. It has been used as an effective tool for diversified nonlinear problems arising in mathematical physics. Through the study of publication exhibits that exp-function method is extremely reliable and has been effective for differential equations.

After He et al. Mohyud-Din enlarged the exp-function method and used this algorithm to find soliton wave solutions of differential equations; Oziz used same technique for Fisher's equation; Yusufoglu for MBBN equations; Momani for travelling wave solutions of KdV equation of fractional order; Zhu for discrete mKdV lattice and the Hybrid-Lattice system; Kudryashov for soliton solutions of the generalized evolution equations arising in wave dynamics ;Wu et al. for the expansion of compaction-like solitary and periodic solutions; Zhang for high-dimensional nonlinear differential equations, see the references [10-34]. It is to be noticed that after applying under study technique i.e. exp-function method to any nonlinear ordinary differential equation Ebaid proved that  $e = f$  and  $s = r$  are the only relations of the variables that can be acquired [25].

This article is keen to the soliton like solutions of nonlinear fractional fifth order evolution equation [32] by applying a novel technique. The applications of under study nonlinear equation are very vast in different areas of physical sciences and engineering. Additionally, such type of equation found in different physical phenomenon related to fluid mechanic, astrophysics, solid state physics, chemical kinematics, ion acoustic waves in plasma, nonlinear optics, control and optimization theory etc.

## 2. PRELIMINARIES AND NOTATION

Some important results of fractional calculus are discussed in under study section. The fractional integral and derivatives defined on  $[a, b]$  are given below:

**Definition 1.** A real function  $f(x), x > 0$ , is said to be in the space  $C_\mu, \mu \in R$ , If there exists a real number ( $p > \mu$ ) such that  $f(x) = x^p f_1(x)$ , where  $f_1(x) = C(0, \infty)$  and it is said to be in the space  $C_\mu^m$  if  $f^m \in C_\mu, m \in N$ .

**Definition 2.** The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$  of a function  $f \in C_\mu, \mu \geq -1$ , is defined as

$$J^\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \alpha > 0, x > 0. \quad (1)$$

$$J^0(x) = f(x).$$

Some properties of the operator  $J^a$  are discussed in the following

For  $f \in C\mu, \mu \geq -1, \alpha, \beta \geq 0$  and  $\gamma \geq -1$

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) &= J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{aligned} \quad (2)$$

There arise some demerits of Riemann-Liouville derivative when we apply it to model real world problems in the form of fractional differential equations. There is a need to overcome this deficiency. For this M. Caputo introduce modified version of fractional differential operator which we used in our article.

**Definition 3.** Caputo time fractional derivative operator is defined below, for the smallest integer  $n$  that exceeds.

$$D_t^\beta h(x) = \begin{cases} \frac{\partial^\beta w(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} h(t) dt, & -1 << n, n \in N, \\ \frac{\partial^\beta w(x,t)}{\partial t^\beta}, & \beta = n. \end{cases}$$

### Chain rule

In this segment we used a complex fractional transformation to convert differential equation of fractional order into classical differential equation. We apply the following chain rule

$$\frac{\partial^\beta w}{\partial t^\beta} = \frac{\partial w}{\partial q} \cdot \frac{\partial^\beta q}{\partial t^\beta} \quad (3)$$

It is a Jumarie's modification of Riemann-Liouville derivative. There are few results, which are very important and useful

$$\begin{aligned} D_t^\beta w &= \sigma'_t \frac{dw}{d\xi} D_t^\beta \xi \\ D_x^\beta w &= \sigma'_x \frac{dw}{d\xi} D_t^\beta \xi \end{aligned}$$

The value of  $\sigma_s$  is determined by assuming a special case given below

$$q = t^\beta \text{ and } w = q^n$$

We have

$$\frac{\partial^\beta w}{\partial t^\beta} = \frac{\Gamma(1+n\beta)t^{n\beta-\beta}}{\Gamma(1+n\beta-\beta)} = \sigma \cdot \frac{\partial w}{\partial q} = \sigma n t^{n\beta-\beta}$$

Thus we can calculate  $\sigma_q$  as

$$\sigma_q = \frac{\Gamma(1+n\beta)}{\Gamma(1+n\beta-\beta)}$$

Other fractional indexes  $(\sigma'_x, \sigma'_y, \sigma'_z)$  can determine in similar way. Li and He [2-8] proposed following fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus.

$$u(x, t) = u(\eta), \eta = \frac{kx^\beta}{\Gamma(1+\beta)} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)} + \frac{Mx^\gamma}{\Gamma(1+\gamma)}. \quad (4)$$

Where  $k, \omega$  and  $M$  are constants.

### 3. ANALYSIS OF TECHNIQUE

We suppose the nonlinear FPDE of the form

$$Q(w, w_t, w_x, w_{xx}, \dots, D_t^\beta w, D_x^\beta w, D_{xx}^\beta w, \dots) = 0, 0 < \beta \leq 1. \quad (5)$$

Where  $D_t^\beta w, D_x^\beta w, D_{xx}^\beta w$  are the modified Riemann-Liouville derivative of  $w$  with respect to  $t$  and  $x$  respectively.

Invoking the transformation

$$w(x, t) = w(\xi), \xi = \frac{kx^\gamma}{\Gamma(1+\gamma)} + \frac{\omega t^\beta}{\Gamma(1+\beta)} + \frac{My^\alpha}{\Gamma(1+\alpha)}. \quad (6)$$

Here  $k, \omega$  and  $M$  are all constants.

We can write equation (5) again in the form of following nonlinear ODE

$$Q(w, w', w'', w''', \dots) = 0. \quad (7)$$

Where prime signify the derivative of  $w$  with respect to  $\xi$ . In proportion to Exp-function method, we suppose that the wave solution can be written in the form given below

$$w(\xi) = \frac{\sum_{m=e}^f b_m \exp(m\xi)}{\sum_{n=s}^r a_n \exp(n\xi)}. \quad (8)$$

Where  $s, r, e$  and  $f$  are positive integers which we have to find,  $b_m$  and  $a_n$  are unknown constants. We can write equation (8) again in the following equivalent form

$$w(\xi) = \frac{b_e \exp(f\xi) + \dots + b_{-f} \exp(-e\xi)}{a_s \exp(s\xi) + \dots + a_{-r} \exp(-r\xi)}. \quad (9)$$

This equivalent transformation plays a fundamental and important part to solve the problem for analytical solution. To find the value of  $s$  and  $r$  by using [25], we have

$$s = e, r = f.$$

#### 4. APPLICATIONS OF THE TECHNIQUE

Exp-function technique is applied to attain soliton wave solutions of fifth order fractional evolution equation [32]. The obtained results are very efficient and encouraging.

Consider the following fractional partial differential equation

$$D_t^{3\beta} w - (D_t^\beta w)_{xxx} - (D_t^\beta w)_{xx} - 4(w_x (D_t^\beta w)_x)_x = 0. \quad (10)$$

Using (6) equation (10) can be converted into an ODE of the form

$$-\omega^3 w''' + \omega k^4 w^{(v)} + k^2 \omega w''' + 4\omega k^3 w' w''' = 0. \quad (11)$$

Where prime signify the derivative w.r.t  $\xi$ . Equation (9) is the expressed solution of the equation (11). To find the value of  $r$  and  $s$ , by using [25]

$$s = e, r = f.$$

**Case 1.** We can frequently select the parameters  $r$  and  $s$ , for directness, we set  $s = e = 1$  and  $r = f = 1$  equation (9) becomes

$$w(\xi) = \frac{b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi]}{a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]}. \quad (12)$$

Using equation (12) into equation (11), we have

$$\frac{1}{B} [l_4 \exp(4\xi) + l_3 \exp(3\xi) + l_2 \exp(2\xi) + l_1 \exp(\xi) + l_0 + l_{-1} \exp(-\xi) + l_{-2} \exp(-2\xi) + l_{-3} \exp(-3\xi) + l_{-4} \exp(-4\xi)].$$

Where  $B = (a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi))^4$ ,  $c_i$  are constants whose values are attained by Maple 16. By equalizing the coefficients of  $\exp(m\xi)$  to zero, we attain

$$l_j = 0, j = 0, \pm 1, \pm 2, \pm 3, \pm 4. \quad (13)$$

The given equation (10) is satisfied by following solution sets.

##### 1st Solution set

$$\{\omega = -\sqrt{k^2 + k}, b_{-1} = \frac{a_{-1}(3ka_0 + b_0)}{a_0}, b_0 = b_0, b_1 = 0, b_{-1} = b_{-1}, b_0 = b_0, b_1 = 0\}$$

The attain generalized soliton solution  $w(x, t)$  is given as,

$$w(x, t) = \frac{\frac{1}{a_0} \left( -b_{-1} (3ka_0 + b_0) \exp\left(-kx - \frac{\sigma\sqrt{k^2 + kt^\beta}}{\Gamma(1+\beta)}\right) \right) + b_0}{b_{-1} \exp\left(-kx - \frac{\sigma\sqrt{k^2 + kt^\beta}}{\Gamma(1+\beta)}\right) + a_0}.$$

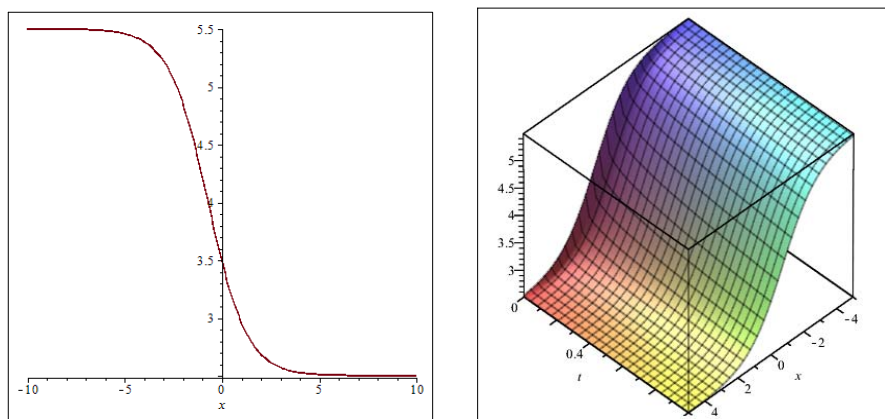


Figure 1a. 2D and 3D representation of solution for  $\beta=0.25$ .

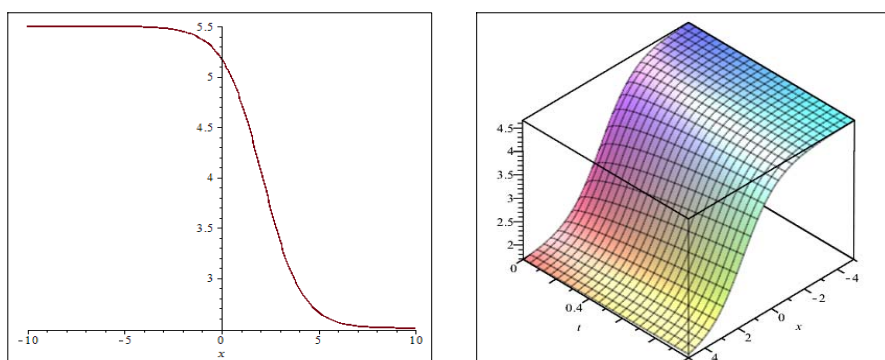


Figure 1b. 2D and 3D representation of solution for  $\beta=0.50$ .

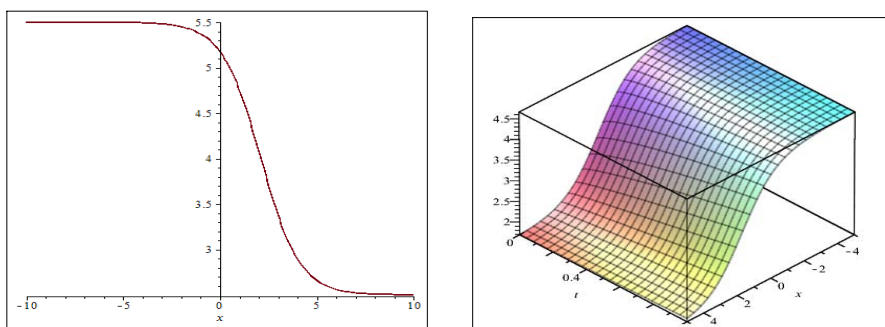


Figure 1c. 2D and 3D representation of solution for  $\beta=0.75$ .

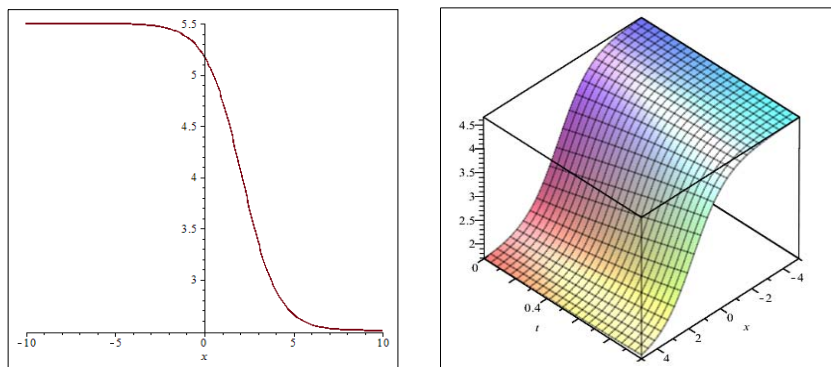


Figure 1d. 2D and 3D representation of solution for  $\beta=1$ .

2nd solution set

$$\left\{ \omega = -\sqrt{4l^2 + l}, b_{-1} = \frac{a_{-1}(6la_1 + b_1)}{a_1}, b_0 = b_0, b_1 = b_1, a_{-1} = a_{-1} = a_{-1}, a_0 = 0, a_1 = a_1 \right\}$$

We attain soliton solution  $w(x, t)$  as,

$$w(x, t) = \frac{-a_{-1}(6la_1 + b_1) \exp\left(-lx - \frac{\sigma\sqrt{4l^2 + l}t^\beta}{\Gamma(1+\beta)}\right) + b_1 \exp\left(lx - \frac{\sigma\sqrt{4l^2 + l}t^\beta}{\Gamma(1+\beta)}\right)}{b_{-1} \exp\left(-lx + \frac{\sigma\sqrt{4k^2 + kt^\beta}}{\Gamma(1+\beta)}\right) + a_1 \exp\left(lx - \frac{\sigma\sqrt{4l^2 + l}t^\beta}{\Gamma(1+\beta)}\right)}.$$

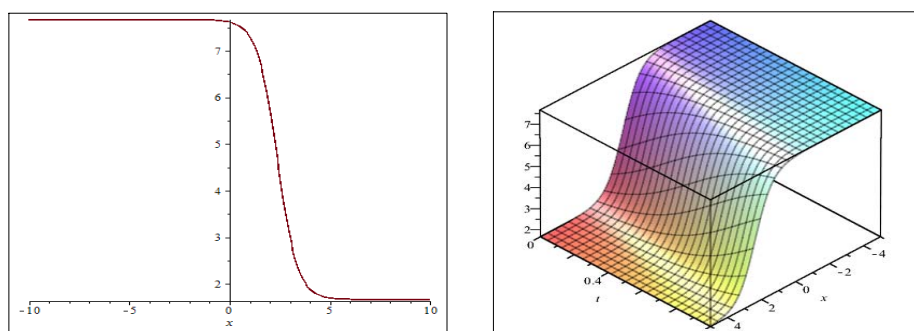


Figure 2a. 2D and 3D representation of solution for  $\beta=0.25$ .

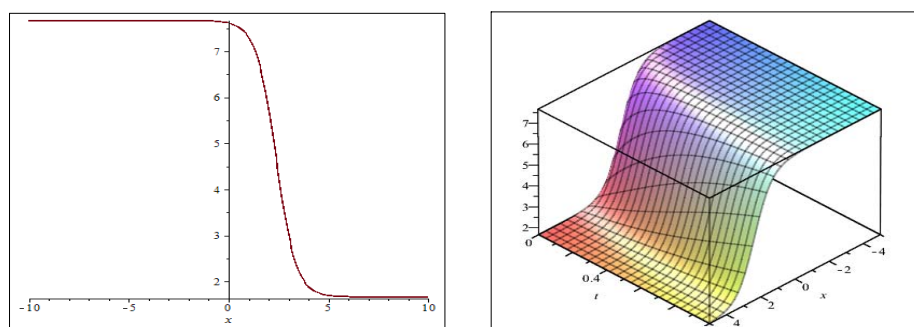


Figure 2b. 2D and 3D representation of solution for  $\beta=0.50$ .

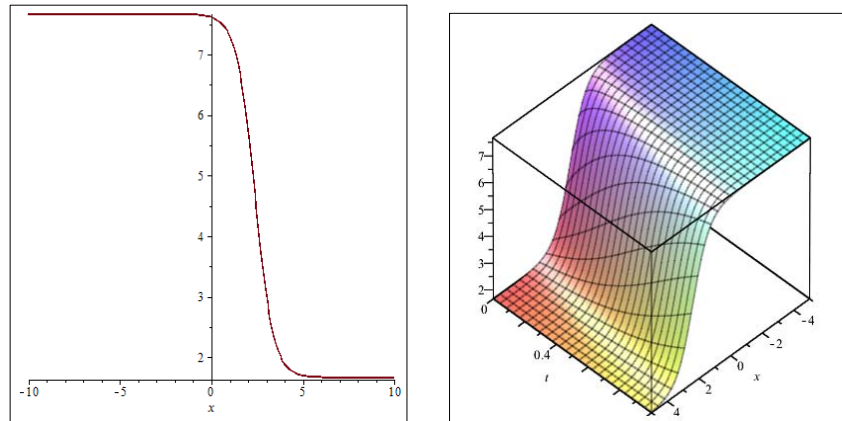


Figure 2c. 2D and 3D representation of solution for  $\beta=0.75$ .

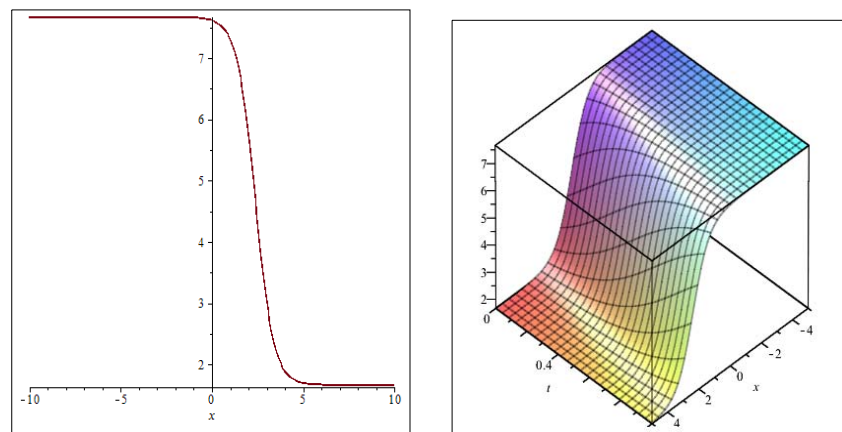


Figure 2d. 2D and 3D representation of solution for  $\beta=1$ .

### 3rd Solution set

$$\left\{ \omega = \sqrt{l^2 + l}, b_{-1} = \frac{a_{-1}(3la_0 + b_1)}{a_1}, b_0 = b_0, b_1 = 0, a_{-1} = a_{-1}, a_0 = a_0, a_1 = 0 \right\}$$

We get generalized solitary wave solution  $w(x, t)$

$$w(x, t) = \frac{\frac{a_{-1}(3la_0 + b_0)}{a_0} \exp\left(-lx - \frac{\sigma\sqrt{l^2 + l}t^\beta}{\Gamma(1+\beta)}\right) + b_0}{b_{-1} \exp\left(-lx + \frac{\sigma^2 + lt^\beta}{\Gamma(1+\beta)}\right) + a_0}.$$



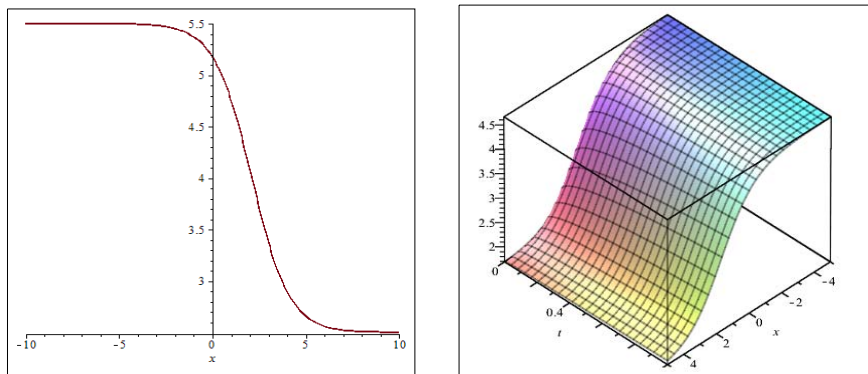


Figure 3a. 2D and 3D representation of solution for  $\beta=0.25$ .

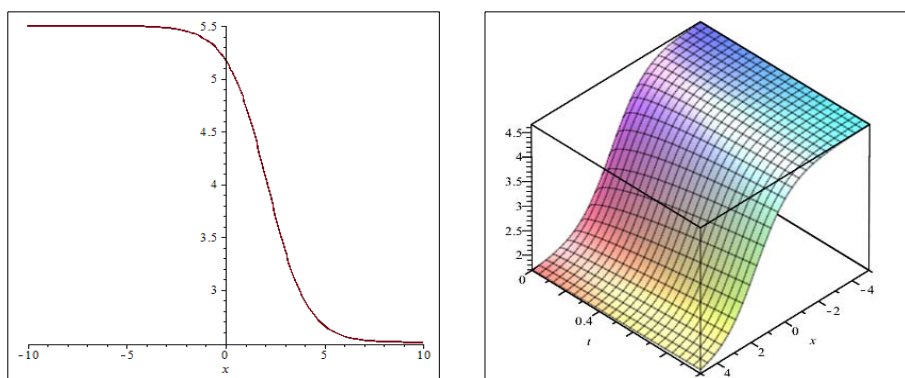


Figure 3b. 2D and 3D representation of solution for  $\beta=0.50$ .

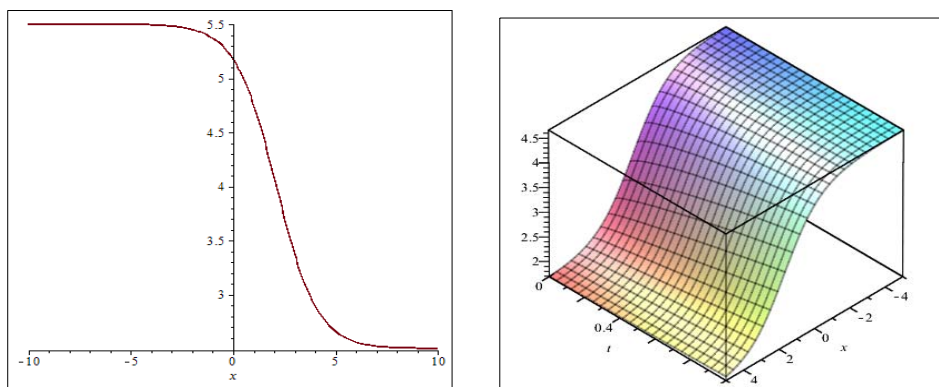


Figure 3c. 2D and 3D representation of solution for  $\beta=0.75$ .

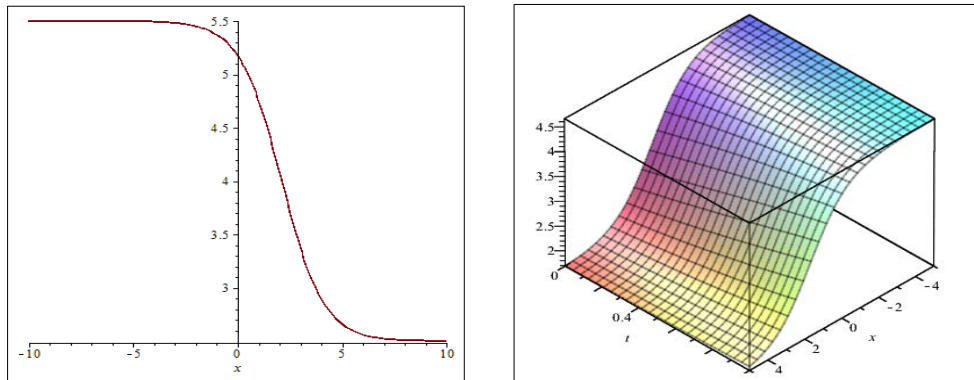


Figure 3d. 2D and 3D representation of solution for  $\beta=1$ .

4th Solution set

$$\left\{ \omega = -\sqrt{4l^2 + l}, b_{-1} = \frac{a_{-1}(6la_1 + b_1)}{a_1}, b_0 = b_0, b_1 = b_1, a_{-1} = a_{-1} = a_{-1}, a_0 = 0, a_1 = a_1 \right\}$$

We obtained the following generalized solitary solution  $w(x, t)$

$$w(x, t) = \frac{\frac{a_{-1} \exp\left(-lx + \frac{\sigma\sqrt{l^2 + lt^\beta}}{\Gamma(1+\beta)}\right) + b_1 \exp\left(lx - \frac{\sigma\sqrt{l^2 + lt^\beta}}{\Gamma(1+\beta)}\right)}{a_0}}{a_{-1} \exp\left(-lx + \frac{\sigma\sqrt{l^2 + lt^\beta}}{\Gamma(1+\beta)}\right) + a_1 \exp\left(lx - \frac{\sigma\sqrt{l^2 + kt^\beta}}{\Gamma(1+\beta)}\right)}$$

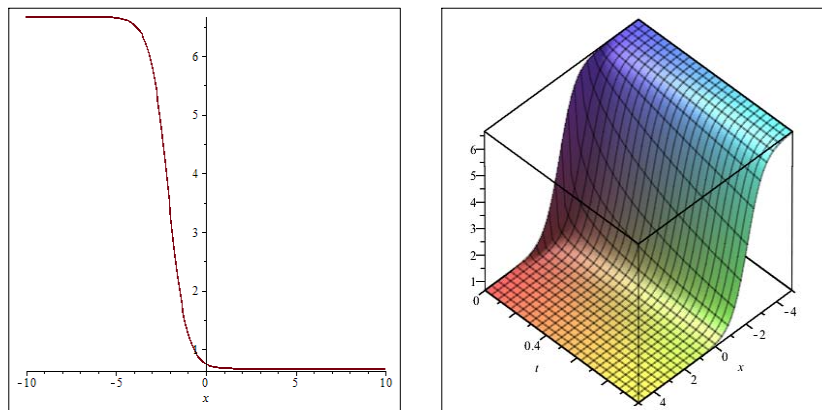


Figure 4a. 2D and 3D representation of solution for  $\beta=0.25$ .

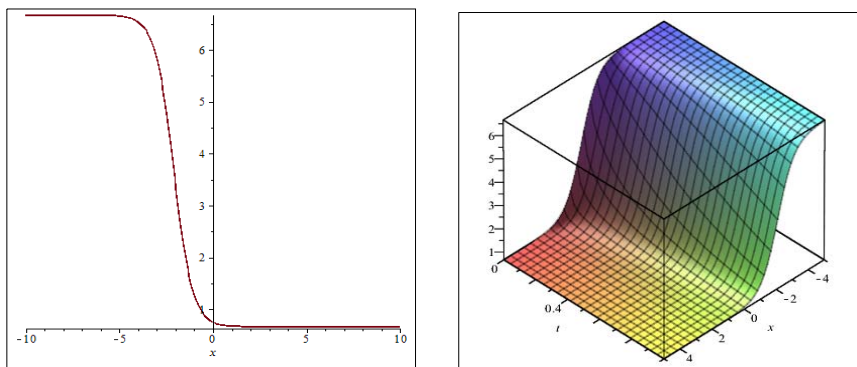


Figure 4b. 2D and 3D representation of solution for  $\beta=0.50$ .

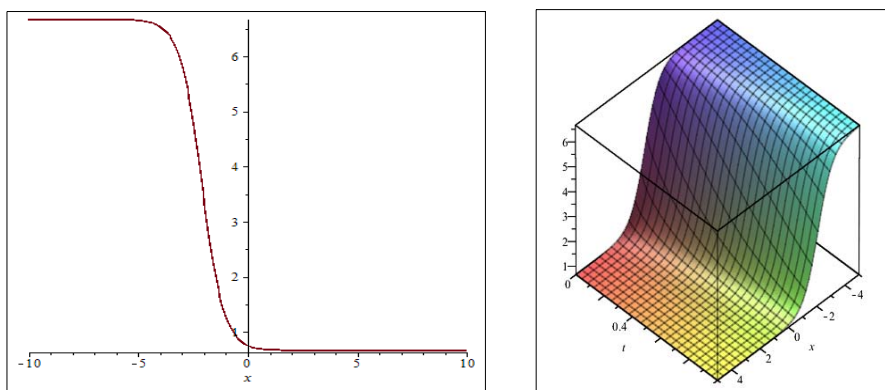


Figure 4c. 2D and 3D representation of solution for  $\beta=0.75$ .

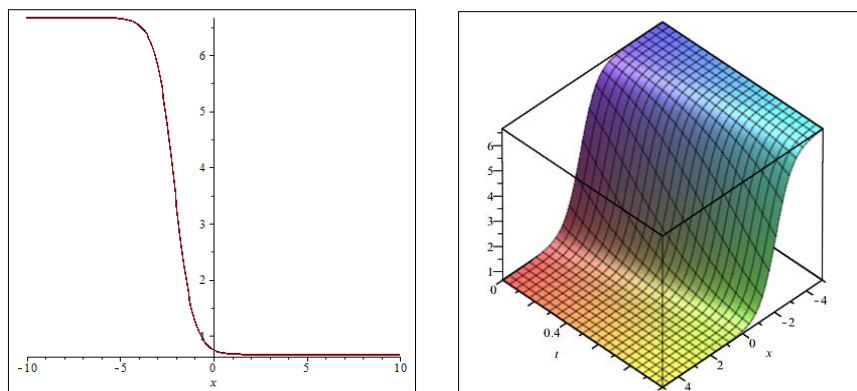


Figure 4d. 2D and 3D representation of solution for  $\beta=1$ .

**Case II.** By setting  $s = e = 2$  and  $r = f = 1$  the trial solution takes the form

$$w(\xi) = \frac{b_2 \exp[2\xi] + b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi]}{a_2 \exp[2\xi] + a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]} \tag{14}$$

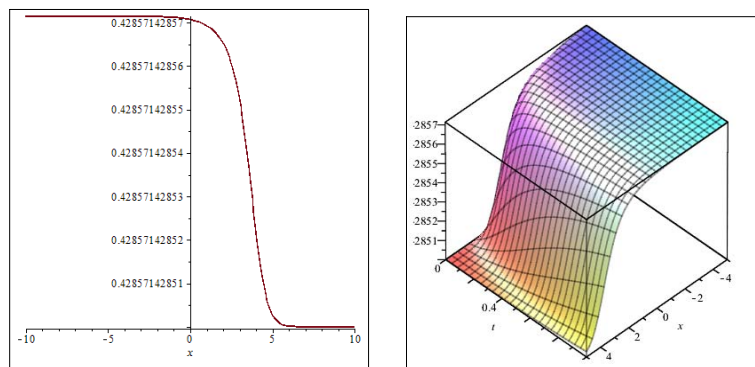
Proceeding as earlier, we attain

**5<sup>th</sup> solution set**

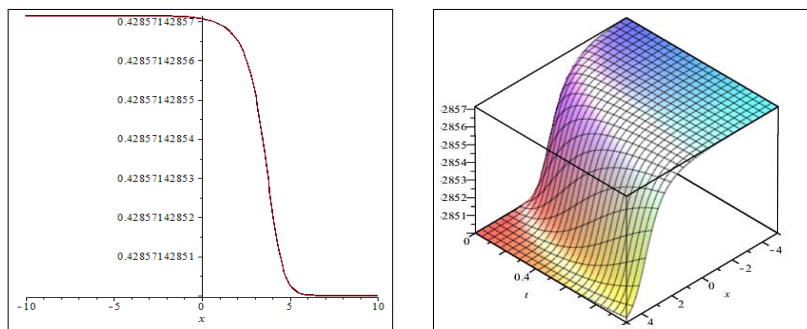
$$\left\{ b_{-1} = b_{-1}, b_0 = \frac{b_{-1}a_0}{a_{-1}}, b_1 = \frac{b_{-1}a_1}{a_{-1}}, b_2 = \frac{b_{-1}a_2}{a_{-1}} \right. \\ \left. b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1, a_2 = a_2 \right\}$$

We attain the solution  $w(x,t)$  as,

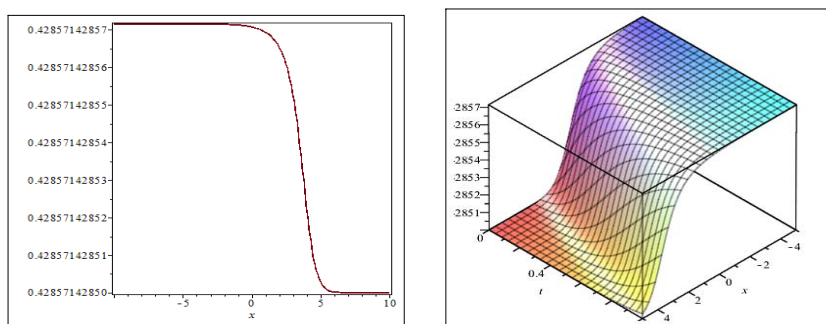
$$w(x,t) = \frac{a_{-1} \exp\left(-lx + \frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right) + \frac{b_{-1}a_0}{a_{-1}} + \frac{b_{-1}a_1}{a_{-1}} \exp\left(lx - \frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right) + \frac{b_{-1}a_2}{a_{-1}} \exp\left(2lx - 2\frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right)}{b_{-1} \exp\left(-lx + \frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right) + b_0 + b_1 \exp\left(lx - \frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right) + a_2 \exp\left(2lx - 2\frac{\sigma_0 t^\beta}{\Gamma(1+\beta)}\right)}$$



**Figure 5a. 2D and 3D representation of solution for  $\beta=0.25$ .**



**Figure 5b. 2D and 3D representation of solution for  $\beta=0.50$ .**



**Figure 5c. 2D and 3D representation of solution for  $\beta=0.75$ .**

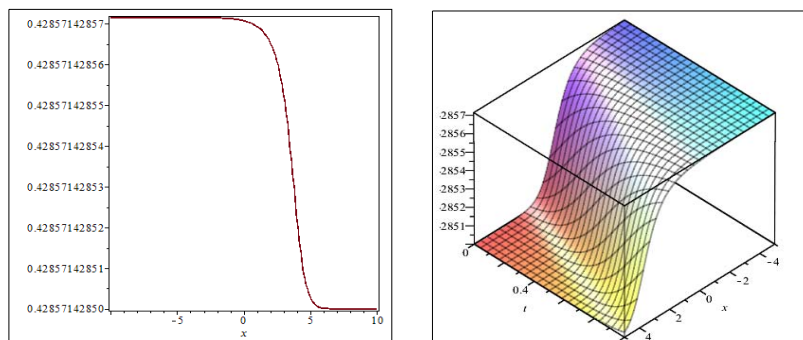


Figure 5d. 2D and 3D representation of solution for  $\beta=1$ .

## 5. RESULTS AND DISCUSSION

From the above figures we note that soliton is a wave which preserves its shape after it collides with another wave of the same kind. By solving nonlinear fractional fifth order evolution equation, we attain desired kink wave solutions for  $t = 1$  and different value of  $\beta$  such as 0.25, 0.5, 0.75 and 1. The solitary wave moves toward right if the velocity is positive or left directions if the velocity is negative and the amplitudes and velocities are controlled by various parameters. Solitary waves show more complicated behaviors which are controlled by various parameters. Figures signify graphical representation for different values of parameters. In both cases, for various values of parameters  $e, f, s$  and  $r$  we attain identical solitary wave solutions which obviously comprehend that final solution does not effectively based upon these parameters. So we can choose arbitrary values of such parameters. Since the solutions depend on arbitrary functions, we choose different parameters as input to our simulations.

## 6. CONCLUSION

This article is devoted to attain, test and analyze the novel soliton wave solutions and physical properties of nonlinear partial differential equation. For this, fractional order nonlinear evolution equation is considered and we apply Exp function method. We attain desired soliton solutions of various types for different values of parameters. It is guaranteed the accuracy of the attain results by backward substitution into the original equation with Maple 16. The scheming procedure of this method is simplest, straight and productive. We observed that the under study technique is more reliable and have minimum computational task, so widely applicable. In precise we can say this method is quite competent and much operative for evaluating exact solution of NLEEs. The validity of given algorithm is totally hold up with the help of the computational work, the graphical representations and successive results. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate the solitary solutions.

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