ORIGINAL PAPER

THE EIGENVALUES AND THE EIGENFUNCTIONS OF THE STURM-LIOUVILLE FUZZY PROBLEM WITH FUZZY COEFFICIENT BOUNDARY CONDITIONS

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Abstract. *In this paper, the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy problem with fuzzy coefficient boundary conditions are examined under the approach of Hukuhara differentiability. Because of the boundary conditions have fuzzy coefficient, it is found that the eigenvalues depend on α. Therefore, different eigenvalues are obtained for each α on* $[0,1)$.

Keywords: *Fuzzy boundary value problems, Second-order fuzzy differential equations, Hukuhara differentiability.*

1. INTRODUCTION

The study of fuzzy differential equation forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervade. The term “fuzzy differential equation” was introduced in 1987 by Kandel and Byatt [10]. There are many suggestions to define a fuzzy derivative. The first and the most popular approach is using the Hukuhara differentiability for fuzzy-value functions [6, 15]. Hüllermeier interpreted fuzzy differential equation as a family of differential inclusions [9]. Another approach to solve fuzzy differential equation is known as Zadeh’s extension principle [14,16]. The strongly generalized differentiability was introduced in [2] and studied in [3, 4, 5].

In this paper, a investigation is made on the eigenvalues and the eigenfunctions of Sturm-Liouville fuzzy problem with fuzzy coefficient boundary conditions by using Hukuhara differentiability.

As the fuzzy boundary value problem is given as the form

$$Ly = p(x)y'' + q(x)y,\ p'(x) = 0,$$

$$Ly + \lambda y = 0,\ x \in (a,b) \tag{1.1}$$

$$[A]^{\alpha} y(a) = [B]^{\alpha} y'(a) \tag{1.2}$$

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$$[C]^{\alpha}\, y(b)=[D]^{\alpha}\, y'(b) \tag{1.3}$$

the eigenvalues and the eigenfunctions are found, where $\lambda>0$, $[A]^{\alpha}=\left[\underline{A}_{\alpha},\overline{A}_{\alpha}\right]$, $[B]^{\alpha}=\left[\underline{B}_{\alpha},\overline{B}_{\alpha}\right]$, $[C]^{\alpha}=\left[\underline{C}_{\alpha},\overline{C}_{\alpha}\right]$, $[D]^{\alpha}=\left[\underline{D}_{\alpha},\overline{D}_{\alpha}\right]$ are symmetric triangular fuzzy numbers.

2. PRELIMINARIES

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. [11] A fuzzy number is a function $u:\mathbb{R}\rightarrow[0,1]$ satisfying the following properties:

1) u is normal,
2) u is convex fuzzy set,
3) u is upper semi-continuous on $\mathbb{R}$,
4) $cl\{x\epsilon\mathbb{R}|u(x)>0\}$ is compact where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2. [12] Let $u\in\mathbb{R}_F$. The α-level set of u, denoted $[u]^{\alpha}, 0<\alpha\leq 1$, is $[u]^{\alpha}=\{x\epsilon\mathbb{R}|u(x)\geq\alpha\}$. If $\alpha=0$, the support of u is defined $[u]^{0}=cl\{x\epsilon\mathbb{R}|u(x)>0\}$. The notation, $[u]^{\alpha}=[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ is a valid α-level set.

Remark 2.3. [11] The sufficient and necessary conditions for $[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ to define the parametric form of a fuzzy number as follows:

1) $\underline{u}_{\alpha}$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha=0$,

2) $\overline{u}_{\alpha}$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha=0$,

3) $\underline{u}_{\alpha}\leq\overline{u}_{\alpha}$, $0\leq\alpha\leq 1$.

Definition 2.4. [12] For $u,v\,\epsilon\,\mathbb{R}_F$ *and* $\lambda\,\epsilon\,\mathbb{R}$, the sum $u+v$ and the product λu are defined by $[u+v]^{\alpha}=[u]^{\alpha}+[v]^{\alpha}$, $[\lambda u]^{\alpha}=\lambda[u]^{\alpha}$, $\forall\alpha\in[0,1]$, where $[u]^{\alpha}+[v]^{\alpha}$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^{\alpha}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D:\mathbb{R}_F\times\mathbb{R}_F\rightarrow\mathbb{R}_{+}\cup\{0\},$$

by

$$D(u,v)=\sup_{\alpha\in[0,1]}\max\left\{\left|\underline{u}_{\alpha}-\underline{v}_{\alpha}\right|,\left|\overline{u}_{\alpha}-\overline{v}_{\alpha}\right|\right\} \text{ [11].}$$

Definition 2.5. [13] Let $u, v \in \mathbb{R}_F$. If there exist $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted $u \underset{H}{-} v$.

Definition 2.6. [11] Let I=(a,b), for $a, b \in \mathbb{R}$, and $F: I \to \mathbb{R}_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0 \in I$ if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal $F'(t_0)$. Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Theorem 2.7. [7] *Let $f: I \to \mathbb{R}_F$ be a function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$, for each $\alpha \in [0,1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and* $[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)]$.

Definition 2.8. [8] If $p'(x) = 0$, $r(x) = 1$ and

$$Ly = p(x)y'' + q(x)y$$

in the fuzzy differential equation

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0,$$

where $p(x), p'(x), q(x)$, $r(x)$ are continuous functions and are positive on $[a,b]$, the fuzzy differential equation

$$Ly + \lambda y = 0 \tag{2.1}$$

is called a fuzzy Sturm-Liouville equation.

Definition 2.9. [8] $[y(x,\lambda_0)]^\alpha = [\underline{y}(x,\lambda_0), \overline{y}(x,\lambda_0)] \neq 0$, we say that $\lambda = \lambda_0$ is eigenvalue of (2.1) if the fuzzy differential equation (2.1) has the nontrivial solutions $\underline{y}(x,\lambda_0) \neq 0$, $\overline{y}(x,\lambda_0) \neq 0$.

3. THE EIGENVALUES AND THE EIGENFUNCTIONS OF THE STURM-LIOUVILLE FUZZY PROBLEM WITH FUZZY COEFFICIENT BOUNDARY CONDITION

Consider the eigenvalues of the fuzzy boundary value problem (1.1)-(1.3). Let be $[y]^\alpha = [\underline{y}_\alpha, \overline{y}_\alpha]$ the general solution of the fuzzy differential equation (1.1). From the boundary condition (1.2), we have

$$\left[\underline{A}_\alpha,\overline{A}_\alpha\right]\left[\underline{y}_\alpha(a,\lambda),\overline{y}_\alpha(a,\lambda)\right]=\left[\underline{B}_\alpha,\overline{B}_\alpha\right]\left[\underline{y}'_\alpha(a,\lambda),\overline{y}'_\alpha(a,\lambda)\right].$$

Using the fuzzy arithmetic

$$\underline{\phi}_\alpha=\min\left\{\underline{A}_\alpha\underline{y}_\alpha(a,\lambda),\underline{A}_\alpha\overline{y}_\alpha(a,\lambda),\overline{A}_\alpha\underline{y}_\alpha(a,\lambda),\overline{A}_\alpha\overline{y}_\alpha(a,\lambda)\right\},$$
$$\overline{\phi}_\alpha=\max\left\{\underline{A}_\alpha,\underline{y}_\alpha(a,\lambda),\underline{A}_\alpha\overline{y}_\alpha(a,\lambda),\overline{A}_\alpha\underline{y}_\alpha(a,\lambda),\overline{A}_\alpha\overline{y}_\alpha(a,\lambda)\right\},$$
$$\underline{\psi}_\alpha=\min\left\{\underline{B}_\alpha\underline{y}'_\alpha(a,\lambda),\underline{B}_\alpha\overline{y}'_\alpha(a,\lambda),\overline{B}_\alpha\underline{y}'_\alpha(a,\lambda),\overline{B}_\alpha\overline{y}'_\alpha(a,\lambda)\right\},$$
$$\overline{\psi}_\alpha=\max\left\{\underline{B}_\alpha,\underline{y}'_\alpha(a,\lambda),\underline{B}_\alpha\overline{y}'_\alpha(a,\lambda),\overline{B}_\alpha\underline{y}'_\alpha(a,\lambda),\overline{B}_\alpha\overline{y}'_\alpha(a,\lambda)\right\},$$
$$\left[\underline{\phi}_\alpha,\overline{\phi}_\alpha\right]=\left[\underline{\psi}_\alpha,\overline{\psi}_\alpha\right]$$

are obtained. Similarly, from the boundary condition (1.3), we obtained

$$\underline{\varphi}_\alpha=\min\left\{\underline{C}_\alpha\underline{y}_\alpha(b,\lambda),\underline{C}_\alpha\overline{y}_\alpha(b,\lambda),\overline{C}_\alpha\underline{y}_\alpha(b,\lambda),\overline{C}_\alpha\overline{y}_\alpha(b,\lambda)\right\},$$
$$\overline{\varphi}_\alpha=\max\left\{\underline{C}_\alpha,\underline{y}_\alpha(b,\lambda),\underline{C}_\alpha\overline{y}_\alpha(b,\lambda),\overline{C}_\alpha\underline{y}_\alpha(b,\lambda),\overline{C}_\alpha\overline{y}_\alpha(b,\lambda)\right\},$$
$$\underline{\mu}_\alpha=\min\left\{\underline{D}_\alpha\underline{y}'_\alpha(b,\lambda),\underline{D}_\alpha\overline{y}'_\alpha(b,\lambda),\overline{D}_\alpha\underline{y}'_\alpha(b,\lambda),\overline{D}_\alpha\overline{y}'_\alpha(b,\lambda)\right\},$$
$$\overline{\mu}_\alpha=\max\left\{\underline{D}_\alpha,\underline{y}'_\alpha(b,\lambda),\underline{D}_\alpha\overline{y}'_\alpha(b,\lambda),\overline{D}_\alpha\underline{y}'_\alpha(b,\lambda),\overline{D}_\alpha\overline{y}'_\alpha(b,\lambda)\right\},$$
$$\left[\underline{\varphi}_\alpha,\overline{\varphi}_\alpha\right]=\left[\underline{\mu}_\alpha,\overline{\mu}_\alpha\right].$$

Consequently, there is a max-min problem. Therefore, firstly, certain solution of the fuzzy Sturm-Liouville problem (1.1)-(1.3) is found for $\alpha=1$. Let be $y(x,\lambda)$ solution function. Taking

1) If $y'(x,\lambda)\geq 0, y''(x,\lambda)\geq 0$,

$$\left[y(x,\lambda)\right]^\alpha=\left[\underline{y}_\alpha(x,\lambda),\overline{y}_\alpha(x,\lambda)\right],$$
$$\left[y'(x,\lambda)\right]^\alpha=\left[\underline{y}'_\alpha(x,\lambda),\overline{y}'_\alpha(x,\lambda)\right],$$
$$\left[y''(x,\lambda)\right]^\alpha=\left[\underline{y}''_\alpha(x,\lambda),\overline{y}''_\alpha(x,\lambda)\right],$$

2) If $y'(x,\lambda)\geq 0, y''(x,\lambda)\leq 0$,

$$\left[y(x,\lambda)\right]^\alpha=\left[\underline{y}_\alpha(x,\lambda),\overline{y}_\alpha(x,\lambda)\right],$$
$$\left[y'(x,\lambda)\right]^\alpha=\left[\underline{y}'_\alpha(x,\lambda),\overline{y}'_\alpha(x,\lambda)\right],$$
$$\left[y''(x,\lambda)\right]^\alpha=\left[\overline{y}''_\alpha(x,\lambda),\underline{y}''_\alpha(x,\lambda)\right],$$

3) If $y'(x,\lambda)\leq 0, y''(x,\lambda)\geq 0$,

$$\left[y(x,\lambda)\right]^{\alpha}=\left[\underline{y}_{\alpha}(x,\lambda),\overline{y}_{\alpha}(x,\lambda)\right],$$
$$\left[y'(x,\lambda)\right]^{\alpha}=\left[\overline{y}'_{\alpha}(x,\lambda),\underline{y}'_{\alpha}(x,\lambda)\right],$$
$$\left[y''(x,\lambda)\right]^{\alpha}=\left[\underline{y}''_{\alpha}(x,\lambda),\overline{y}''_{\alpha}(x,\lambda)\right],$$

4) If $y'(x,\lambda)\le 0, y''(x,\lambda)\le 0$,

$$\left[y(x,\lambda)\right]^{\alpha}=\left[\underline{y}_{\alpha}(x,\lambda),\overline{y}_{\alpha}(x,\lambda)\right],$$
$$\left[y'(x,\lambda)\right]^{\alpha}=\left[\overline{y}'_{\alpha}(x,\lambda),\underline{y}'_{\alpha}(x,\lambda)\right],$$
$$\left[y''(x,\lambda)\right]^{\alpha}=\left[\overline{y}''_{\alpha}(x,\lambda),\underline{y}''_{\alpha}(x,\lambda)\right],$$

fuzzy solution of investigated problem is found [1].

Example 3.1. Consider the fuzzy Sturm-Liouville problem

$$y''+\lambda y=0,\ y(0)=0,\ [1]^{\alpha}\, y(1)=y'(1), \tag{3.1}$$

where $[1]^{\alpha}=[\alpha,2-\alpha]$ is symmetric triangular fuzzy number.

Firstly, let find certain solution of the fuzzy Sturm-Liouville problem (3.1) for $\alpha=1$. The fuzzy Sturm-Liouville problem (3.1) transform the Sturm-Liouville problem

$$y''+\lambda y=0,\ \ y(0)=0,\ y(1)=y'(1) \tag{3.2}$$

for $\alpha=1$. We write $\lambda=k^2$, k>0. The solution of Sturm-Liouville diferential equation $y''+\lambda y=0$ is

$$y(x,\lambda)=c_1\cos(kx)+c_2\sin(kx).$$

From boundary conditions, we have

$$y(0,\lambda)=c_1=0,$$
$$y(1,\lambda)=c_2\sin(k)=kc_2\cos(k)=y'(1,\lambda),$$
$$\Rightarrow c_2(\sin(k)-k\cos(k))=0,$$
$$\Rightarrow c_2\ne 0,\ \sin(k)-k\cos(k)=0.$$

Computing the values k satisfying last equation, we have

$$k_0=4.493409458,\ k_1=7.725251837,\ k_2=10.90412166,\ k_3=14.06619391,\ldots.$$

We show that this values are k_n, n=0,1,...then,

$$y_n(x)=\sin(k_n x)$$

is obtained. According to the this, yields $y'_n(x) = k_n \cos(k_n x)$, $y''_n(x) = -k_n^2 \sin(k_n x)$.

Consequently,

i) $k_n x \in \left[0, \frac{\pi}{2}\right] \Rightarrow y' \geq 0,\ y'' \leq 0$,

ii) $k_n x \in \left[\frac{\pi}{2}, \pi\right] \Rightarrow y' \leq 0,\ y'' \leq 0$,

iii) $k_n x \in \left[\pi, \frac{3\pi}{2}\right] \Rightarrow y' \leq 0,\ y'' \geq 0$,

iv) $k_n x \in \left[\frac{3\pi}{2}, 2\pi\right] \Rightarrow y' \geq 0,\ y'' \geq 0$.

Now, let solve the fuzzy Sturm-Liouville problem (3.1).

i) Let be $k_n x \in \left[0, \frac{\pi}{2}\right]$. Since $y' \geq 0,\ y'' \leq 0$, the fuzzy Sturm-Liouville problem which is investigated is

$$\left[\overline{y}''_\alpha(x,\lambda), \underline{y}''_\alpha(x,\lambda)\right] + \lambda\left[\underline{y}_\alpha(x,\lambda), \overline{y}_\alpha(x,\lambda)\right] = 0 \qquad (3.3)$$

$$\left[\underline{y}_\alpha(0,\lambda), \overline{y}_\alpha(0,\lambda)\right] = 0\ ,\ [\alpha, 2-\alpha]\left[\underline{y}_\alpha(1,\lambda), \overline{y}_\alpha(1,\lambda)\right] = \left[\underline{y}'_\alpha(1,\lambda), \overline{y}'_\alpha(1,\lambda)\right].$$

Using the fuzzy arithmetic, the fuzzy differential equation (3.3) is transformed into a linear system of real-valued differential equations

$$\begin{cases} \underline{y}''_\alpha(x,\lambda) + k^2 \overline{y}_\alpha(x,\lambda) = 0 \\ \overline{y}''_\alpha(x,\lambda) + k^2 \underline{y}_\alpha(x,\lambda) = 0 \end{cases},$$

where $\lambda = k^2,\ k>0$. Solving this linear system, the solution of the fuzzy differential equation (3.3) is obtained as

$$\left[y(x,\lambda)\right]^\alpha = \left[\underline{y}_\alpha(x,\lambda), \overline{y}_\alpha(x,\lambda)\right], \qquad (3.4)$$

where

$$\underline{y}_\alpha(x,\lambda) = -c_1(\alpha)\cosh(kx) - c_2(\alpha)\sinh(kx) + c_3(\alpha)\cos(kx) + c_4(\alpha)\sin(kx), \qquad (3.5)$$

$$\overline{y}_\alpha(x,\lambda) = c_1(\alpha)\cosh(kx) + c_2(\alpha)\sinh(kx) + c_3(\alpha)\cos(kx) + c_4(\alpha)\sin(kx). \qquad (3.6)$$

From the boundary condition

$$\left[\underline{y}_\alpha(0,\lambda), \overline{y}_\alpha(0,\lambda)\right] = 0,$$

we have $c_1(\alpha) = c_3(\alpha) = 0$.

From the boundary condition

$$[\alpha, 2-\alpha]\left[\underline{y}_\alpha(1,\lambda), \overline{y}_\alpha(1,\lambda)\right] = \left[\underline{y}'_\alpha(1,\lambda), \overline{y}'_\alpha(1,\lambda)\right],$$

we get

$$\alpha\left(-c_2(\alpha)\sinh(k) + c_4(\alpha)\sin(k)\right) = -kc_2(\alpha)\cosh(k) + kc_4(\alpha)\cos(k)$$
$$\Rightarrow c_2(\alpha)\left(-\alpha\sinh(k) + k\cosh(k)\right) + c_4(\alpha)\left(\alpha\sin(k) - k\cos(k)\right) = 0, \tag{3.7}$$

$$(2-\alpha)\left(c_2(\alpha)\sinh(k) + c_4(\alpha)\sin(k)\right) = kc_2(\alpha)\cosh(k) + kc_4(\alpha)\cos(k)$$
$$\Rightarrow c_2(\alpha)\left((2-\alpha)\sinh(k) - k\cosh(k)\right) + c_4(\alpha)\left((2-\alpha)\sin(k) - k\cos(k)\right) = 0. \tag{3.8}$$

As the determinant of coefficients matrix

$$W(\alpha,k) = \begin{vmatrix} -\alpha\sinh(k) + k\cosh(k) & \alpha\sin(k) - k\cos(k) \\ (2-\alpha)\sinh(k) - k\cosh(k) & (2-\alpha)\sin(k) - k\cos(k) \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow W(\alpha,k) = {} & \cosh(k)\cos(k)k^2 - \left(\sinh(k)\cos(k) + \cosh(k)\sin(k)\right)k \\ & + \alpha(2-\alpha)\sinh(k)\sin(k) = 0\end{aligned} \tag{3.9}$$

the homogeneous equation system (3.7)-(3.8) has non-zero solution. Computing the values k satisfying the equation (3.9) for each $\alpha \in [0,1)$, we get infinitely many values as

$$\alpha = 0 \Rightarrow k_{0,0} = 4.428621078,\ k_{0,1} = 7.705951193,\ k_{0,2} = 10.89485363, ...$$
$$\alpha = \frac{1}{2} \Rightarrow k_{\frac{1}{2},0} = 4.477416681,\ k_{\frac{1}{2},1} = 7.720436505,\ k_{\frac{1}{2},2} = 10.90180622, ...$$
$$\alpha = \frac{3}{4} \Rightarrow k_{\frac{3}{4},0} = 4.489424654,\ k_{\frac{3}{4},1} = 7.724048642,\ k_{\frac{3}{4},2} = 10.90354290, ...$$

....

We show that this values are $k_{\alpha,n}$, for each $\alpha \in [0,1)$. Then, we obtain

$$[y_n(x)]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right], \tag{3.10}$$

where

$$\underline{y}_{n\alpha}(x) = -c_2(\alpha)\sinh(k_{\alpha,n}x) + c_4(\alpha)\sin(k_{\alpha,n}x), \tag{3.11}$$

$$\overline{y}_{n\alpha}(x) = c_2(\alpha)\sinh(k_{\alpha,n}x) + c_4(\alpha)\sin(k_{\alpha,n}x), \tag{3.12}$$

$$c_4(\alpha) = \frac{\alpha\sinh(k_{\alpha,n}) - k_{\alpha,n}\cosh(k_{\alpha,n})}{\alpha\sin(k_{\alpha,n}) - k_{\alpha,n}\cos(k_{\alpha,n})} c_2(\alpha).$$

As

$$\frac{\partial \underline{y}_{n\alpha}(x)}{\partial\alpha} \geq 0,\ \frac{\partial \overline{y}_{n\alpha}(x)}{\partial\alpha} \leq 0 \text{ and } \underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x) \tag{3.13}$$

for each $\alpha\in[0,1)$, $\left[y_n(x)\right]^\alpha$ is valid $\alpha-$level set. Consequently, for $\underline{y}_{n\alpha}(x)$, $\overline{y}_{n\alpha}(x)$ satisfying the inequalities (3.13), the eigenvalues are $\lambda_{\alpha,n}=\left(k_{\alpha,n}\right)^2$, with associated eigenfunctions (3.10)-(3.12).

ii) Let be $k_n x\in\left[\frac{\pi}{2},\pi\right]$. Since $y'\le 0,\ y''\le 0$, the fuzzy Sturm-Liouville problem which is investigated is transformed into the fuzzy differential equation (3.3) and the boundary conditions

$$\left[\underline{y}_\alpha(0,\lambda),\overline{y}_\alpha(0,\lambda)\right]=0,\ [\alpha,2-\alpha]\left[\underline{y}_\alpha(1,\lambda),\overline{y}_\alpha(1,\lambda)\right]=\left[\overline{y}'_\alpha(1,\lambda),\underline{y}'_\alpha(1,\lambda)\right].$$

The solution of the fuzzy differential equation (3.3) is (3.4)-(3.6). From the first boundary condition $c_1(\alpha)=c_3(\alpha)=0$ is obtained. From the second boundary condition

$$c_2(\alpha)\left(-\alpha\sinh(k)-k\cosh(k)\right)+c_4(\alpha)\left(\alpha\sin(k)-k\cos(k)\right)=0, \tag{3.14}$$

$$c_2(\alpha)\left(\left(2-\alpha\right)\sinh(k)+k\cosh(k)\right)+c_4(\alpha)\left(\left(2-\alpha\right)\sin(k)-k\cos(k)\right)=0 \tag{3.15}$$

are obtained. As the determinant of coefficients matrix

$$\begin{aligned}\phi(\alpha,k)=&\cosh(k)\cos(k)k^2+\left(\sinh(k)\cos(k)+\cosh(k)\sin(k)\right)k\\ &-\alpha(2-\alpha)\sinh(k)\sin(k)=0\end{aligned} \tag{3.16}$$

the homogeneous equation system (3.14)-(3.15) has non-zero solution. Computing the values k satisfying the equation (3.16) for each $\alpha\in[0,1)$, we get infinitely many values as

$$\alpha=0\Rightarrow k_{0,0}=1.080631911,\ k_{0,1}=4.533596965,\ k_{0,2}=7.740061341,...$$

$$\alpha=\frac{1}{2}\Rightarrow k_{\frac{1}{2},0}=0.5887825541,\ k_{\frac{1}{2},1}=4.503516432,\ k_{\frac{1}{2},2}=7.728959180,...$$

$$\alpha=\frac{3}{4}\Rightarrow k_{\frac{3}{4},0}=0.3029512695,\ k_{\frac{3}{4},1}=4.495939762,\ k_{\frac{3}{4},2}=7.726178972,...$$

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We show that this values are $k_{\alpha,n}$, for each $\alpha\in[0,1)$. Then, we obtain

$$\left[y_n(x)\right]^\alpha=\left[\underline{y}_{n\alpha}(x),\overline{y}_{n\alpha}(x)\right], \tag{3.17}$$

where

$$\underline{y}_{n\alpha}(x)=-c_2(\alpha)\sinh(k_{\alpha,n}x)+c_4(\alpha)\sin(k_{\alpha,n}x), \tag{3.18}$$

$$\overline{y}_{n\alpha}(x)=c_2(\alpha)\sinh(k_{\alpha,n}x)+c_4(\alpha)\sin(k_{\alpha,n}x), \tag{3.19}$$

$$c_4(\alpha)=\frac{\alpha\sinh(k_{\alpha,n})+k_{\alpha,n}\cosh(k_{\alpha,n})}{\alpha\sin(k_{\alpha,n})-k_{\alpha,n}\cos(k_{\alpha,n})}c_2(\alpha).$$

As the inequalities (3.13) is satisfied for each $\alpha\in[0,1)$, $\left[y_n(x)\right]^\alpha$ is valid $\alpha-$level set. Consequently, for $\underline{y}_{n\alpha}(x)$, $\overline{y}_{n\alpha}(x)$ satisfying the inequalities (3.13), the eigenvalues are $\lambda_{\alpha,n}=\left(k_{\alpha,n}\right)^2$, with associated eigenfunctions (3.17)-(3.19).

iii) Let be $k_n x\in\left[\pi,\frac{3\pi}{2}\right]$. Since $y'\le 0,\ y''\ge 0$, the fuzzy Sturm-Liouville problem which is investigated is

$$\left[\underline{y}''_\alpha(x,\lambda),\overline{y}''_\alpha(x,\lambda)\right]+\lambda\left[\underline{y}_\alpha(x,\lambda),\overline{y}_\alpha(x,\lambda)\right]=0, \tag{3.20}$$
$$\left[\underline{y}_\alpha(0,\lambda),\overline{y}_\alpha(0,\lambda)\right]=0\ ,\ [\alpha,2-\alpha]\left[\underline{y}_\alpha(1,\lambda),\overline{y}_\alpha(1,\lambda)\right]=\left[\overline{y}'_\alpha(1,\lambda),\underline{y}'_\alpha(1,\lambda)\right].$$

From the fuzzy differential equation (3.20), the fuzzy differential equations

$$\underline{y}''_\alpha(x,\lambda)+k^2\underline{y}_\alpha(x,\lambda)=0,\ \overline{y}''_\alpha(x,\lambda)+k^2\overline{y}_\alpha(x,\lambda)=0$$

are obtained, where $\lambda=k^2$, k>0. Solving these differential equations, the solution of the fuzzy differential equation (3.20) is

$$\left[y(x,\lambda)\right]^\alpha=\left[\underline{y}_\alpha(x,\lambda),\overline{y}_\alpha(x,\lambda)\right], \tag{3.21}$$

where

$$\underline{y}_\alpha(x,\lambda)=c_1(\alpha)\cos(kx)+c_2(\alpha)\sin(kx), \tag{3.22}$$

$$\overline{y}_\alpha(x,\lambda)=c_3(\alpha)\cos(kx)+c_4(\alpha)\sin(kx). \tag{3.23}$$

From the first boundary condition, we have

$$\underline{y}_\alpha(0,\lambda)=c_1(\alpha)=0,\ \overline{y}_\alpha(0,\lambda)=c_3(\alpha)=0$$

From the second boundary condition,

$$\alpha\underline{y}_\alpha(1,\lambda)=\alpha\left(c_2(\alpha)\sin(k)\right)=kc_4(\alpha)\cos(k)=\overline{y}'_\alpha(1,\lambda)$$
$$\Rightarrow\alpha\sin(k)c_2(\alpha)-k\cos(k)c_4(\alpha)=0 \tag{3.24}$$

$$(2-\alpha)\overline{y}_\alpha(1,\lambda)=(2-\alpha)\left(c_4(\alpha)\sin(k)\right)=kc_2(\alpha)\cos(k)=\underline{y}'_\alpha(1,\lambda)$$
$$\Rightarrow -k\cos(k)c_2(\alpha)+(2-\alpha)\sin(k)c_4(\alpha)=0 \tag{3.25}$$

are obtained. Then, the determinant of coefficients matrix of the homogeneous equation system (3.24)-(3.25) must be

$$\psi(\alpha,k)=\begin{vmatrix}\alpha\sin(k) & -k\cos(k)\\ -k\cos(k) & (2-\alpha)\sin(k)\end{vmatrix}=0$$

$$\Rightarrow \psi(\alpha,k)=\alpha(2-\alpha)\sin^2(k)-k^2\cos(k)=0 \tag{3.26}$$

Computing the values k satisfying the equation (3.26) for each $\alpha\in[0,1)$, we get infinitely many values as

$$\alpha=\frac{1}{2}\Rightarrow k_{\frac{1}{2},0}=0.6255550451,\ k_{\frac{1}{2},1}=1.982625293,\ k_{\frac{1}{2},2}=4.523216051,...$$

$$\alpha=\frac{3}{4}\Rightarrow k_{\frac{3}{4},0}=0.3076681487,\ k_{\frac{3}{4},1}=2.018132026\ ,\ k_{\frac{3}{4},2}=4.500476165,...$$

$$\alpha=\frac{9}{10}\Rightarrow k_{\frac{9}{10},0}=0.1225668056,\ k_{\frac{9}{10},1}=2.027092783\ ,\ k_{\frac{9}{10},2}=4.494524994,\dots$$

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We show that this values are $k_{\alpha,n}$, for each $\alpha\in[0,1)$. Then, we get

$$[y_n(x)]^\alpha=\left[\underline{y}_{n\alpha}(x),\overline{y}_{n\alpha}(x)\right], \tag{3.27}$$

where

$$\underline{y}_{n\alpha}(x)=c_2(\alpha)\sin(k_{\alpha,n}x), \tag{3.28}$$

$$\overline{y}_{n\alpha}(x)=c_4(\alpha)\sin(k_{\alpha,n}x), \tag{3.29}$$

$$c_4(\alpha)=\frac{\alpha\sin(k_{\alpha,n})}{k_{\alpha,n}\cos(k_{\alpha,n})}c_2(\alpha).$$

Consequently, for $\underline{y}_{n\alpha}(x)$, $\overline{y}_{n\alpha}(x)$ satisfying the inequalities (3.13), the eigenvalues are $\lambda_{\alpha,n}=\left(k_{\alpha,n}\right)^2$, with associated eigenfunctions (3.27)-(3.29).

iv) Let be $k_n x\in\left[\frac{3\pi}{2},2\pi\right]$. Since $y'\geq 0,\ y''\geq 0$, the fuzzy Sturm-Liouville problem which is investigated is transformed into the fuzzy differential equation (3.20) and the boundary conditions

$$\left[\underline{y}_\alpha(0,\lambda),\overline{y}_\alpha(0,\lambda)\right]=0,\ [\alpha,2-\alpha]\left[\underline{y}_\alpha(1,\lambda),\overline{y}_\alpha(1,\lambda)\right]=\left[\underline{y}'_\alpha(1,\lambda),\overline{y}'_\alpha(1,\lambda)\right].$$

The solution of the fuzzy differential equation (3.20) is (3.21)-(3.23). From the first boundary condition $c_1(\alpha)=c_3(\alpha)=0$ is obtained. From the second boundary condition, we get

$$\alpha \underline{y}_\alpha(1,\lambda) = \alpha\left(c_2(\alpha)\sin(k)\right) = kc_2(\alpha)\cos(k) = \underline{y}'_\alpha(1,\lambda)$$
$$\Rightarrow c_2(\alpha)\left(\alpha\sin(k) - k\cos(k)\right) = 0,$$
$$(2-\alpha)\overline{y}_\alpha(1,\lambda) = (2-\alpha)\left(c_4(\alpha)\sin(k)\right) = kc_4(\alpha)\cos(k) = \overline{y}'_\alpha(1,\lambda)$$
$$\Rightarrow c_4(\alpha)\left((2-\alpha)\sin(k) - k\cos(k)\right) = 0.$$

Let be $c_2(\alpha) \neq 0$, $c_4(\alpha) \neq 0$. Then, the equations

$$\alpha\sin(k) - k\cos(k) = 0 \tag{3.30}$$

$$(2-\alpha)\sin(k) - k\cos(k) = 0 \tag{3.31}$$

are obtained. Computing the values k satisfying the equation (3.30) for each $\alpha \in [0,1)$, we get infinitely many values as

$$\alpha = \frac{1}{2} \Rightarrow k_{\frac{1}{2},0} = 1.165561185,\ k_{\frac{1}{2},1} = 4.604216777,\ k_{\frac{1}{2},2} = 7.789883751,...$$
$$\alpha = \frac{3}{4} \Rightarrow k_{\frac{3}{4},0} = 0.8447308435,\ k_{\frac{3}{4},1} = 4.548987100,\ k_{\frac{3}{4},2} = 7.757601811,...$$
$$\alpha = \frac{9}{10} \Rightarrow k_{\frac{9}{10},0} = 0.5422808854,\ k_{\frac{9}{10},1} = 4.515660438,\ k_{\frac{9}{10},2} = 7.738195665,\dots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and computing the values k satisfying the equation (3.31) for each $\alpha \in [0,1)$, we get infinitely many values as

$$\alpha = \frac{1}{2} \Rightarrow k_{\frac{1}{2},0} = 4.382625527,\ k_{\frac{1}{2},1} = 7.660621452,\ k_{\frac{1}{2},2} = 10.85829994,...$$
$$\alpha = \frac{3}{4} \Rightarrow k_{\frac{3}{4},0} = 4.437833300,\ k_{\frac{3}{4},1} = 7.692901959,\ k_{\frac{3}{4},2} = 10.88119861,...$$
$$\alpha = \frac{9}{10} \Rightarrow k_{\frac{9}{10},0} = 4.471158516,\ k_{\frac{9}{10},1} = 7.712308011,\ k_{\frac{9}{10},2} = 10.89495107,\dots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We show that the values satisfying the equation (3.30) are $k_{\alpha,n}$ and the values satisfying the equation (3.31) are $k_{\alpha,m}$ for each $\alpha \in [0,1)$. Then,

$$\left[y_{n,m}(x)\right]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{m\alpha}(x)\right] \tag{3.32}$$

is obtained, where

$$\underline{y}_{n\alpha}(x) = c_2(\alpha)\sin(k_{\alpha,n}x), \tag{3.33}$$

$$\overline{y}_{m\alpha}(x) = c_4(\alpha)\sin(k_{\alpha,m}x). \tag{3.34}$$

As

$$\frac{\partial \underline{y}_{n\alpha}(x)}{\partial \alpha} \geq 0 \;\; , \;\; \frac{\partial \overline{y}_{m\alpha}(x)}{\partial \alpha} \leq 0 \text{ and } \underline{y}_{n\alpha}(x) \leq \overline{y}_{m\alpha}(x) \tag{3.35}$$

for each $\alpha \in [0,1)$, $\left[y_{n,m}(x)\right]^{\alpha}$ is valid $\alpha-$level set. Consequently, for $\underline{y}_{n\alpha}(x)$ satisfying the inequalities (3.35), the eigenvalues are $\lambda_{\alpha,n} = \left(k_{\alpha,n}\right)^2$, with associated eigenfunctions (3.32)-(3.34) and for $\overline{y}_{m\alpha}(x)$ satisfying the inequalities (3.35), the eigenvalues are $\lambda_{\alpha,m} = \left(k_{\alpha,m}\right)^2$, with associated eigenfunctions (3.32)-(3.34).

4. CONCLUSIONS

We investigate the eigenvalues and the eigenfunctions of Sturm-Liouville fuzzy problem with fuzzy coefficient boundary conditions under the approach of Hukuhara differentiability. Because of the boundary conditions have fuzzy coefficient, it is found that the eigenvalues depend on α. Therefore, different eigenvalues are obtained for each α on $[0,1)$.

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