*ORIGINAL PAPER*

# AN EXTENSION OF A CLASS OF GENERATING FUNCTIONS FOR THE CESÀRO POLYNOMIALS OF THREE VARIABLES

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***Abstract.*** *The present study deals with some new properties for the Cesàro polynomials of three variables. The results obtained here include various families of multilinear and multilateral generating functions and their miscellaneous properties. We also derive an application giving certain families of bilateral generating functions for the Cesàro polynomials and the generalized Lauricella functions. At the end, we discuss some special cases for this theorem.*

***Keywords:*** *Cesàro polynomials, generating function, multilinear and multilateral generating function, recurrence relation, hypergeometric function.*

## 1. INTRODUCTION

The Cesàro polynomials $g_n^{(m)}(x)$ are defined by ([3,4])

$$g_n^{(m)}(x) = \binom{m+n}{n} {}_2F_1\left[-n, 1; -m-n; x\right],$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Agarwal and Manocha obtained the following generating function for the Cesàro polynomials [4]:

$$\sum_{n=0}^{\infty} g_n^{(m)}(x)t^n = (1-t)^{-m-1}(1-xt)^{-1}.$$

In [8], Malik defined the Cesàro polynomials of three variables $g_n^{(m)}(x,y,z)$ as follows:

$$g_n^{(m)}(x,y,z) = \binom{m+n}{n} {}_2F_1\left[\begin{array}{lll} -n & :: -;-;-: & 1;1;1; \\ & & & x,y,z \\ -m-n & :: -;-;-: & -;-;-; \end{array}\right]$$

$$= \frac{(m+n)!}{m!n!} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}}{(-m-n)_{r+s+k}} x^r y^s z^k, \tag{1.1}$$

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where $(\lambda)_n$ denotes the Pochhammer symbol. These polynomials have the following generating function:

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z)t^n = (1-t)^{-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}. \tag{1.2}$$

The next formula holds true for the polynomials $g_n^{(m)}(x,y,z)$:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(m)}(x,y,z)t^n = (1-t)^{-m-1-k}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1} \times g_k^{(m)}\left(\frac{x(1-t)}{1-xt}, \frac{y(1-t)}{1-yt}, \frac{z(1-t)}{1-zt}\right). \tag{1.3}$$

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables is due to Srivastava and Daoust who defined the generalized Lauricella (or the Srivastava-Daoust) function as follows [2]:

$$F_{C:D^{(1)};\dots;D^{(n)}}^{A:B^{(1)};\dots;B^{(n)}}\left(\begin{matrix}\left[(a):\ \theta^{(1)},\dots,\theta^{(n)}\right]:\ \left[(b^{(1)}):\ \varphi^{(1)}\right];\ \dots;\ \left[(b^{(n)}):\ \varphi^{(n)}\right]; \\ \left[(c):\ \psi^{(1)},\dots,\psi^{(n)}\right]:\ \left[(d^{(1)}):\ \delta^{(1)}\right];\ \dots;\ \left[(d^{(n)}):\ \delta^{(n)}\right];\end{matrix}\ z_1,\dots,z_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \Omega(m_1,\dots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1,\dots,m_n) := \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\dots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\dots+m_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}} \cdots \frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\theta_j^{(k)}\ (j=1,\dots,A;\ k=1,\dots,n), \text{ and } \phi_j^{(k)}\ (j=1,\dots,B^{(k)};\ k=1,\dots,n),$$

$$\psi_j^{(k)}\ (j=1,\dots,C;\ k=1,\dots,n), \text{ and } \delta_j^{(k)}\ (j=1,\dots,D^{(k)};\ k=1,\dots,n)$$

are real constants and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\ (j=1,\dots,B^{(k)};\ k=1,\dots,n)$$

with similar interpretations for other sets of parameters [1].

For a suitable bounded non-vanishing multiple sequence $\{\Omega(m_1,...,m_s)\}_{m_1,...,m_s \in N_0}$ having real or complex parameters, let $\varphi_n(u_1;u_2,...,u_s)$ of $s$ (real or complex) variables $u_1;u_2,...,u_s$ be defined by

$$\phi_n(u_1;u_2,...,u_s) := \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\varphi}}{((d))_{m_1\delta}} \times\Omega\left(f(m_1,m_2,...,m_s),m_2,...,m_s\right)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}, \tag{1.4}$$

where, for convenience,

$$((b))_{m_1\phi} = \prod_{j=1}^{B}(b_j)_{m_1\phi_j} \text{ and } ((d))_{m_1\delta} = \prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

The main object of this paper is to study several properties of the Cesàro polynomials of three variables $g_n^{(m)}(x,y,z)$. Various families of multilinear and multilateral generating functions and miscellaneous properties for these polynomials are obtained. In addition, we derive a theorem giving certain families of bilateral generating functions for the Cesàro polynomials $g_n^{(m)}(x,y,z)$ and the generalized Lauricella functions.

## 2. BILINEAR AND BILATERAL GENERATING FUNCTIONS

In this section, we study a number of families of bilinear and bilateral generating functions for the Cesàro polynomials of three variables $g_n^{(m)}(x,y,z)$ by using a similar method considered in the papers [5-7].

We begin by stating the following theorem.

**Theorem 1.** *Corresponding to an identically non-vanishing function $\Omega_\mu(s_1,...,s_r)$ of $r$ complex variables $s_1,...,s_r$ $(r \in \mathrm{N})$ and of complex order $\mu,\psi$, let*

$$\Lambda_{\mu,\psi}(s_1,...,s_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(s_1,...,s_r)\zeta^k \quad (a_k \neq 0)$$

*and*

$$\Theta_{n,p}^{\mu,\psi}\left(x,y,z;s_1,...,s_r;\xi\right) := \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(m)}(x,y,z)\Omega_{\mu+\psi k}(s_1,...,s_r)\xi^k.$$

*Then, for $p \in \mathrm{N}$, we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y,z;s_1,\dots,s_r;\frac{\eta}{t^p}\right)t^n$$
$$=(1-t)^{-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}\Lambda_{\mu,\psi}(s_1,\dots,s_r;\eta) \tag{2.1}$$

*provided that each member of (2.1) exists.*

*Proof.* For convenience, let $S$ denote the first member of the assertion (2.1). Then, we get

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(m)}(x,y,z)\Omega_{\mu+\psi k}(s_1,\dots,s_r)\eta^k t^{n-pk}.$$

Replacing $n$ by $n+pk$, we may write that

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, g_n^{(m)}(x,y,z)\Omega_{\mu+\psi k}(s_1,\dots,s_r)\eta^k t^n$$

$$=\sum_{n=0}^{\infty}g_n^{(m)}(x,y,z)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(s_1,\dots,s_r)\eta^k$$

$$=(1-t)^{-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}\Lambda_{\mu,\psi}(s_1,\dots,s_r;\eta)$$

which completes the proof.

By using a similar idea, we also get the next result.

**Theorem 2** *Corresponding to an identically non-vanishing function* $\Omega_\mu(s_1,\dots,s_r)$ *of* $r$ *complex variables* $s_1,\dots,s_r$ $(r\in\mathbb{N})$ *and of complex order* $\mu$, *let*

$$\Lambda_{\mu,p,q}[x,y,z;s_1,\dots,s_r;t]:=\sum_{n=0}^{\infty}a_n g_{m+qn}^{(\lambda)}(x,y,z)\Omega_{\mu+pn}(s_1,\dots,s_r)t^n,$$

*where* $a_n\neq 0$ *and*

$$\theta_{n,p,q}(s_1,\dots,s_r;z):=\sum_{k=0}^{[n/q]}a_k\binom{m+n}{n-qk}\Omega_{\mu+pk}(s_1,\dots,s_r)z^k.$$

*Then, for* $p,q\in\mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} g_{n+m}^{(\lambda)}(x,y,z)\theta_{n,p,q}(s_1,...,s_r;z)t^n$$

$$=(1-t)^{-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1} \tag{2.2}$$

$$\times\Lambda_{\mu,p,q}\left(\frac{x(1-t)}{1-xt},\frac{y(1-t)}{1-yt},\frac{z(1-t)}{1-zt};s_1,...,s_r;z(\frac{t}{1-t})^q\right)$$

*provided that each member of (2.2) exists.*

*Proof.* For convenience, let $T$ denote the first member of the assertion (2.2). Then,

$$T=\sum_{n=0}^{\infty} g_{n+m}^{(\lambda)}(x,y,z)\sum_{k=0}^{[n/q]} a_k \binom{m+n}{n-qk}\Omega_{\mu+pk}(s_1,...,s_r)z^k t^n .$$

Replacing $n$ by $n+qk$ and then using (1.3), we may write that

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+n+qk}{n} g_{n+m+qk}^{(\lambda)}(x,y,z)a_k\,\Omega_{\mu+pk}(s_1,...,s_r)z^k t^{n+qk}$$

$$=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{m+n+qk}{n} g_{n+m+qk}^{(\lambda)}(x,y,z)t^n\right)a_k\Omega_{\mu+pk}(s_1,...,s_r)(zt^q)^k$$

$$=\sum_{k=0}^{\infty} a_k(1-t)^{-\lambda-1-m-qk}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times g_{m+qk}^{(\lambda)}\left(\frac{x(1-t)}{1-xt},\frac{y(1-t)}{1-yt},\frac{z(1-t)}{1-zt}\right)\Omega_{\mu+pk}(s_1,...,s_r)(zt^q)^k .$$

Then we get

$$T=(1-t)^{-\lambda-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}\sum_{k=0}^{\infty} a_k(1-t)^{-qk}$$

$$\times g_{m+qk}^{(\lambda)}\left(\frac{x(1-t)}{1-xt},\frac{y(1-t)}{1-yt},\frac{z(1-t)}{1-zt}\right)\Omega_{\mu+pk}(s_1,...,s_r)(zt^q)^k$$

$$=(1-t)^{-\lambda-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times\Lambda_{\mu,p,q}\left(\frac{x(1-t)}{1-xt},\frac{y(1-t)}{1-yt},\frac{z(1-t)}{1-zt};s_1,...,s_r;z(\frac{t}{1-t})^q\right),$$

which gives the desired result.

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$ is expressed in terms of simpler functions of one and more variables, we can obtain lots of applications of the Theorem 1 and Theorem 2. Furthermore, for every suitable choice of the coefficients $a_k$ $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r \in \mathbb{N}$ are expressed as an appropriate product of several simpler functions, the assertions of these theorems can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the Cesàro polynomials of three variables $g_n^{(m)}(x,y,z)$ given explicitly by (1.1).

## 3. MISCELLANEOUS PROPERTIES

In this section we give some properties for the Cesàro polynomials of three variables.

**Theorem 3** *The Cesàro polynomials in three variables* $g_n^{(m)}(x,y,z)$ *have the following integral representation:*

$$g_n^{(m)}(x,y,z) = \frac{1}{n!\Gamma(m+1)} \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(u_1+u_2+u_3+u_4)}$$

$$\times u_1^m (u_1 + xu_2 + yu_3 + zu_4)^n \, du_1 du_2 du_3 du_4,$$

*where* $\Gamma(m+1) > 0$.

*Proof. If we use the identity*

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt \qquad (\operatorname{Re}(v) > 0)$$

*on the left-hand side of the generating function (1.2), we have*

$$\sum_{n=0}^\infty g_n^{(m)}(x,y,z) t^n = (1-t)^{-m-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$= \frac{1}{\Gamma(m+1)} \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(u_1+u_2+u_3+u_4)} u_1^m$$

$$\times e^{t(u_1+xu_2+yu_3+zu_4)} du_1 du_2 du_3 du_4$$

$$= \frac{1}{\Gamma(m+1)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Big( e^{-(u_1+u_2+u_3+u_4)} u_1^m$$

$$\times \sum_{n=0}^{\infty} (u_1 + xu_2 + yu_3 + zu_4)^n \frac{t^n}{n!} \Big) du_1 du_2 du_3 du_4,$$

*which gives*

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z) t^n$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{n!\Gamma(m+1)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(u_1+u_2+u_3+u_4)} u_1^m (u_1 + xu_2 + yu_3 + zu_4)^n du_1 du_2 du_3 du_4 \right) t^n.$$

*From the coefficents of $t^n$ on the both sides of the last equality, one can get the desired result.*

We now discuss some miscellaneous recurrence relations of the Cesàro polynomials of three variables. By differentiating each member of the generating function relation (1.2) with respect to $x, y, z$ and using

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k),$$

we arrive at the following (differential) recurrence relations for these polynomials:

$$\frac{\partial}{\partial x} g_n^{(m)}(x,y,z) = \sum_{p=0}^{n-1} x^p g_{n-p-1}^{(m)}(x,y,z) \quad n \geq p+1.$$

$$\frac{\partial}{\partial y} g_n^{(m)}(x,y,z) = \sum_{p=0}^{n-1} y^p g_{n-p-1}^{(m)}(x,y,z) \quad n \geq p+1.$$

$$\frac{\partial}{\partial z} g_n^{(m)}(x,y,z) = \sum_{p=0}^{n-1} z^p g_{n-p-1}^{(m)}(x,y,z) \quad n \geq p+1.$$

Besides, by differentiating each member of the generating function relation (1.2) with respect to $t$, we have the following another recurrence relation:

$$(n+1) g_{n+1}^{(m)}(x,y,z) = (1+m) \sum_{k=0}^{n} g_{n-k}^{(m)}(x,y,z) + \sum_{k=0}^{n} g_{n-k}^{(m)}(x,y,z) \left\{ x^{k+1} + y^{k+1} + z^{k+1} \right\}.$$

## 4. THE GENERALIZED LAURICELLA FUNCTIONS

In the present section, we derive various families of bilateral generating functions for the Cesàro polynomials of three variables and the generalized Lauricella (or the Srivastava-Daoust) functions.

**Theorem 4** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(\lambda)}(x,y,z)\phi_n(u_1;u_2,...,u_s)t^n$$

$$=(1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times\sum_{m_1,r,s,k,m_2,...,m_s=0}^{\infty}\frac{((b))_{(m_1+r+s+k)\varphi}(\lambda+1)_{m_1}}{((d))_{(m_1+r+s+k)\delta}}$$

$$\times\Omega\left(f((m_1+r+s+k),m_2,...,m_s);m_2,...,m_s\right)$$

$$\times\frac{(-\frac{u_1t}{1-t})^{m_1}}{m_1!}\left(-\frac{u_1xt}{1-xt}\right)^r\left(-\frac{u_1yt}{1-yt}\right)^s(-\frac{u_1zt}{1-zt})^k\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_s}}{m_s!},$$

*where* $\varphi_n(u_1;u_2,...,u_s)$ *is given by (1.4).*

*Proof*. By using (1.3), we observe that

$$\sum_{n=0}^{\infty} g_n^{(\lambda)}(x,y,z)\varphi_n(u_1;u_2,...,u_s)t^n$$

$$=\sum_{n=0}^{\infty} g_n^{(\lambda)}(x,y,z)\sum_{m_1=0}^{n}\sum_{m_2,...,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}$$

$$\times\Omega(f(m_1,...,m_s),m_2,...,m_s)\frac{u_1^{m_1}}{m_1!}...\frac{u_s^{m_s}}{m_s!}t^n$$

$$=\sum_{m_1,m_2,...,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega(f(m_1,...,m_s),m_2,...,m_s)(-u_1t)^{m_1}$$

$$\times\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_s}}{m_s!}(1-t)^{-\lambda-m_1-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times g_{m_1}^{(\lambda)}\left(\frac{x(1-t)}{1-xt},\frac{y(1-t)}{1-yt},\frac{z(1-t)}{1-zt}\right)$$

Then we get

$$\sum_{n=0}^{\infty} g_n^{(\lambda)}(x,y,z)\phi_n(u_1;u_2,...,u_s)t^n$$

$$=(1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times \sum_{m_1,m_2,...,m_s=0}^{\infty} \frac{((b))_{m_1\varphi}}{((d))_{m_1\delta}} \Omega\left(f(m_1,...,m_s),m_2,...,m_s\right)\left(-\frac{u_1 t}{1-t}\right)^{m_1}$$

$$\times \frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\frac{(\lambda+m_1)!}{\lambda! m_1!}\sum_{r=0}^{m_1}\sum_{s=0}^{m_1-r}\sum_{k=0}^{m_1-r-s}\frac{(-m_1)_{r+s+k}(1)_r(1)_s(1)_k}{(-\lambda-m_1)_{r+s+k}\, r!s!k!}$$

$$\times\left(\frac{x(1-t)}{1-xt}\right)^r\left(\frac{y(1-t)}{1-yt}\right)^s\left(\frac{z(1-t)}{1-zt}\right)^k$$

$$=(1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}\sum_{m_1,r,s,k,m_2,...,m_s=0}^{\infty}(\lambda+1)_{m_1}$$

$$\times\frac{((b))_{(m_1+r+s+k)\varphi}}{((d))_{(m_1+r+s+k)\delta}}\Omega\left(f((m_1+r+s+k),m_2,...,m_s);m_2,...,m_s\right)$$

$$\times\frac{(-\frac{u_1 t}{1-t})^{m_1}}{m_1!}\left(-\frac{u_1 xt}{1-xt}\right)^r\left(-\frac{u_1 yt}{1-yt}\right)^s(-\frac{u_1 zt}{1-zt})^k\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!},$$

whence the result.

By appropriately choosing the multiple sequence $\Omega(m_1,...,m_s)$ in Theorem 4, we obtain several interesting results as follows which give bilateral generating functions for the Cesàro polynomials in three variables $g_n^{(m)}(x,y,z)$ and the generalized Lauricella (or the Srivastava-Daoust) functions:

**I**. By letting

$$\Omega\left(f(m_1,...,m_s),m_2,...,m_s\right)$$

$$=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_s\theta_j^{(s)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_s\psi_j^{(s)}}}\frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\varphi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}\cdots\frac{\prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{m_s\varphi_j^{(s)}}}{\prod_{j=1}^{D^{(s)}}(d_j^{(s)})_{m_s\delta_j^{(s)}}}$$

in Theorem 4, we obtain the next result.

***Corollary*** *The following bilateral generating function relation holds true:*

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z) F_{E:D;D^{(2)};\dots;D^{(s)}}^{A:B+1;B^{(2)};\dots;B^{(s)}}\left( \begin{array}{l} \left[(a):\ \theta^{(1)},\dots,\theta^{(s)}\right]:\ [-n:1],\ [(b):\ \varphi];\ [(b^{(2)}):\ \varphi^{(2)}];\ \dots;\ [(b^{(s)}):\ \varphi^{(s)}]; \\ \left[(c):\ \psi^{(1)},\dots,\psi^{(s)}\right]:\ [(d):\ \delta];\ [(d^{(2)}):\ \delta^{(2)}];\ \dots;\ [(d^{(s)}):\ \delta^{(s)}]; \end{array} u_1,u_2,\dots,u_s \right) t^n$$

$$= (1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times F_{E+D:0;0;0;0;D^{(2)};\dots;D^{(s)}}^{A+B:1;1;1;1;B^{(2)};\dots;B^{(s)}}\left( \begin{array}{l} \left[(e):\ \phi^{(1)},\dots,\phi^{(s+3)}\right]:\ [\lambda+1:1];\ [1:1];\ [1:1];\ [1:1]; \\ [(f):\ \xi^{(1)},\dots,\xi^{(s+3)}]:\ -;\ -;\ -;\ -; \end{array} \right.$$
$$\left. \begin{array}{l} [(b^{(2)}):\ \varphi^{(2)}];\ \dots;\ [(b^{(s)}):\ \varphi^{(s)}]; \\ [(d^{(2)}):\ \delta^{(2)}];\ \dots;\ [(d^{(s)}):\ \delta^{(s)}]; \end{array} -\frac{u_1 t}{1-t}, -\frac{u_1 xt}{1-xt}, -\frac{u_1 yt}{1-yt}, -\frac{u_1 zt}{1-zt}, u_2,\dots,u_s \right),$$

*where the coefficients* $e_j$, $f_j$, $\varphi_j^{(s)}$ *and* $\xi_j^{(s)}$ *are given by*

$$e_j = \begin{cases} a_j, & (1 \le j \le A) \\ b_{j-A}, & (A < j \le A+B) \end{cases}$$

$$f_j = \begin{cases} c_j, & (1 \le j \le E) \\ d_{j-E}, & (E < j \le E+D), \end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)}, & (1 \le j \le A;\ 1 \le r \le 4) \\ \theta_j^{(r-1)}, & (1 \le j \le A;\ 4 < r \le s+3) \\ \phi_{j-A}, & (A < j \le A+B;\ 1 \le r \le 4) \\ 0, & (A < j \le A+B;\ 4 < r \le s+3), \end{cases}$$

*And*

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)}, & (1 \le j \le E;\ 1 \le r \le 4) \\ \psi_j^{(r-1)}, & (1 \le j \le E;\ 4 < r \le s+3) \\ \delta_{j-E}, & (E < j \le E+D;\ 1 \le r \le 4) \\ 0, & (E < j \le E+D;\ 4 < r \le s+3). \end{cases}$$

**II**. Upon setting

$$\Omega\left(f(m_1,...,m_s),m_2,...,m_s\right)=\frac{(a)_{m_1+...+m_s}(b_2)_{m_2}...(b_s)_{m_s}}{(c_1)_{m_1}...(c_s)_{m_s}}$$

and

$$\phi=\delta=0 \text{ (that is, } \phi_1=...=\phi_B=\delta_1=...=\delta_D=0)$$

in Theorem 4, we obtain the next result.

***Corollary*** *The following bilateral generating function relation holds true:*

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z)F_A^{(s)}\left[a,-n,b_2,...,b_s;c_1,...,c_s;u_1,u_2,...,u_s\right]t^n$$

$$=(1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$

$$\times F^{1:1;1;1;1;1;...;1}_{1:0;0;0;0;1;...;1}\left(\begin{matrix}[(a):1,...,1]: & [\lambda+1:1]; & [1:1]; & [1:1]; & [1:1];\\ [(c_1):\psi^{(1)},...,\psi^{(s+3)}]: & -; & -; & -; & -;\end{matrix}\right.$$

$$\left.\begin{matrix}[b_2:1]; & ...; & [b_s:1];\\ [c_2:1]; & ...; & [c_s:1];\end{matrix}-\frac{u_1t}{1-t},-\frac{u_1xt}{1-xt},-\frac{u_1yt}{1-yt},-\frac{u_1zt}{1-zt},u_2,...,u_s\right),$$

*where the coefficients* $\psi^{(\eta)}$ *are given by*

$$\psi^{(\eta)}=\begin{cases}1, & (1\le\eta\le 4)\\ 0, & \left(4<\eta\le s+3\right)\end{cases}.$$

**III.** If we take

$$\Omega\left(f(m_1,...,m_s);m_2,...,m_s\right)=\frac{(a_1^{(1)})_{m_2}...(a_1^{(s-1)})_{m_s}(a_2^{(1)})_{m_2}...(a_2^{(s-1)})_{m_s}}{(c)_{m_1+\cdots+m_s}}$$

and

$$B=1,\ b_1=b,\ \phi_1=1 \text{ and } \delta=0$$

in Theorem 4, then we get the next result.

***Corollary*** *The following bilateral generating function relation holds true:*

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z) F_B^{(s)}\left[-n, a_1^{(1)},...,a_1^{(s-1)}, b, a_2^{(1)},...,a_2^{(s-1)}; c; u_1, u_2,...,u_s\right] t^n$$
$$= (1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$
$$\times F_{1:0;0;0;0;0;...;0}^{1:1;1;1;1;2;...;2}\left( \begin{array}{l} \left[(b) : \theta^{(1)},...,\theta^{(s+4)}\right] : \ [\lambda+1 : 1]; \ [1:1]; \ [1:1]; \ [1:1]; \\ [(c) : 1,...,1] : \quad -; \quad -; \quad -; \quad -; \end{array} \right.$$
$$\left. \begin{array}{l} \left[a^{(1)} : 1\right]; \ ...; \ \left[a^{(s-1)} : 1\right]; \\ -; \quad ...; \quad -; \end{array} -\frac{u_1 t}{1-t}, -\frac{u_1 xt}{1-xt}, -\frac{u_1 yt}{1-yt}, -\frac{u_1 zt}{1-zt}, u_2,...,u_s \right),$$

*where the coefficients* $\theta^{(\eta)}$ *are given by*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \le \eta \le 4) \\ 0, & (4 < \eta \le s+4) \end{cases}.$$

**IV**. Letting

$$\Omega(f(m_1,...,m_s), m_2,...,m_s) = \frac{(a)_{m_1+...+m_s}(b_2)_{m_2}...(b_s)_{m_s}}{(c)_{m_1+...+m_s}}$$

and

$$\phi = \delta = 0$$

in Theorem 4, we obtain the following result.

***Corollary*** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z) F_D^{(s)}\left[a, -n, b_2,...,b_s; c; u_1, u_2,...,u_s\right] t^n$$
$$= (1-t)^{-\lambda-1}(1-xt)^{-1}(1-yt)^{-1}(1-zt)^{-1}$$
$$\times F_D^{(s+3)}\left[a, \lambda+1, 1,1,1, b_2,...,b_s; c; -\frac{u_1 t}{1-t}, -\frac{u_1 xt}{1-xt}, \right.$$
$$\left. -\frac{u_1 yt}{1-yt}, -\frac{u_1 zt}{1-zt}, u_2,...,u_s \right].$$

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