ORIGINAL PAPER

EIGENVALUES AND EIGENFUNCTIONS OF Q-DIRAC SYSTEM

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Abstract. *In this paper, we deal with a q-Dirac system. We investigate some spectral properties and the asymptotic behavior of the eigenvalues and the eigenfunctions of this q-Dirac system.*

Keywords: *q-Dirac system, eigenvalues and eigenfunctions, eigenfunction expansions*

1. INTRODUCTION

We consider a q-Dirac system which consists of the system of q-differential equations

$$\begin{cases} -\frac{1}{q} D_{q^{-1}} y_2(x) + p(x) y_1(x) = \lambda y_1(x), \\ D_q y_1(x) + r(x) y_2(x) = \lambda y_2(x), \end{cases} \tag{1}$$

and the boundary conditions

$$B_1(y) := k_{11} y_1(0) + k_{12} y_2(0) = 0, \tag{2}$$

$$B_2(y) := k_{21} y_1(a) + k_{22} y_2(aq^{-1}) = 0, \tag{3}$$

where $k_{ij}(i, j = 1,2)$ are real numbers, λ is a complex eigenvalue parameter, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $0 \le x \le a < \infty$, $p(x)$ and $r(x)$ are real-valued functions defined on $[0, a]$ and continuous at zero and $p(x), r(x) \in L_q^1(0, a)$ (see[1]).

In [1], the authors introduced a q-analog of one-dimensonal Dirac equation (1) and they investigated the existence and uniqueness of the solution of this equation and also gave some spectral properties of the problem (1)-(3). Dissipative, accumulative, self-adjoint for the same q-Dirac equation were described in [2].

In this paper, we study similar spectral properties and obtain the asymptotic formulas of the eigenvalues and the eigenfunctions of the problem (1)-(3) in the light of the theory of q-(basic) Sturm-Liouville problems [3, 4].

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2. PRELIMINARIES

In this section we introduce some of the required q-notations and results. Throughout this paper q is a positive number with $0<q<1$.

A set $A\subseteq\mathbb{R}$ is called q-geometric if, for every $x\in A,\ qx\in A$. Let f be a real or complex-valued function defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(x) := \frac{f(x)-f(qx)}{x(1-q)},\ x\neq 0. \tag{4}$$

If $0\in A$, the q-derivative at zero is defined to be

$$D_q f(0) := \lim_{n\to\infty}\frac{f(xq^n)-f(0)}{xq^n},\ x\in A, \tag{5}$$

if the limit exists and does not depend on x. Also, for $x\in A$, $D_{q^{-1}}$ is defined to be

$$D_{q^{-1}} f(x) := \begin{cases} \dfrac{f(x)-f(q^{-1}x)}{x(1-q^{-1})}, & x\in A\setminus\{0\}, \\ D_q f(0), & x=0, \end{cases} \tag{6}$$

provided that $D_q f(0)$ exists. The following relation can be verified directly from the definition

$$D_{q^{-1}} y(x) = (D_q y)(xq^{-1}). \tag{7}$$

A right inverse, q-integration, of the q-difference operator D_q is defined by Jackson [5] as

$$\int_0^x f(t)\,d_q t := x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),\ x\in A, \tag{8}$$

provided that the series converges. A q-analog of the fundamental theorem of calculus is given by

$$D_q\int_0^x f(t)\,d_q t = f(x),\quad \int_0^x D_q f(t)\,d_q t = f(x)-\lim_{n\to\infty} f(xq^n), \tag{9}$$

where $\lim_{n\to\infty} f(xq^n)$ can be replaced by $f(0)$ if f is q-regular at zero, that is, if $\lim_{n\to\infty} f(xq^n) = f(0)$, for all $x\in A$. Throughout this paper, we deal only with functions q-regular at zero.

The q-type product formula is given by

$$D_q(fg)(x) = g(x)D_q f(x) + f(qx)D_q g(x), \tag{10}$$

and hence the q-integration by parts is given by

$$\int_0^a g(x)D_q f(x)d_q x = (fg)(a) - (fg)(0) - \int_0^a D_q g(x) f(qx)d_q x, \tag{11}$$

where f and g are q-regular at zero.

For more results and properties in q-calculus, readers are referred to the recent works [6-9].

The basic trigonometric functions $\cos(z;q)$ and $\sin(z;q)$ are defined on $\mathbb{C}$ by

$$\cos(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (z(1-q))^{2n}}{(q;q)_{2n}}, \tag{12}$$

$$\sin(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (z(1-q))^{2n+1}}{(q;q)_{2n+1}}, \tag{13}$$

and they are q-analogs of the cosine and sine functions [7, 10, 11].

Theorem 2.1. ([12]) If $\{x_m\}$ and $\{y_m\}$ are the positive zeros of $\cos(z;q)$ and $\sin(z;q)$, respectively, then we have for sufficiently large m,

$$\{x_m\} = q^{-m+1/2}(1-q)^{-1}\left(1+O\left(q^m\right)\right), \tag{14}$$

$$\{y_m\} = q^{-m}(1-q)^{-1}\left(1+O\left(q^m\right)\right). \tag{15}$$

Corollary 2.1. ([12, Corollaries 3.2 and 3.4]) For $r := |z| \to \infty$ we have

$$M\left(r;\cos(z;q)\right) = O\left(\exp\left(-\frac{(\log r(1-q))^2}{\log q}\right)\right), \tag{16}$$

$$M\left(r;\sin(z;q)\right) = O\left(\exp\left(-\frac{(\log r(1-q))^2}{\log q}\right)\right). \tag{17}$$

3. SPECTRAL PROPERTIES AND ASYMPTOTIC FORMULAS

In this section we give some spectral properties similar to [1, 13], then we obtain asymptotic formulas for the eigenvalues and the eigenfunctions of the problem (1)-(3).

Lemma 3.1. The eigenfunctions $y(x,\lambda_1)$ and $z(x,\lambda_2)$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal, i.e.,

$$\int_0^a y^{\perp} z d_q x = \int_0^a \left\{ y_1(x,\lambda_1) z_1(x,\lambda_2) + y_2(x,\lambda_1) z_2(x,\lambda_2) \right\} d_q x = 0, \tag{18}$$

where $y^{\perp} = (y_1, y_2)$, $\perp$ denotes the matrix transpose.

Proof. Since $y(x,\lambda_1)$ and $z(x,\lambda_2)$ are solutions of the q-system (1),

$$\begin{cases} -\dfrac{1}{q} D_{q^{-1}} y_2 + \left\{ p(x) - \lambda_1 \right\} y_1 = 0, \\ D_q y_1 + \left\{ r(x) - \lambda_1 \right\} y_2 = 0, \end{cases}$$
$$\begin{cases} -\dfrac{1}{q} D_{q^{-1}} z_2 + \left\{ p(x) - \lambda_2 \right\} z_1 = 0, \\ D_q z_1 + \left\{ r(x) - \lambda_2 \right\} z_2 = 0. \end{cases}$$

Multiplying by $z_1, z_2, -y_1$ and $-y_2$, respectively, and adding together and also using the formulas (7) and (10), we obtain

$$\begin{aligned} &D_q \left(y_1(x,\lambda_1) z_2(xq^{-1},\lambda_2) - y_2(xq^{-1},\lambda_1) z_1(x,\lambda_2) \right) \\ &\qquad = (\lambda_1 - \lambda_2) \left\{ y_1(x,\lambda_1) z_1(x,\lambda_2) + y_2(x,\lambda_1) z_2(x,\lambda_2) \right\}. \end{aligned} \tag{19}$$

Applying the q-integration to (19), we have

$$(\lambda_1 - \lambda_2) \int_0^a y^{\perp}(x,\lambda_1) z(x,\lambda_2) d_q x = \left\{ y_1(x,\lambda_1) z_2(xq^{-1},\lambda_2) - y_2(xq^{-1},\lambda_1) z_1(x,\lambda_2) \right\}_0^a. \tag{20}$$

It follows from the boundary conditions (2) and (3) the right hand side vanishes. Therefore, we get

$$(\lambda_1 - \lambda_2) \int_0^a y^{\perp}(x,\lambda_1) z(x,\lambda_2) d_q x = 0. \tag{21}$$

The lemma is thus proved, since $\lambda_1 \neq \lambda_2$. □

Lemma 3.2. The eigenvalues of the problem (1)-(3) are real.

Proof. Assume the contrary that λ_0 is a nonreal eigenvalue of the problem (1)-(3). Let $y(x,\lambda_0)$ be a corresponding (nontrivial) eigenfunction. $\overline{\lambda_0}$ is also an eigenvalue, corresponding to the eigenfunction $\overline{y}(x,\lambda_0)$. Since $\lambda_0 \neq \overline{\lambda_0}$ by the previous lemma,

$$\int_0^a \left(\left| y_1\left(x,\lambda_0\right) \right|^2 + \left| y_2\left(x,\lambda_0\right) \right|^2 \right) d_q x = 0 . \tag{22}$$

Hence $y\left(x,\lambda_0\right) \equiv 0$ and this is a contradiction. Consequently, λ_0 must be real. □

Now, we will construct a special system of solution of q-system (1). Let $\phi\left(x,\lambda\right) = \begin{pmatrix} \phi_1\left(.,\lambda\right) \\ \phi_2\left(.,\lambda\right) \end{pmatrix}$ be a solution of the q-system (1) that satisfies the initial conditions

$$\phi_1\left(0,\lambda\right) = k_{12}, \quad \phi_2\left(0,\lambda\right) = -k_{11} . \tag{23}$$

The existence and uniqueness of this solution for the q-system (1) were presented in [1]. It is obvious that $\phi\left(x,\lambda\right)$ satisfies the boundary condition (2).

Theorem 3.1. The following integral equations hold for the solution of $\phi\left(x,\lambda\right)$

$$\begin{aligned} \phi_1\left(x,\lambda\right) &= k_{12} \cos\left(\lambda x;q\right) - k_{11} \sin\left(\lambda x;q\right) \\ &+ q\int_0^x \left\{ \sin\left(\lambda x;q\right)\cos\left(\lambda qt;q\right) - \cos\left(\lambda x;q\right)\sin\left(\lambda qt;q\right) \right\} p\left(qt\right)\phi_1\left(qt,\lambda\right) d_q t \\ &- \int_0^x \left\{ \cos\left(\lambda x;q\right)\cos\left(\lambda\sqrt{q}t;q\right) + \sqrt{q}\sin\left(\lambda x;q\right)\sin\left(\lambda\sqrt{q}t;q\right) \right\} r\left(t\right)\phi_2\left(t,\lambda\right) d_q t, \end{aligned} \tag{24}$$

$$\begin{aligned} \phi_2\left(x,\lambda\right) &= -k_{12}\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right) - k_{11}\cos\left(\lambda\sqrt{q}x;q\right) \\ &+ q\int_0^x \left\{ \cos\left(\lambda\sqrt{q}x;q\right)\cos\left(\lambda qt;q\right) + \sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right)\sin\left(\lambda qt;q\right) \right\} p\left(qt\right)\phi_1\left(qt,\lambda\right) d_q t \\ &+ \sqrt{q}\int_0^x \left\{ \sin\left(\lambda\sqrt{q}x;q\right)\cos\left(\lambda\sqrt{q}t;q\right) - \cos\left(\lambda\sqrt{q}x;q\right)\sin\left(\lambda\sqrt{q}t;q\right) \right\} r\left(t\right)\phi_2\left(t,\lambda\right) d_q t. \end{aligned} \tag{25}$$

Proof. Let us construct two solutions of the q-system (1) as

$$\varphi_1\left(.,\lambda\right) = \begin{pmatrix} \varphi_{11}\left(x,\lambda\right) \\ \varphi_{12}\left(x,\lambda\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\lambda x;q\right) \\ -\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right) \end{pmatrix}, \ \varphi_2\left(.,\lambda\right) = \begin{pmatrix} \varphi_{21}\left(x,\lambda\right) \\ \varphi_{22}\left(x,\lambda\right) \end{pmatrix} = \begin{pmatrix} \sin\left(\lambda x;q\right) \\ \cos\left(\lambda\sqrt{q}x;q\right) \end{pmatrix}, \tag{26}$$

for $p\left(x\right) = r\left(x\right) \equiv 0$, with the q-Wronskian

$$W\left(\varphi_1,\varphi_2\right)\left(x,\lambda\right) = \varphi_{11}\left(x,\lambda\right)\varphi_{22}\left(xq^{-1},\lambda\right) - \varphi_{21}\left(x,\lambda\right)\varphi_{12}\left(xq^{-1},\lambda\right) = 1. \tag{27}$$

The function

$$y(.,\lambda)=\begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}=\begin{pmatrix} c_1\cos(\lambda x;q)+c_2\sin(\lambda x;q) \\ -c_1\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right)+c_2\cos\left(\lambda\sqrt{q}x;q\right) \end{pmatrix}, \tag{28}$$

is a fundamental set of the q-system (1) for $p(x)=r(x)\equiv 0$. Using q-analogue of the method of variation of constants, a particular solution of the q-system (1) may be given by

$$\begin{cases} y_1(x,\lambda)=c_1(x)\cos(\lambda x;q)+c_2(x)\sin(\lambda x;q), \\ y_2(x,\lambda)=-c_1(x)\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right)+c_2(x)\cos\left(\lambda\sqrt{q}x;q\right). \end{cases} \tag{29}$$

Hence the functions $c_i(x)(i=1,2)$ satisfy the q-linear system of equations

$$\begin{cases} \cos(\lambda x;q)D_{q^{-1}}c_1(x)+\sin(\lambda x;q)D_{q^{-1}}c_2(x)=-r\left(xq^{-1}\right)y_2\left(xq^{-1},\lambda\right), \\ \sqrt{q}\sin\left(\lambda q^{-1/2}x;q\right)D_{q^{-1}}c_1(x)-\cos\left(\lambda q^{-1/2}x;q\right)D_{q^{-1}}c_2(x)=-qp(x)y_1(x,\lambda). \end{cases} \tag{30}$$

Since the equality (27) satisfies, then (30) has a unique solution which leads

$$\begin{aligned} D_{q^{-1}}c_1(x)&=-r\left(xq^{-1}\right)\cos\left(\lambda q^{-1/2}x;q\right)y_2\left(xq^{-1},\lambda\right)-qp(x)\sin(\lambda x;q)y_1(x,\lambda), \\ D_{q^{-1}}c_2(x)&=qp(x)\cos(\lambda x;q)y_1(x,\lambda)-r\left(xq^{-1}\right)\sqrt{q}\sin\left(\lambda q^{-1/2}x;q\right)y_2\left(xq^{-1},\lambda\right). \end{aligned} \tag{31}$$

Using the formula (7) and replacing x by xq in (31), then we obtain

$$\begin{aligned} c_1(x)&=c_1-q\int_0^x p(qt)\sin(\lambda qt;q)y_1(qt,\lambda)d_qt-\int_0^x r(t)\cos\left(\lambda\sqrt{q}t;q\right)y_2(t,\lambda)d_qt, \\ c_2(x)&=c_2+q\int_0^x p(qt)\cos(\lambda qt;q)y_1(qt,\lambda)d_qt-\int_0^x r(t)\sqrt{q}\sin\left(\lambda\sqrt{q}t;q\right)y_2(t,\lambda)d_qt, \end{aligned} \tag{32}$$

when $c_i(x)(i=1,2)$ are q-regular at zero (here $c_1:=c_1(0),c_2:=c_2(0)$). That is the general solution $y(x,\lambda)=\begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$ of the q-system (1) is obtained to be

$$\begin{aligned} y_1(x,\lambda)&=c_1\cos(\lambda x;q)+c_2\sin(\lambda x;q) \\ &\quad+q\int_0^x\left\{\sin(\lambda x;q)\cos(\lambda qt;q)-\cos(\lambda x;q)\sin(\lambda qt;q)\right\}p(qt)y_1(qt,\lambda)d_qt \\ &\quad-\int_0^x\left\{\cos(\lambda x;q)\cos\left(\lambda\sqrt{q}t;q\right)+\sqrt{q}\sin(\lambda x;q)\sin\left(\lambda\sqrt{q}t;q\right)\right\}r(t)y_2(t,\lambda)d_qt, \end{aligned} \tag{33}$$

$$y_2(x,\lambda) = -c_1\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right) + c_2\cos\left(\lambda\sqrt{q}x;q\right)$$
$$+q\int_0^x\left\{\cos\left(\lambda\sqrt{q}x;q\right)\cos(\lambda qt;q) + \sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right)\sin(\lambda qt;q)\right\}p(qt)\,y_1(qt,\lambda)\,d_qt \quad (34)$$
$$+\sqrt{q}\int_0^x\left\{\sin\left(\lambda\sqrt{q}x;q\right)\cos\left(\lambda\sqrt{q}t;q\right) - \cos\left(\lambda\sqrt{q}x;q\right)\sin\left(\lambda\sqrt{q}t;q\right)\right\}r(t)\,y_2(t,\lambda)\,d_qt.$$

It is easy to determine c_1, c_2 for which $\phi(x,\lambda)$ satisfies the q-system (1) and the conditions (23), then we obtain (24) and (25). □

Theorem 3.2. As $|\lambda|\to\infty$, the function $\phi(x,\lambda)$ has the following asymptotic relations

$$\phi_1(x,\lambda) = k_{12}\cos(\lambda x;q) - k_{11}\sin(\lambda x;q) + O\left(|\lambda|^{-1}\exp\left(\frac{-\left(\log|\lambda|\,x(1-q)\right)^2}{\log q}\right)\right), \quad (35)$$

$$\phi_2(x,\lambda) = -k_{12}\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right) - k_{11}\cos\left(\lambda\sqrt{q}x;q\right)$$
$$+O\left(|\lambda|^{-1}\exp\left(\frac{-\left(\log|\lambda|\,q^{1/2}x(1-q)\right)^2}{\log q}\right)\right), \quad (36)$$

where for each $x\in(0,a]$ the O-terms are uniform on $\left\{xq^n : n\in\mathbb{N}\right\}$.

Proof. Similar to asymptotic relations for q-Sturm-Liouville problems in [4] and from Corollary (2.1), (35) and (36) can be obtained easily. □

Lemma 3.3. The eigenvalues of the problem (1)-(3) are simple.

Proof. The solution $\phi(x,\lambda)$ defined above is a nontrivial solution of the q-system (1) satisfying the boundary condition (2). To find the eigenvalues of the problem (1)-(3), we have to insert this function into the boundary condition (3) and find the roots of the obtained equation. So, putting the function $\phi(x,\lambda)$ into the boundary condition (3) we get the following equation

$$\Delta(\lambda) = k_{21}\phi_1(a,\lambda) + k_{22}\phi_2\left(aq^{-1},\lambda\right). \quad (37)$$

Then $\dfrac{d\Delta(\lambda)}{d\lambda} = k_{21}\dfrac{\partial\phi_1(a,\lambda)}{\partial\lambda} + k_{22}\dfrac{\partial\phi_2\left(aq^{-1},\lambda\right)}{\partial\lambda}$. Let λ_0 be a double eigenvalue, and $\phi^0(x,\lambda_0)$ one of the corresponding eigenfunctions. Then the conditions $\Delta(\lambda_0)=0,\ \dfrac{d\Delta(\lambda_0)}{d\lambda}=0$ should be fulfilled simultaneously, i.e.,

$$\begin{aligned} &k_{21}\phi_1^0\left(a,\lambda_0\right)+k_{22}\phi_2^0\left(aq^{-1},\lambda_0\right)=0,\\ &k_{21}\frac{\partial}{\partial\lambda}\phi_1^0\left(a,\lambda_0\right)+k_{22}\frac{\partial}{\partial\lambda}\phi_2^0\left(aq^{-1},\lambda_0\right)=0. \end{aligned} \tag{38}$$

Since k_{21} and k_{22} cannot vanish simultaneously, it follows from (38) that

$$\phi_1^0\left(a,\lambda_0\right)\frac{\partial\phi_2^0\left(aq^{-1},\lambda_0\right)}{\partial\lambda}-\phi_2^0\left(aq^{-1},\lambda_0\right)\frac{\partial\phi_1^0\left(a,\lambda_0\right)}{\partial\lambda}=0. \tag{39}$$

Now, differentiating the q-system (1) with respect to λ, we obtain

$$\begin{cases} -\dfrac{1}{q}D_{q^{-1}}\left(\dfrac{\partial y_2}{\partial\lambda}\right)+\left\{p(x)-\lambda\right\}\dfrac{\partial y_1}{\partial\lambda}=y_1,\\ D_q\left(\dfrac{\partial y_1}{\partial\lambda}\right)+\left\{r(x)-\lambda\right\}\dfrac{\partial y_2}{\partial\lambda}=y_2. \end{cases} \tag{40}$$

Multiplying the q-system (1) and (40) by $\frac{\partial y_1}{\partial\lambda},\frac{\partial y_2}{\partial\lambda},-y_1$ and $-y_2$, respectively, adding them together and integrating with respect to x from 0 and a, we obtain

$$\left\{y_2\left(xq^{-1},\lambda\right)\frac{\partial y_1(x,\lambda)}{\partial\lambda}-y_1(x,\lambda)\frac{\partial y_2\left(xq^{-1},\lambda\right)}{\partial\lambda}\right\}_0^a=\int_0^a\left\{y_1^2(x,\lambda)+y_2^2(x,\lambda)\right\}d_qx. \tag{41}$$

Putting $\lambda=\lambda_0$, taking into account that $\left.\frac{\partial\phi_1^0(x,\lambda_0)}{\partial\lambda}\right|_{x=0}=\left.\frac{\partial\phi_2^0(x,\lambda_0)}{\partial\lambda}\right|_{x=0}=0$, and using the equality (39), we obtain the relation

$$\int_0^a\left\{\left(\phi_1^0(x,\lambda_0)\right)^2+\left(\phi_2^0(x,\lambda_0)\right)^2\right\}d_qx=0. \tag{42}$$

Hence $\phi_1^0(x,\lambda_0)=\phi_2^0(x,\lambda_0)\equiv 0$, which is impossible. Consequently λ_0 must be a simple eigenvalue. □

Theorem 3.3. As $|\lambda|\to\infty$ the function $\Delta(\lambda)$ has the following asymptotic relation

$$\begin{aligned} \Delta(\lambda)=k_{21}&\left\{k_{12}\cos(\lambda a;q)-k_{11}\sin(\lambda a;q)+O\left(|\lambda|^{-1}\exp\left(\frac{-\left(\log|\lambda|a(1-q)\right)^2}{\log q}\right)\right)\right\}\\ &+k_{22}\left\{-k_{12}\sqrt{q}\sin\left(\lambda q^{-1/2}a;q\right)-k_{11}\cos\left(\lambda q^{-1/2}a;q\right)+O\left(|\lambda|^{-1}\exp\left(\frac{-\left(\log|\lambda|aq^{-1/2}(1-q)\right)^2}{\log q}\right)\right)\right\}. \end{aligned} \tag{43}$$

Proof. The proof is immediate by substituting (35) and (36) into the relation

$$\Delta(\lambda) = k_{21}\phi_1(a,\lambda) + k_{22}\phi_2(aq^{-1},\lambda). \qquad \square$$

Theorem 3.4. The eigenvalues $\{\lambda_m\}$ are the zeros of $\Delta(\lambda)$ has the following asymptotic relations as $m \to \infty$:

Case 1. $k_{12} \neq 0, k_{11} = 0$;

$$i)\ \lambda_m = \frac{q^{-m+1/2}}{a(1-q)}\left(1 + O\left(q^{m/2}\right)\right),\ k_{21} = 0, \tag{44}$$

$$ii)\ \lambda_m = \frac{q^{-m+1/2}}{a(1-q)}\left(1 + O\left(q^{m}\right)\right),\ k_{22} = 0, \tag{45}$$

Case 2. $k_{12} = 0, k_{11} \neq 0$;

$$i)\ \lambda_m = \frac{q^{-m+1}}{a(1-q)}\left(1 + O\left(q^{m/2}\right)\right),\ k_{21} = 0, \tag{46}$$

$$ii)\ \lambda_m = \frac{q^{-m}}{a(1-q)}\left(1 + O\left(q^{m}\right)\right),\ k_{22} = 0. \tag{47}$$

Proof. Similar to asymptotic relation for q-Sturm-Liouville problems [4] and from Theorem 2.1, the asymptotic relations (44)-(47) can be obtained easily. $\square$

Then from (35) and (36) and above theorem, the asymptotic relations of the eigenfunctions of the problem (1)-(3) is given by

Case 1. $k_{12} \neq 0, k_{11} = 0$;

$$\phi(x,\lambda_m) = \begin{pmatrix} \phi_1(x,\lambda_m) \\ \phi_2(x,\lambda_m) \end{pmatrix}$$
$$= \begin{cases} k_{12}\cos(\lambda_m x;q) + O\left(|\lambda_m|^{-1}\exp\left(\dfrac{-\left(\log|\lambda_m|x(1-q)\right)^2}{\log q}\right)\right), \\ -k_{12}\sqrt{q}\sin\left(\lambda_m\sqrt{q}x;q\right) + O\left(|\lambda_m|^{-1}\exp\left(\dfrac{-\left(\log|\lambda_m|q^{1/2}x(1-q)\right)^2}{\log q}\right)\right), \end{cases}$$

Case 2. $k_{12}=0, k_{11}\neq 0$;

$$\phi(x,\lambda_m)=\begin{pmatrix}\phi_1(x,\lambda_m)\\ \phi_2(x,\lambda_m)\end{pmatrix}$$

$$=\begin{cases}-k_{11}\sin(\lambda_m x;q)+O\left(|\lambda_m|^{-1}\exp\left(\dfrac{-\left(\log|\lambda_m|x(1-q)\right)^2}{\log q}\right)\right),\\ -k_{11}\cos\left(\lambda_m\sqrt{q}x;q\right)+O\left(|\lambda_m|^{-1}\exp\left(\dfrac{-\left(\log|\lambda_m|q^{1/2}x(1-q)\right)^2}{\log q}\right)\right).\end{cases}$$

REFERENCES

[1] Allahverdiev, B.P., Tuna, H., *Math. Meth. Appl. Sci.*, **40**, 7287, 2017.
[2] Allahverdiev, B.P., Tuna, H., *Quaestiones Mathematicae*, **41**, 239, 2017.
[3] Annaby, M.H., Mansour, *J. Phys. A:Math. Gen.*, **38**, 3775, 2005.
[4] Annaby, M.H., Mansour, Z.S., *Math. Nachr.*, **284**, 443, 2011.
[5] Jackson, F.H., *Q.J. Pure Appl. Math.*, **41**,193, 1910.
[6] Kac, V., Cheung, P., *Quantum Calculus*, Springer, New York, 2002.
[7] Gasper, G., Rahman, M., *Basic Hypergeometric Series*, Cambridge Univ. Press, New York, 1990.
[8] Bangerezako, G., *An Introduction to q-Difference Equations*, preprint, 2008. https://cdn.uclouvain.be/public/Exports%20reddot/math/documents/RAPSEM354.pdf
[9] Annaby, M.H., Mansour, Z.S., *q-Fractional Calculus and Equations*, Springer, Berlin, 2012.
[10] Andrews, G.E., Askey, R., Roy, R., *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.
[11] Annaby, M.H., Mansour, Z.S., *J. Math. Anal. Appl*., **323**, 1091, 2006.
[12] Annaby, M.H., Mansour, Z.S., *Math. Z.*, **258**, 363, 2008.
[13] Levitan, B.M., Sargsjan, I.S., *Sturm-Liouville and Dirac Operators*, Kluwer, Dordrecht, 1991.