ORIGINAL PAPER

ON SOME FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS USING DIFFERENTIAL TRANSFORM METHOD

KUBRA EHSAN[1], JAMSHAD AHMAD[1]

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Abstract. *In this paper, we have given the procedure and implement the differential transform method (DTM) to solve some linear fractional differential equations under fuzzy initial value conditions. The derivative is in the sense of the Caputo fractional derivative. The solutions of considered model examples are computed in the form of convergent series with easily computable components. To display the strength and applicability of this proposed method few illustrative examples are solved and the obtained results show that the said approach is easy to implement.*

Keywords: *Caputo Fractional Differentiation, Fuzzy Fractional Differential Equation, Differential Transform Method*

1. INTRODUCTION

A powerful tool to deal uncertainty and processing subjective or vague status in mathematics is fuzzy set theory. This has been dealt with many real situations, for example, medicine [1], the golden mean [2], systems of practical [3], quantum optics and gravity [4], and engineering phenomenon. First Zadeh [5] introduced fuzzy sets. Then the fuzzy number was defined and its use in fuzzy control [6] and reasoning approximate problems [7-10]. Later Dubois and Prade [11-12], Mizumoto and Tanaka [13-14], Nahmias [15], and Ralescu [16] developed the basic arithmetics for fuzzy numbers. They considered the fuzzy number in the form of a collection of intervals i-e, δ-levels, $0 \leq \delta \leq 1$ [17]. Real life situations with differential equations of fractional type are of much importance, because fractional differential equations sum the complete information about the situation in weighted form. Applications can be seen in chemistry, physics, engineering, etc. So, we require a method to solve these equations, adequately, easy to apply and used in different problems. For the solution of fractional differential equations of fuzzy kind, we will apply the differential transform method. This method was first used in the engineering field in [18]. The solution of electric circuit problems was also found using this method. Now days, the significance of differential transform method is enlarged to approximate analytical solutions of fractional order differential equations [19-20], and fractional order integro-differential equations [21-23]. The purpose of this paper is to describe and apply differential transform method to linear fractional differential equations with fuzzy initial conditions.

[1] University of Gujrat, Faculty of Science, Department of Mathematics, Pakistan.
E-mail: jamshadahmadm@gmail.com.

2. BASIC CONCEPTS

Definition 1: Naming universal set as *U*. A set '*A' is fuzzy* in *U* defined with the membership function $A(x)$ that relate every element in *U* to the interval $[0, 1]$.

Definition 2: A fuzzy set '*A'* is called a fuzzy number if

(i) for at least one $x_0 \in \mathbb{R}$, $A(x_0) = 1$;

(ii) for all $x_1, x_2 \in \mathbb{R}$, $0 \leq \lambda \leq 1$, it holds that

$min\,(A(x_1), A(x_2)) \leq A(\lambda x_1 + (1-\lambda)x_2)$;

(iii) for any $x_0 \in \mathbb{R}$, it always holds that

$$A(x_0) \geq \lim_{x \to x_0^{\pm}} A(x);$$

(iv) $[A]^0 = \overline{\{x \in \mathbb{R} \mid A(x) > 0\}}$ is a compact set contained in $\mathbb{R}$.

Definition 3: [24] Riemann-Liouville fractional integration

$$I_{y_0}^{q}\, g(y) = \frac{1}{\Gamma(q)} \int_{y_0}^{y} (y-r)^{q-1}\, g(r) dr,\ q, y > 0 \tag{1}$$

and for q = 0,

$$I_y^0\, g(y) = g(y) \tag{2}$$

and Γ(q) is the Gamma function defined below

$$\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} dt. \tag{3}$$

Definition 4: [25] Caputo fractional derivative is

$${}_0^C D_y^q g = \frac{1}{\Gamma(m-q)} \int_0^y (y-r)^{m-q-1} g(r) dr,\ 0 < r, q \text{ and } m = [q] + 1. \tag{4}$$

3. METHOD PROCEDURE

The differential transform method based on the expansion of Taylor series. This gives a polynomial answer through an iterative process. Let $v(y)$ is continuously differentiable function, then $v(y)$ can be expressed in Taylor's series as

$$v(y) = \sum_{t=0}^{\infty} \frac{1}{t!} \frac{d^t v(y_0)}{dy^t} (y - y_0)^k \tag{5}$$

We define differential transform of $v(y)$ of order t, denoted by $V(t)$ as

$$V(t) = \frac{1}{t!} \frac{d^t v(y)}{dy^t}\bigg|_{y=y_0} \tag{6}$$

Form Eq. (5) and Eq. (6), we get

$$v(y) = \sum_{t=0}^{\infty} V(t)(y - y_0)^t. \tag{7}$$

4. FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

Let the fractional power series of the analytical function v is

$$v(y) = \sum_{t=0}^{\infty} V(t)(y - y_0)^{t/\alpha}, \tag{8}$$

where α is order and $V(t)$ is differential transform of fractional type of v. To bypass the initial and boundary conditions of fractional form, we define the Caputo fractional derivative as

$$D_{*y_0}^{q} v(y) = \frac{1}{\Gamma(\mathrm{m-q})} \frac{d^m}{dy^m} \int_{y_0}^{y} \frac{f(r) - \sum_{t=0}^{m-1} \frac{1}{t!}(y-y_0)^t v^t(y_0)}{(y-r)^{1+q-m}} dr. \tag{9}$$

The transformations of the initial conditions for $t = 0, 1, \dots, (\delta q - 1)$, can be obtained as

$$V(y) = \begin{cases} 0 & , \frac{t}{\alpha} \notin \mathbb{Z}^+ \\ \frac{1}{(t/\delta)!} [\frac{d^{t/\alpha}}{dy^{t/\alpha}} v(y)]|_{y=y_0}, & \frac{t}{\alpha} \in \mathbb{Z}^+ \end{cases} \tag{10}$$

with q is order of fractional DE.

5. NUMERICAL EXAMPLES

In this subdivision, the solution of fuzzy fractional differential equations with initial conditions is determined under Caputo fractional derivative using differential transform method.

Example 5.1 Consider the following linear fractional differential equation with fuzzy initial conditions

$${}_0^C D_0^q \tilde{x}(t) = \tilde{x}(t), t\epsilon[0, 1] \tag{11}$$

subject to the condition

$$\tilde{x}(0) = (0.75 + 0.25\mathrm{\alpha}, 1.125 - 0.125\mathrm{\alpha}), 0 < \alpha \leq 1 \tag{12}$$

where $q\epsilon(0, 1), t > 0$.

The solution will be of the form$[\underline{x}, \overline{x}]$.

To find $\underline{x}$, we take

$$ {}_0^CD_0^q \underline{x}(t) = \underline{x}(t), t\epsilon[0,1] \quad (13)$$

with the initial condition

$$\underline{x}(0) = 0.75 + 0.25\alpha. \quad (14)$$

By the application of the differential transform method to Eq. (13), we have

$$\underline{X}(k+1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\underline{X}(k). \quad (15)$$

and Eq. (14) as

$$\underline{X}(0) = 0.75 + 0.25\alpha \quad (16)$$

By using (15) and (16), we get

$$\underline{X}(1) = \frac{0.75+0.25\alpha}{\Gamma(\frac{3}{2})}, \underline{X}(2) = 0.75 + 0.25\alpha, \underline{X}(3) = \frac{0.75+0.25\alpha}{\Gamma(\frac{5}{2})}, \underline{X}(4) = \frac{0.75+0.25\alpha}{2},$$
$$\underline{X}(5) = \frac{0.75+0.25\alpha}{\Gamma(\frac{7}{2})} \quad (17)$$

By using Eq. (8) up to five terms, we have

$$\underline{x}(t) = (0.75 + 0.25\alpha)\left\{1 + \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + t + \frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + \frac{t^2}{2} + \frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}\right\}. \quad (18)$$

Now similarly, to find $\overline{x}(t)$, consider

$$ {}_0^CD_0^q \overline{x}(t) = \overline{x}(t), \ t\epsilon[0,1] \quad (19)$$

with the initial condition

$$\overline{x}(0) = 1.125 - 0.125\alpha. \quad (20)$$

By the application of the differential transform method to Eq. (19) and Eq. (20), we have

$$\overline{X}(k+1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\overline{X}(k). \quad (21)$$

and

$$\overline{X}(0) = 1.125 - 0.125\alpha. \quad (22)$$

By using (21) and (22), we get

$$\overline{X}(1) = \frac{1.125-0.125\alpha}{\Gamma(\frac{3}{2})}, \overline{X}(2) = 1.125 - 0.125\alpha, \overline{X}(3) = \frac{1.125-0.125\alpha}{\Gamma(\frac{5}{2})},$$

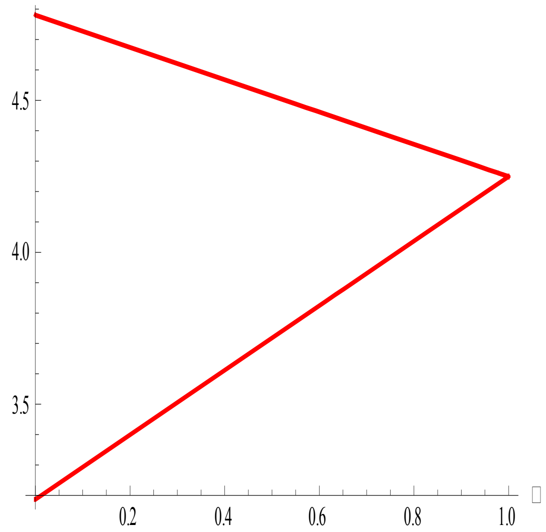
$$\overline{X}(4) = \frac{1.125-0.125\alpha}{2}, \overline{X}(5) = \frac{1.125-0.125\alpha}{\Gamma(\frac{7}{2})}, \tag{23}$$

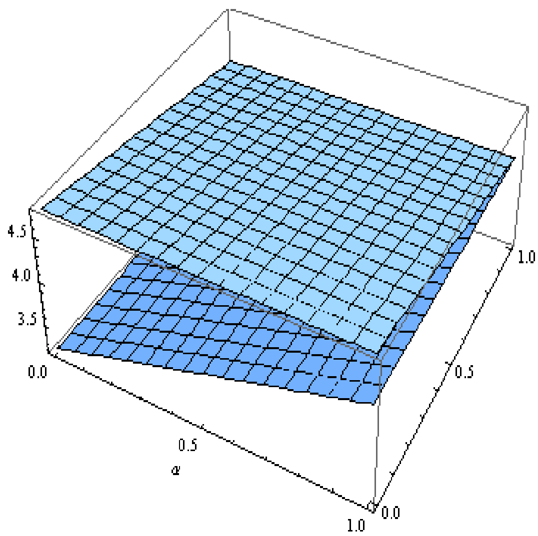
and so on.

By using (8) upto five terms, we have

$$\overline{x}(t) = (1.125 - 0.125\alpha)\left\{1 + \frac{t^{1/2}}{\Gamma(\frac{3}{2})} + t + \frac{t^{3/2}}{\Gamma(\frac{5}{2})} + \frac{t^2}{2} + \frac{t^{5/2}}{\Gamma(\frac{7}{2})}\right\}. \tag{24}$$

Hence, the required solution is given by Eq. (18) and Eq. (24).





(a) **(b)**

Figure 1. (a) 2D and (b) 3D plot of fuzzy solution of Example 5.1.

Example 5.2 Consider the following linear fractional differential equation

$${}_0^C D_0^q \tilde{x}(t) = -\tilde{x}(t), t\epsilon[0,1] \tag{25}$$

subject to the condition

$$\tilde{x}(0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha), 0 < \alpha \leq 1 \tag{26}$$

where $q\epsilon(0,1)$, $t > 0$.

The solution will be of the form $[\underline{x}, \overline{x}]$.

To find $\underline{x}$, we take

$${}_0^C D_0^q \underline{x}(t) = -\underline{x}(t), t\epsilon[0,1] \tag{27}$$

with the initial condition

$$\underline{x}(0) = 0.75 + 0.25\alpha. \tag{28}$$

According to described above DTM procedure, we get

$$\underline{X}(k+1)=\frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left[-\underline{X}(k)\right], \tag{29}$$

and

$$\underline{X}(0)=0.75+0.25\alpha. \tag{30}$$

By using (29) and (30), we have

$$\underline{X}(1)=\frac{-[0.75+0.25\alpha]}{\Gamma(\frac{3}{2})}, \underline{X}(2)=0.75+0.25\alpha, \underline{X}(3)=\frac{-[0.75+0.25\alpha]}{\Gamma(\frac{5}{2})},$$
$$\underline{X}(4)=\frac{0.75+0.25\alpha}{2}, \underline{X}(5)=\frac{-[0.75+0.25\alpha]}{\Gamma(\frac{7}{2})}, \tag{31}$$

and so on.
By using Eq. (8) up to five terms, we have

$$\underline{x}(t)=(0.75+0.25\alpha)\left\{1-\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+t-\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{t^2}{2}-\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}\right\}. \tag{32}$$

Now similarly, to find $\overline{x}(t)$, the solving problem must be solved

$${}_0^CD_0^q\overline{x}(t)=-\overline{x}(t),\ \ t\epsilon[0,1] \tag{33}$$

with the initial condition

$$\overline{x}(0)=1.125-0.125\alpha, \tag{34}$$

According to described above DTM procedure, we have

$$\overline{X}(k+1)=\frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left[-\overline{X}(k)\right], \tag{35}$$

and

$$\overline{X}(0)=1.125-0.125\alpha. \tag{36}$$

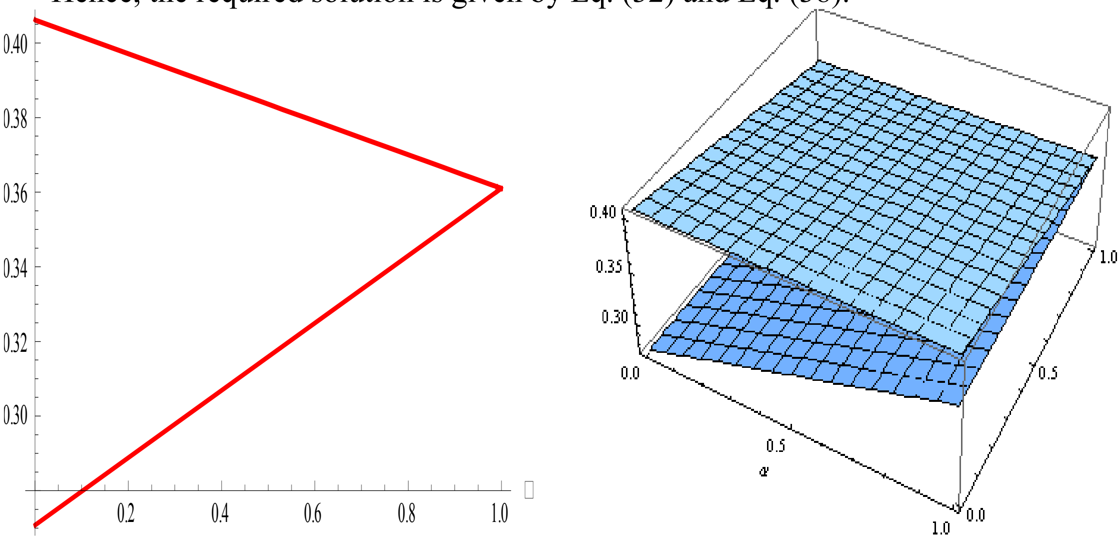
By using Eq. (35) and Eq. (36), we get

$$\overline{X}(1)=\frac{-[1.125-0.125\alpha]}{\Gamma(\frac{3}{2})}, \overline{X}(2)=1.125-0.125\alpha, \overline{X}(3)=\frac{-[1.125-0.125\alpha]}{\Gamma(\frac{5}{2})},$$
$$\overline{X}(4)=\frac{1.125-0.125\alpha}{2}, \overline{X}(5)=\frac{-[1.125-0.125\alpha]}{\Gamma(\frac{7}{2})}, \tag{37}$$

and so on.
By using Eq. (8) up to five terms, we have

$$\overline{x}(t)=(1.125-0.125\alpha)\left\{1-\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+t-\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+\frac{t^2}{2}-\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}\right\}. \tag{38}$$

Hence, the required solution is given by Eq. (32) and Eq. (38).



(a) (b)

Figure 2. (a) 2D and (b) 3D plot of fuzzy solution of Example 5.2.

Example 5.3 Consider the following linear fractional differential equation

$$ {}_0^C D_0^{\alpha}\tilde{\theta}(t) = \xi\tilde{\theta}(t),\ t\epsilon[0,1] \quad (39)$$

Subject to the fuzzy initial conditions

$$\tilde{\theta}(0) = (1+\mathrm{r}, 3-\mathrm{r}),\ 0 < r \le 1,\ \xi\epsilon\mathbb{R} \quad (40)$$

where $\alpha\epsilon(0,1)$, $t > 0$.

Suppose $\xi = 1$. The solution will be of the form $[\underline{\theta}, \overline{\theta}]$.

To get $\underline{\theta}$, we take the problem

$$ {}_0^C D_0^{q}\underline{\theta}(t) = \underline{\theta}(t),\ t\epsilon[0,1] \quad (41)$$

with the condition

$$\underline{\theta}(0) = 1+\mathrm{r}. \quad (42)$$

According to described above DTM procedure, we have

$$\underline{\Theta}(k+1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}[\underline{\Theta}(k)] \quad (43)$$

with the initial condition

$$\underline{\Theta}(0) = 1+\mathrm{r} \quad (44)$$

By using Eq. (43) and Eq. (44), we get

$$\underline{\Theta}(1) = \frac{1+\mathrm{r}}{\Gamma(\frac{3}{2})}, \underline{\Theta}(2) = 1 + \mathrm{r}, \underline{\Theta}(3) = \frac{1+\mathrm{r}}{\Gamma(\frac{5}{2})},\ \underline{\Theta}(4) = \frac{1+\mathrm{r}}{2}, \underline{\Theta}(5) = \frac{1+\mathrm{r}}{\Gamma(\frac{7}{2})}, \tag{45}$$

and so on.

By Eq. using (8) up to five terms, we have

$$\underline{\theta}(t) = (1 + \mathrm{r}\,)\left\{1 + \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + t + \frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + \frac{t^2}{2} + \frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}\right\}. \tag{46}$$

Now similarly, to find $\overline{\theta}(t)$, the solving problem must be solved

$${}_0^C D_0^q \overline{\theta}(t) = \overline{\theta}(t),\ t\epsilon[0, 1] \tag{47}$$

with the condition

$$\overline{\theta}(0) = 1.125 - 0.125\alpha. \tag{48}$$

and application of the differential transforms method to Eq. (47) and Eq. (48), gives

$$\overline{\Theta}(k + 1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left[\overline{\Theta}(k)\right], \tag{49}$$

and

$$\overline{\Theta}(0) = 3 - \mathrm{r}. \tag{50}$$

By using Eq. (49) and Eq. (50), we get

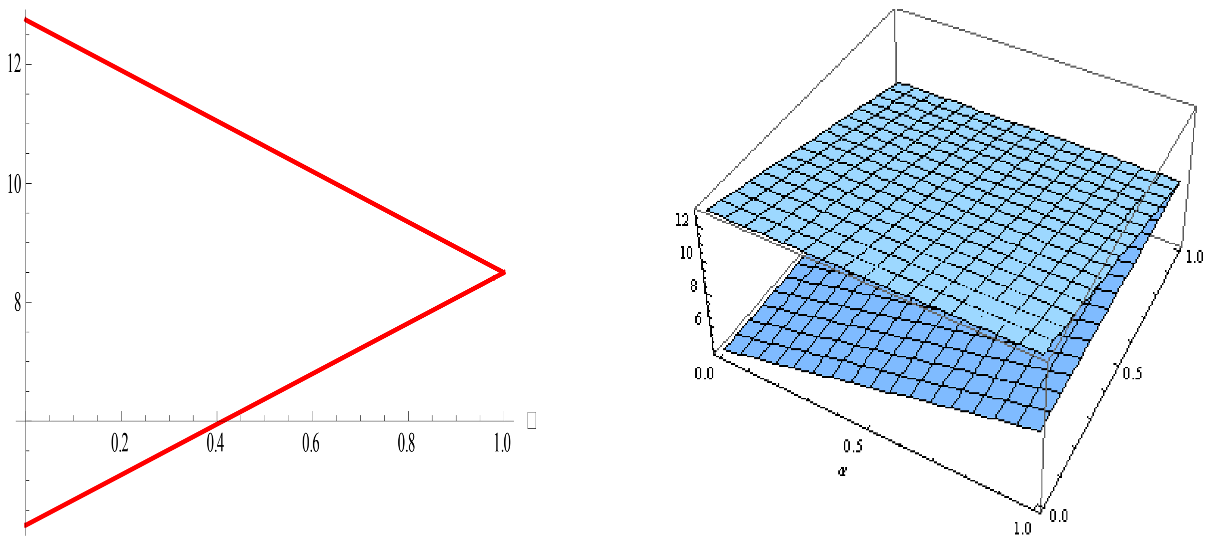
$$\overline{\Theta}(1) = \frac{3-\mathrm{r}}{\Gamma(\frac{3}{2})}, \overline{\Theta}(2) = 3 - \mathrm{r}, \overline{\Theta}(3) = \frac{3-\mathrm{r}}{\Gamma(\frac{5}{2})}, \overline{\Theta}(4) = \frac{3-\mathrm{r}}{2}, \overline{\Theta}(5) = \frac{3-\mathrm{r}}{\Gamma(\frac{7}{2})}, \tag{51}$$

and so on.

By using Eq. (8) up to five terms, we have

$$\overline{\theta}(t) = (3 - \mathrm{r})\{1 + \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + t + \frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + \frac{t^2}{2} + \frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}\}. \tag{52}$$

Hence, the required solution is given by Eq. (46) and Eq. (52).



(a) **(b)**

Figure 3. (a) 2D and (b) 3D plot of fuzzy solution of Example 5.3.

Example 5.4 Consider the following linear fractional differential equation

$$ {}_0^C D_0^{1/2}\tilde{q}(t) = \tilde{q}(t) + 1, \tag{53}$$

Subject to fuzzy initial conditions

$$\tilde{q}(0) = (0.2, 0.4, 0.6). \tag{54}$$

The solution will be of the form $[\underline{q}, \overline{q}]$.
For $\underline{q}$, we will solve the problem

$$ {}_0^C D_0^{1/2}\underline{q}(t) = \underline{q}(t) + 1, \tag{55}$$

with the initial condition

$$\underline{q}(0) = 0.2 + 0.2\alpha. \tag{56}$$

According to described above DTM procedure, we have

$$\underline{Q}(k+1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left[\underline{Q}(k) + \delta(k)\right], \tag{57}$$

with the initial condition

$$\underline{Q}(0) = \underline{q}(0) = 0.2 + 0.2\alpha. \tag{58}$$

By using Eq. (57) and Eq. (58), we get

$$\underline{Q}(1)=\frac{1.2+0.2\alpha}{\Gamma(\frac{3}{2})},\underline{Q}(2)=1.2+0.2\alpha,\underline{Q}(3)=\frac{1.2+0.2\alpha}{\Gamma(\frac{5}{2})},\underline{Q}(4)=\frac{1.2+0.2\alpha}{2},\underline{Q}(5)=\frac{1.2+0.2\alpha}{\Gamma(\frac{7}{2})}, \quad (59)$$

and so on.

By using Eq. (8) upto five terms, we have

$$\underline{q}(t)=0.2+0.2\alpha+(1.2+0.2\alpha)\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+(1.2+0.2\alpha)t+(1.2+0.2\alpha)\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+ (1.2+0.2\alpha)\frac{t^2}{2}+(1.2+0.2\alpha)\frac{t^{5/2}}{\Gamma(\frac{7}{2})}. \quad (60)$$

Now similarly, to find $\overline{q}(t)$, the solving problem must be solved

$${}_0^C D_0^q \overline{q}(t)=\overline{q}(t), \quad (61)$$

with the initial condition

$$\overline{q}(0)=0.6-0.2\alpha. \quad (62)$$

Now applying differential transform method to Eq. (61) and Eq. (62), gives

$$\overline{Q}(k+1)=\frac{\Gamma\left(1+\frac{k}{2}\right)}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left[\overline{Q}(k)+\delta(k)\right], \quad (63)$$

with the initial condition

$$\overline{Q}(0)=\overline{q}(0)=0.6-0.2\alpha. \quad (64)$$

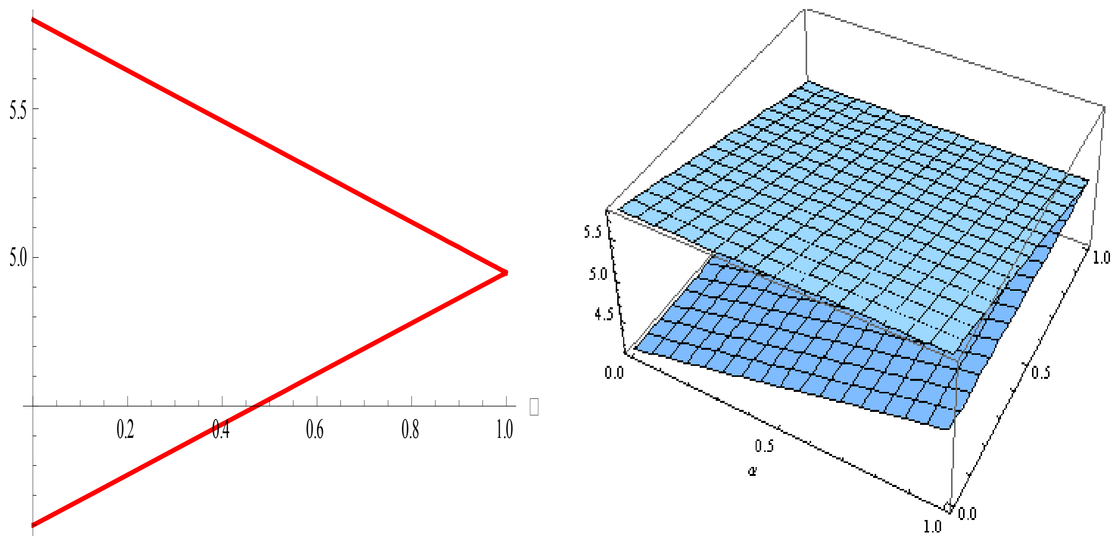
By using Eq. (63) and Eq. (64), we get

$$\overline{X}(1)=\frac{1.6-0.2\alpha}{\Gamma(\frac{3}{2})},\overline{X}(2)=1.6-0.2\alpha,\overline{X}(3)=\frac{1.6-0.2\alpha}{\Gamma(\frac{5}{2})},\overline{X}(4)=\frac{1.6-0.2\alpha}{2},\overline{X}(5)=\frac{1.6-0.2\alpha}{\Gamma(\frac{7}{2})}, \quad (65)$$

By using Eq. (8) up to five terms, we have

$$\overline{q}(t)=0.6-0.2\alpha+(1.6-0.2\alpha)\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}+(1.6-0.2\alpha)t+(1.6-0.2\alpha)\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}+ (1.6-0.2\alpha)\frac{t^2}{2}+(1.6-0.2\alpha)\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)}. \quad (66)$$

Hence, the required solution is given by Eq. (60) and Eq. (66).



(a) **(b)**

Figure 4. (a) 2D and (b) 3D plot of fuzzy solution of Example 5.4.

Example 5.5 Finally, consider the following linear fractional differential equation

$$ {}_0^C D_{0+}^{v} y(x) + y(x) = x e^{-x}, \tag{67}$$

with fuzzy initial conditions

$$ y^r(0) = [-1 + \mathrm{r}, 1 - \mathrm{r}], 0 < v \leq 1, 0 \leq x \leq 1. \tag{68}$$

The results will be of the form $[y_-^r, y_+^r]$.
To calculate y_-^r, we have to solve the problem

$$ {}_0^C D_{0+}^{v} y_-^r(x) + y_-^r(x) = x e^{-x}, \tag{69}$$

with the condition

$$ y_-^r(0) = -1 + \mathrm{r}. \tag{70}$$

According to described above DTM procedure, Eq. (69) and Eq. (70), we have

$$ Y_-^r(k+1) = \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})} [\textstyle\sum_{m=0}^{k} m e^{m-k} - Y_-^r(k)]. \tag{71}$$

and the initial conditions

$$ Y_-^r(0) = -1 + \mathrm{r}. \tag{72}$$

By using Eq. (71) and Eq. (72), we get

$$ Y_-^r(1) = \frac{1-\mathrm{r}}{\Gamma(\frac{3}{2})},\ Y_-^r(2) = \Gamma\left(\frac{3}{2}\right) - 1 + \mathrm{r},\ Y_-^r(3) = \frac{e^{-1} + 3 - \Gamma\left(\frac{3}{2}\right) - r}{\Gamma(\frac{5}{2})},\ Y_-^r(4) = \frac{1}{2}[\Gamma\left(\frac{5}{2}\right) e^{-2} + \left(2\Gamma\left(\frac{5}{2}\right) - 1\right) e^{-1} + 3\Gamma\left(\frac{5}{2}\right) - 3 + \Gamma\left(\frac{3}{2}\right) + r], $$

$$Y_-^r(5) = \frac{1}{\Gamma\left(\frac{7}{2}\right)}[2e^{-3} + 4e^{-2} + 6e^{-1} + 8 - \Gamma\left(\frac{5}{2}\right)e^{-2} - (2\Gamma\left(\frac{5}{2}\right) - 1)e^{-1} - 3\Gamma\left(\frac{5}{2}\right) + 3 - \Gamma\left(\frac{3}{2}\right) - r], \quad (73)$$

By using Eq. (8) up to five terms, we have

$$y_-^r(x) = -1 + \mathrm{r} + \frac{1-\mathrm{r}}{\Gamma\left(\frac{3}{2}\right)}x^{1/2} + \left[\Gamma\left(\frac{3}{2}\right) - 1 + \mathrm{r}\right]x + [\frac{e^{-1}+3-\Gamma\left(\frac{3}{2}\right)-r}{\Gamma\left(\frac{5}{2}\right)}]x^{3/2} + \frac{1}{2}[\Gamma\left(\frac{5}{2}\right)e^{-2} + \left(2\Gamma\left(\frac{5}{2}\right) - 1\right)e^{-1} + 3\Gamma\left(\frac{5}{2}\right) - 3 + \Gamma\left(\frac{3}{2}\right) + r]x^2 + \frac{1}{\Gamma\left(\frac{7}{2}\right)}[2e^{-3} + 4e^{-2} + 6e^{-1} + 8 - \Gamma\left(\frac{5}{2}\right)e^{-2} - (2\Gamma\left(\frac{5}{2}\right) - 1)e^{-1} - 3\Gamma\left(\frac{5}{2}\right) + 3 - \Gamma\left(\frac{3}{2}\right) - r]x^{5/2}. \quad (74)$$

Now similarly, to get y_+^r, we have to solve the problem

$${}_0^C D_{0+}^v y_+^r(x) + y_+^r(x) = xe^{-x}, \quad (75)$$

with the condition

$$y_+^r(0) = 1 - \mathrm{r}, \quad (76)$$

and the application of the differential transform method to Eq. (75) and Eq. (76), we have

$$Y_+^r(k+1) = \frac{\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{k}{2}\right)}[\sum_{m=0}^{k} me^{m-k} - Y_+^r(k)]. \quad (77)$$

with the initial conditions

$$Y_+^r(0) = 1 - \mathrm{r}. \quad (78)$$

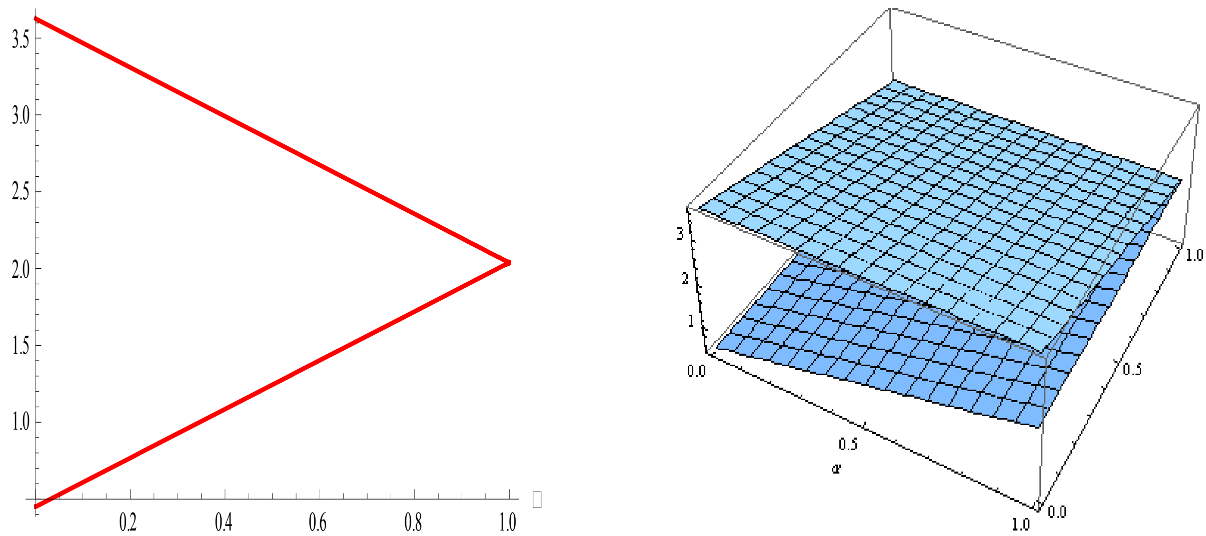
By using Eq. (77) and Eq. (78), we get

$$Y_+^r(1) = \frac{r-1}{\Gamma\left(\frac{3}{2}\right)},\ Y_+^r(2) = \Gamma\left(\frac{3}{2}\right) - \mathrm{r} + 1,\ Y_+^r(3) = \frac{e^{-1}+1-\Gamma\left(\frac{3}{2}\right)+r}{\Gamma\left(\frac{5}{2}\right)}, Y_+^r(4) = \frac{1}{2}[\Gamma\left(\frac{5}{2}\right)e^{-2} + \left(2\Gamma\left(\frac{5}{2}\right) - 1\right)e^{-1} + 3\Gamma\left(\frac{5}{2}\right) - 1 + \Gamma\left(\frac{3}{2}\right) - r], Y_+^r(5) = \frac{1}{\Gamma\left(\frac{7}{2}\right)}\left[2e^{-3} + 4e^{-2} + 6e^{-1} + 8 - \right.$$
$$\Gamma 52e{-}2{-}2\Gamma 52{-}1e{-}1{-}3\Gamma 52{+}1{-}\Gamma 32{+}r, \quad (79)$$

Finally, we have

$$y_+^r(x) = 1 - \mathrm{r} + \frac{r-1}{\Gamma\left(\frac{3}{2}\right)}x^{1/2} + \left[\Gamma\left(\frac{3}{2}\right) - \mathrm{r} + 1\right]x + [\frac{e^{-1}+1-\Gamma\left(\frac{3}{2}\right)+r}{\Gamma\left(\frac{5}{2}\right)}]x^{3/2} + \frac{1}{2}[\Gamma\left(\frac{5}{2}\right)e^{-2} + \left(2\Gamma\left(\frac{5}{2}\right) - 1\right)e^{-1} + 3\Gamma\left(\frac{5}{2}\right) - 1 + \Gamma\left(\frac{3}{2}\right) - r]x^2 + \frac{1}{\Gamma\left(\frac{7}{2}\right)}\left[2e^{-3} + 4e^{-2} + 6e^{-1} + 8 - \right.$$
$$\Gamma 52e{-}2{-}2\Gamma 52{-}1e{-}1{-}3\Gamma 52{+}1{-}\Gamma 32{+}rx52. \quad (80)$$

Hence, the required solution is given by Eq. (74) and Eq. (80).



(a) (b)

Figure 5. (a) 2D and (b) 3D plot of fuzzy solution of Example 5.5.

6. CONCLUSION

In present paper, we elaborated the differential transform method to solve the fractional differential equations with fuzzy initial conditions. We choose the type of differentiability in the Caputo sense as it has much more realistic in initial condition in real life problems. Differential transform method is an iterative method. This method provides the solutions in terms of convergent series with easily computable components. It can produce valid and accurate results of fuzzy fractional differential equation. The accuracy of the obtained solution can be improved by adding more terms in the solution.

REFERENCES

[1] Abood, M.F., Linkens, D.A., Mahfof, M., *Fuzzy Sets and Systems*, **120**, 331, 2001.
[2] Datta, D.P., *Fuzzy Sets System in Physics and Biology, Chaos, Solutions and Fractals*, **17**, 781, 2003.
[3] El-Nachie, M.S., *International Journal of Nonlinear Science and Numerical Simulation*, **6**, 95, 2005.
[4] El-Nachie, M.S. *Chaos, Solutions and Fractals*, **25**, 969, 2005.
[5] Zadeh, L. A., *Information Control*, **8**, 338, 1965.
[6] Change, S.S.L., Zadeh, L.A., *IEEE Trans. System Man Cyberent*, **2**, 30, 1972.
[7] Zadeh, L.A., *Inform. Sci.*, **8**(3), 199, 1975.
[8] Zadeh, L.A., *Inform. Sci.*, **8**(4), 301, 1975.
[9] Zadeh, L.A., *Inform. Sci.*, **9**(1), 43, 1975.
[10] Zadeh, L.A., *Med. Inform.*, **8**, 173, 1983.
[11] Dubois, D., Prade, H., *J. Systems Sci.*, **9**, 613, 1978.
[12] Dubois, D., Prade H., *Fuzzy Sets and Systems; Theory and Applications*, Academic Press, New York, 1980.

[13] Mizumoto, M., Tanaka, K., *Systems-Comput.-Controls*, **7**(5), 73, 1976.

[14] Mizumoto, M., Tanaka, K., Some Properties of Fuzzy Numbers. In Gupta M.M., Rajade, R.K., Yager, R.R. (Eds.), *Advances in Fuzzy Set Theory and Applications*, North Holland, Amsterdam, p. 156, 1979.

[15] Nahmias, S., *Fuzzy Sets and System*, **1**, 97, 1978.

[16] Ralescu, D., A Survey of the Representation of Fuzzy Concepts and its Applications. In Gupta M.M., Rajade, R.K., Yager, R.R. (Eds.), *Advances in Fuzzy Set Theory and Applications*, North Holland, Amsterdam, p. 77, 1979.

[17] Negoita, C.V., Ralecu, D.A., *Applications of Fuzzy Sets to System Analysis*, Wiley, New York, 1975.

[18] Zhou, J.K., *Differential Transformation and its Applications for Electrical Circuits*, Huazhhong University Press, Wuhan, 1986.

[19] Arikoglu, A., Ozkol, I., *Chaos and Sol. Fract.*, **34**(5), 1473, 2007.

[20] Ertürk, V.S., Momani, S., *J. of Computational and Applied Mathematics*, **215**, 142, 2008.

[21] Arikoglu, A., Ozkol, I., *Chaos and Sol. Fract.*, **40**(2), 521, 2009.

[22] Nazari, D., Shahmorad, S., *J. of Computatinal and Applied Mathematics*, **2345**, 883, 2010.

[23] Tari, A., Shahmorad, S., *Computers and Mathematics with Applications*, **61**(9), 2621, 2011.

[24] Hermann, R., *Fractional Calculus: An Introduction for Physicists*, World Scientific Publishing, New Jersey, p. 54, 2014.

[25] Caputo, M., *Geophysical Journal International*, **13**(5), 529, 1967.