ORIGINAL PAPER

BRONZE DIFFERENTIAL GEOMETRY

PRADEEP KUMAR PANDEY[1], SAMEER[1]

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Abstract. *The object of this paper is to discuss a new structure on manifolds, namely the bronze structure and study connections as bronze structure, integrability and parallelism of bronze structures, and bronze Riemannian manifolds.*
Keywords: *Almost product structure, Bronze structure, Bronze manifold, Bronze Riemannian manifold.*

1. INTRODUCTION

In 1963, by generalizing the almost complex and almost contact structures, Yano [1] introduced the concept of an f-structure which is a $(1,1)$-tensor field of the constant rank on a manifold M, and satisfies the equation $f^3 + f = 0$. Moreover, in 1970, it has been further extended by Goldberg and Yano [2] by introducing the notion of the polynomial structure on a manifold.

Motivated by several interesting properties of the golden ratio, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \ldots$. a positive zero of the algebraic equation $x^2 - x - 1 = 0$; recently a golden structure on manifolds has been introduced and studied by Hretcanu [3], by introducing a corresponding almost product structure. In [4], Crasmareanu et al. studied golden differential geometry and obtained some important results.

Inspired by the silver ratio $\theta = 1 + \sqrt{2} \approx 2.414 \ldots$ a positive root of $x^2 - 2x - 1 = 0$. In 2016, Ozkan et al. [5] introduced a new structure on manifolds known as the silver structure.

Analogous to the notion of golden structure and silver structure on manifolds recently we have studied the notion of a bronze structure on manifolds [12] by using the concept of a bronze ratio $\psi = \frac{3+\sqrt{13}}{2} \approx 3.302 \ldots$ which is a positive zero of the equation $x^2 - 3x - 1 = 0$. Basically, the convergence of the golden ratio is most slow that is the golden ratio is the most irrational among all irrational numbers [6], for this reason the silver ratio and the bronze ratio are much sought after the golden ratio and their study becomes more engaging due to certain fast convergence properties.

Throughout the paper, we assume that all of the manifolds, connections and tensor fields are differentiable and are of class C^∞.

[1] Jaypee University of Information Technology, Department of Mathematics, 173234 Solan, India.
E-mail: pandeypkdelhi@gmail.com; sksameer08@gmail.com.

2. PRELIMINARIES

In this section first, we recall some useful definitions.

Definition 2.1 ([3]). "*Let M denote a C^∞-differentiable manifold. If a tensor field Φ of type $(1,1)$ satisfies the equation*

$$\Phi^2 = \Phi + I \tag{2.1}$$

then Φ is called a golden structure on M and (M,Φ) is the golden manifold".

Definition 2.2 ([5]). "*On a C^∞-differentiable manifold M, a $(1,1)$-tensor field Θ that satisfies the equation*

$$\Theta^2 = 2\Theta + I \tag{2.2}$$

is called a silver structure on M and (M,Θ) is the silver manifold".

It is worthwhile to note that the notion of the silver structure was motivated by the silver ratio $\theta = 1+\sqrt{2} \approx 2.414$ ... a positive root of equation $x^2 - 2x - 1 = 0$.

Now analogous to the notion of golden structure and silver structure, we have the concept of a bronze structure [12] on the manifold M as follows.

Definition 2.3. "*Let M denote a C^∞-differentiable manifold. A $(1,1)$-tensor field Ψ on M which satisfies the equation*

$$\Psi^2 = 3\Psi + I \tag{2.3}$$

where I is the identity $(1,1)$-tensor field on M is called a bronze structure on M".

The above notion of the bronze structure is motivated by the bronze ratio $\psi = \frac{3+\sqrt{13}}{2} \approx 3.302$ ... a positive zero of the equation $x^2 - 3x - 1 = 0$.

Proposition 2.4.
i) *ψ and $3-\psi$ are the eigenvalues of the bronze structure Ψ.*
ii) *The bronze structure Ψ is an isomorphism on T_pM, $\forall\, p \in M$.*
iii) *Consequently, Ψ is invertible and its inverse $\widehat{\Psi} = \Psi^{-1}$ verifies the following:*

$$\widehat{\Psi}^2 = -3\widehat{\Psi} + I \tag{2.4}$$

Now we give the following theorem which establishes a relation between bronze and a almost product structure of manifold M.

Theorem 2.5. *Suppose P denote an almost product structure, then P induces a bronze structure on the manifold M in the following manner*

$$\Psi = \frac{1}{2}(3I + \sqrt{13}P) \tag{2.5}$$

moreover, if Ψ denotes a bronze structure on M, then

$$P = \frac{1}{\sqrt{13}}(2\Psi - 3I) \tag{2.6}$$

gives an almost product structure on manifold M.

3. CONNECTIONS AS BRONZE STRUCTURE

3.1. CONNECTION IN THE PRINCIPAL FIBRE BUNDLE

Suppose $P(M,\pi,G)$ denote a principal fibre bundle, where P is a total space, M a base space, π a projection, and G the structure group. Let V represents a vertical distribution and H represents a horizontal distribution (the complementary distribution, that is $V \oplus H = TP$ and H is G-invariant).

The $(1,1)$-tensor field is given by

$$F = v - h$$

denotes an almost product structure over P, with v and h denoting the projectors of V and H respectively. Now analogous to [4, 5], we may propose that "F represents a principal connection iff following conditions holds":

i) $F(X) = X \Leftrightarrow X$ be a vertical vector field.
ii) $dR_a \circ F_u = F_{ua} \circ dR_a$ for each $a \in G$ and $u \in P$.
Now, on account of (2.5), for the bronze structure, we give following results.

Proposition 3.1. "*The bronze structure Ψ on P is related to a principal connection if and only if following conditions are satisfied*":
i) $X \in \chi\,(P)$ denote an eigenvector of Ψ associated with an eigenvalue $\psi \Leftrightarrow X \in V$.
ii) $dR_a \circ \Psi_u = \Psi_{ua} \circ dR_a$ for each $a \in G$ and $u \in P$.

Proposition 3.2. *The bronze structure Ψ is integrable, i.e. $N_\Psi = 0$ iff principal connection is flat.*
Here the lift $l_\omega\colon \chi(M) \to \chi(P)$, is determined by the principal connection; satisfying

$$[l_\omega\bar{X}, l_\omega\bar{Y}] - l_\omega[\bar{X},\bar{Y}] = N_F(l_\omega\bar{X}, l_\omega\bar{Y})$$

For $\bar{X},\bar{Y} \in \chi(M)$.

Proposition 3.3. *The bronze structure Ψ is integrable, i.e. $N_\Psi = 0$ iff the lift l_ω be a morphism.*

3.2. CONNECTIONS IN THE TANGENT BUNDLES

Suppose M denote a differentiable manifold of dimension n and (TM,π_M,M) be the tangent bundle along with M the base space. Suppose $(U,x^i)_{1\le i\le n}$ denote a local coordinate system on M, and $(\pi_M^{-1}(U),x^i,y^i)_{1\le i\le n}$ be the induced local coordinate system on TM given by $x^i(u) = x^i(\pi_M(u))$ and $y^i(u) = dx^i(u)$ for all $u \in \pi_M^{-1}(U)$.
$V = \{X \in TM\colon \pi_{M*}(X) = 0\}$, is said to be a vertical distribution of M.

A $(1,1)$-tensor field T on M defined as $T = \frac{\partial}{\partial y^i} \otimes dx^i$ is an almost tangent structure on M, that is $T^2 = 0$.

Now parallel to [4, 5], we give following definitions.

Definition 3.1. "*A tensor field* v *of type* $(1,1)$ *is said to be a vertical projector*", *provided*

$$\begin{cases} T \circ v = 0 \\ v \circ T = T \end{cases} \tag{3.1}$$

Definition 3.2. "*The complementary distribution N to V, will be called a non-linear connection*", *if*

$$N \oplus V = \chi(M) \tag{3.2}$$

In fact a vertical projector v is $C^{\infty}(M)$-linear, with $im\, v = V$, we can state the following.

Proposition 3.4. "*The vertical vector* v *induces a non-linear connection, denoted by* $N\,(v)$ *given by relation* $N\,(v) = ker\ v$".

If N is a connection which is non-linear then h_N and v_N denote the horizontal as well as vertical projections associated with the decomposition given by (3.2), consequently we have the following result.

Proposition 3.5. "*A non-linear connection* N *implies that* v_N *is a vertical projector with* $N(v_N) = N$".

Definition 3.3. "*A tensor field* Γ *of type* $(1, 1)$ *is said to be a non-linear connection of an almost product type*", *provided*

$$\begin{cases} \Gamma \circ T = -T \\ T \circ \Gamma = T \end{cases}$$

Proposition 3.6. "*Suppose* Γ *denotes a non-linear connection of an almost product type; then*
i) $v_\Gamma = \frac{1}{2}\left(I_{\chi(M)} - \Gamma\right)$ *is a vertical vector.*
ii) $V\,(M)$ *is the* (-1)*-eigenspace of* Γ *while* $N\,(v_\Gamma)$ *is the* $(+1)$*-eigenspace of* Γ".

Proposition 3.7. "*Let* $\Gamma = I_{\chi(M)} - 2v$ *, where* v *is a vertical vector. Then* Γ *defines an almost product structure on the manifold M*".

Now, for the bronze case we give the following proposition.

Proposition 3.8. "Suppose N *denote a non-linear connection on* M, *given by vertical vector* v. *Then* N *can also be defined by a bronze structure* $\Psi(= \Psi_\Gamma)$

$$\Psi = \psi I_{\chi(M)} - \sqrt{13}\, v$$

with N *the* ψ*-eigenspace and* V *the* $(3 - \psi)$*-eigenspace*".

4. INTEGRABILITY, PARALLELISM OF BRONZE STRUCTURES

Suppose Ψ denote a bronze structure on M. N_Ψ denote Nijenhuis tensor of $(1, 2)$-type tensor field Ψ. Using [7], for $X, Y \in \chi(M)$, we have

$$N_{\Psi}(X,Y) = \Psi^2[X,Y] + [\Psi X, \Psi Y] - \Psi[\Psi X, Y] - \Psi[X, \Psi Y] \quad (4.1)$$

Suppose R and S denote complementary distributions on manifold M associated with ψ and $3-\psi$. Say r and s are corresponding projections. Thus

$$\begin{cases} r^2 = r, \ s^2 = s \\ rs = sr = 0, \ r + s = I \end{cases} \quad (4.2)$$

Now using (2.5), a straightforward computation yields

$$\begin{cases} r = \frac{1}{\sqrt{13}}\Psi - \frac{3-\psi}{\sqrt{13}} I \\ s = -\frac{1}{\sqrt{13}}\Psi + \frac{\psi}{\sqrt{13}} I \end{cases} \quad (4.3)$$

We infer from [7], that

i) Ψ is integrable $\Rightarrow N_{\Psi} = 0$.

ii) Integrable distribution $R \Rightarrow s[rX, rY] = 0$ and integrable distribution $S \Rightarrow r[sX, sY] = 0$, $\forall X, Y \in \chi(M)$.

Clubbing (2.3) and (4.3), yields

$$\begin{cases} \Psi r = r\Psi = \psi r = \frac{\psi}{\sqrt{13}}\Psi + \frac{1}{\sqrt{13}} I \\ \Psi s = s\Psi = (3-\psi)s = \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{1}{\sqrt{13}} I \end{cases} \quad (4.4)$$

Now $\psi r = \psi(\frac{1}{\sqrt{13}}\Psi - \frac{3-\psi}{\sqrt{13}} I)$

$$\psi r = \frac{\psi}{\sqrt{13}}\Psi + \frac{\psi^2 - 3\psi}{\sqrt{13}} I$$

$$\psi r = \frac{\psi}{\sqrt{13}}\Psi + \frac{1}{\sqrt{13}} I \text{ and}$$

$$(3-\psi)s = (3-\psi)(-\frac{1}{\sqrt{13}}\Psi + \frac{\psi}{\sqrt{13}} I)$$

$$(3-\psi)s = \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{\psi^2 - 3\psi}{\sqrt{13}} I$$

$$(3-\psi)s = \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{1}{\sqrt{13}} I$$

Then for a bronze structure, it follows that

$$\begin{cases} 13\, s[rX, rY] = sN_{\Psi}(rX, rY) \\ 13\, r[sX, sY] = rN_{\Psi}(sX, sY) \end{cases} \quad (4.5)$$

Proposition 4.1. "*A bronze structure Ψ is integrable if and only if almost product structure $\Psi = \frac{1}{2}\left(3I + \sqrt{13}P\right)$ is integrable*".

Proposition 4.2. "*Let $X, Y \in \chi(M)$. The distribution R is integrable if and only if $sN_{\Psi}(rX, rY)$ vanishes and distribution S is integrable if and only if $rN_{\Psi}(sX, sY)$ vanishes. Whenever Ψ is integrable, then both of the distributions R and S are integrable*".

Consider a linear connection ∇ on the manifold M. Now corresponding (Ψ, ∇) we associate following new (linear) connections [8].
i) The connection,

$$\widetilde{\nabla}_X Y = r\,(\nabla_X rY) + s\,(\nabla_X sY) \tag{4.6}$$

is called a $Schouten$ connection.
ii) The connection,

$$\breve{\nabla}_X Y = r\,(\nabla_{rX}\, rY) + s\,(\nabla_{sX}\, sY) + r[sX, rY] + s\,[rX, sY] \tag{4.7}$$

is called $Vr\breve{a}nceanu$ connection.

Proposition 4.3. "*For any linear connection ∇ on M, the projectors r and s are parallel in terms of $Schouten$ and $Vr\breve{a}nceanu$ connections. Moreover, Ψ is also parallel in terms of $Schouten$ and $Vr\breve{a}nceanu$ connections*".

Proof. From equation (4.2), $\forall\, X, Y \in \chi(M)$

$$(\widetilde{\nabla}_X r)Y = \widetilde{\nabla}_X rY - r(\widetilde{\nabla}_X Y) = r(\nabla_X rY) - r(\nabla_X rY) = 0$$

$$(\breve{\nabla}_X r)Y = \breve{\nabla}_X rY - r(\breve{\nabla}_X Y) = r(\nabla_{rX}\, rY) + r[sX, rY] - r(\nabla_{rX}\, rY) - r[sX, rY] = 0$$

Hence r is parallel with respect to the $\widetilde{\nabla}$ and $\breve{\nabla}$.
Similarly, it can be shown that the above relations hold for s.
From (4.4), the result follows.
Now analogous to [9], we define the following.

Definition 4.1. "*We say that distribution R is parallel with respect to the linear connection ∇, whenever $\nabla_X Y \in R$ for $X \in \chi(M)$, $Y \in R$*".

Definition 4.2. "*A distribution R is said to be ∇-half parallel, whenever $(\Delta\Psi)(X, Y) \in R$ where*

$$(\Delta\Psi)(X, Y) = \Psi\nabla_X Y - \Psi\nabla_Y X - \nabla_{\Psi X} Y + \nabla_Y(\Psi X)$$

for $X \in R, Y \in \chi(M)$".

Definition 4.3. "*A distribution R is said to be ∇-anti half parallel, whenever $(\Delta\Psi)(X, Y) \in S$ where $X \in R$, $Y \in \chi(M)$*".

Now we give the following result.

Proposition 4.4. "*It follows that distributions R and S are parallel in terms of $Schouten$ and $Vr\breve{a}nceanu$ connections for a linear connection ∇*".

Proof. Suppose $X \in \chi(M), Y \in R$. Hence, $sY = 0$ as well as $rY = Y$.
Combining (4.6) and (4.7), yields

$$\widetilde{\nabla}_X Y = r(\nabla_X Y) \in R,$$
$$\breve{\nabla}_X Y = r(\nabla_{rX} Y) + r[sX, Y] \in R$$

Consequently, R is parallel with respect to the connections $\tilde{\nabla}$ and $\breve{\nabla}$. Moreover, it can be shown that S also verifies similar conditions.

5. ON THE BRONZE RIEMANNIAN MANIFOLDS

Suppose P and g respectively denote an almost product structure and a Riemannian metric on M, given by

$$g\big(P(X), P(Y)\big) = g(X, Y); \ \forall\, X, Y \in \chi(M)$$

in other words, P is a g-symmetric endomorphism, given by

$$g(P(X), Y) = g\big(X, P(Y)\big).$$

In this case, the ordered pair (g, P) is said to be a Riemannian almost product structure [10, 11].

Using (2.5) and (2.6) in conjunction with [4, 5], we give the following.

Proposition 5.1. "*An almost product structure P defines a g-symmetric endomorphism $\Leftrightarrow$ associated bronze structure Ψ also defines a g-symmetric endomorphism*".

Definition 5.1. "*Suppose g denote a Riemannian metric on a manifold M, given by*

$$g(\Psi(X), Y) = g\big(X, \Psi(Y)\big), \text{ for every } X, Y \in \chi(M)\text{''}.$$

Usually the ordered pair (g, Ψ) is called a bronze Riemannian structure and the triple (M, g, Ψ) is said to be a bronze Riemannian manifold.

Corollary 5.1. "*On the bronze Riemannian manifold (M, g, Ψ)", we have*
i) projectors r and s are g-symmetric, i.e.

$$\begin{cases} g\big(X, r(Y)\big) = g(r(X), Y) \\ g\big(X, s(Y)\big) = g(s(X), Y) \end{cases}$$

ii) distributions R and S are g-orthogonal, i.e.

$$g(r(X), s(Y)) \text{ vanishes.}$$

iii) bronze structure Ψ on M is N_Ψ symmetric, i.e.

$$N_\Psi(\Psi(X), Y) = N_\Psi\big(X, \Psi(Y)\big).$$

Proposition 5.2. "*A Riemannian almost product structure is a locally product structure if* P *is parallel with respect to the Levi-Civita connection* $\overset{g}{\nabla}$ *of* g*, i.e.* $\overset{g}{\nabla}P = 0$ *and if* ∇ *is a symmetric linear connection then the Nijenhuis tensor of* P *verifies*

$N_P(X, Y) = (\nabla_{PX}P)Y - (\nabla_{PY}P)X - P(\nabla_X P)Y + P(\nabla_Y P)X$".

Thus in case of a bronze structure, we propose the following.

Proposition 5.3. "*The bronze structure* Ψ *is integrable if* (M, g, Ψ) *is a locally product bronze Riemannian manifold*".

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