ORIGINAL PAPER

ON IDEAL CONVERGENT DOUBLE SEQUENCES OF σ –BOUNDED VARIATION IN 2–NORMED SPACES DEFINED BY A SEQUENCE OF MODULI

VAKEEL A. KHAN[1], SAMEERA A. A. ABDULLAH[1], AYHAN ESI[2], KAMAL M.A.S. ALSHLOOL[1]

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Abstract. *The notion of ideal convergence in* **2***–normed spaces was defined and studied by Gürdal [1] for single real sequences. After Gürdal work Saeed Sarabadan and Sorayya Talebi [2] defined and studied the notion of ideal convergence in* **2***–normed spaces for double sequences. The space of all double sequences of* $\boldsymbol{\sigma}$*–bounded variation has been defined and studied by Vakeel [3]. In this present article we are working on to connect the above two studies and define some new spaces double sequences of* $\boldsymbol{\sigma}$*–bounded variation in* **2***–normed spaces using the moduli* $\boldsymbol{F} = (\boldsymbol{f}_{ij})$ *and some others operators as well. Further, we study basic topological and algebraic properties and prove some inclusion relations on these spaces.*

Keywords: *Invariant mean, sequence of σ–bounded variation, 2–normed space, paranormed space, I–convergence, I–Cauchy, I–bounded double sequence over 2–normed space.*

1. INTRODUCTION

Throughout the article, let $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers respectively. Let us denote ω for the space of all real or complex sequences $x = (x_k)$ where $k \in \mathbb{N}$. Recall that, a sequence $x = (x_k) \in \omega$ is said to be statistically convergent to $\ell \in \mathbb{C}$, if for each $\epsilon > 0$, the natural density of the set $\{k \in \mathbb{N}: |x_k - \ell| \geq \epsilon\}$ equal zero. The notion of ideal convergence was first introduced by Kostyrko et al. [4] as an interesting generalization of the concept we just mentioned above which was introduced by Fast [5] and Steinhaus [6] independently for the real sequences. On other hand, the concept of 2–normed spaces was initially developed by Gähler [7]. Recently ideal convergence in 2–normed spaces has been studied by Gürdal [1]. As a continuation of the work provided by Gürdal, Saeed Sarabadan and Sorayya Talebi [2] defined and studied the notion of ideal convergence in 2–normed spaces for double sequences.

In this article we continue in the same direction and studying the concept of ideal convergence in 2–normed spaces by considering the specific type of sequences, that is double sequence of σ–bounded variation which was defined by Vakeel [3] and define some new spaces of these double sequences by using the double sequence of moduli $F = (f_{ij})$ and some

[1] Aligarh Muslim University, Department of Mathematics, 202002 Aligarh, India.
[2] Adiyaman University, Department of Mathematics, Adiyaman, Turkey.
E-mail: vakhanmaths@gmail.com; sameera173a@gmail.com; aesi23@hotmail.com; kmoto@yahoo.com.

others operators as well. In addition, we study basic topological and algebraic properties and prove some inclusion relations on these spaces.

Definition 1.1 [7] Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2–normed on X is a function $\|\cdot,\cdot\|: X \times X \longrightarrow \mathbb{R}$ which satisfying the following conditions:

(i). $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linearly dependent in X,

(ii). $\| x_1, x_2 \| = \| x_2, x_1 \|$,

(iii). $\| \alpha x_1, x_2 \| = |\alpha| \, \| x_1, x_2 \|$ for any $\alpha \in \mathbb{R}$,

(iv). $\| x_1 + x', x_2 \| \leq \| x_1, x_2 \| + \| x', x_2 \|$

is called a 2–norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2–normed space over $\mathbb{R}$.

Example 1.1 [1] Take $X = \mathbb{R}^2$ being equipped with the 2-norm $\| x, y \|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\| x, y \| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Definition 1.2 [2] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be converge to some $L \in X$ if for every $\epsilon > 0$ there exists a positive integer N such that

$$\| x_{ij} - L, z \| < \epsilon \quad \text{for all } i, j \geq N, \text{ for every } z \in X .$$

Definition 1.3 [2] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be Cauchy with respect the 2–norm if there exist a two positive integers $s = s(\epsilon), t = t(\epsilon)$ such that

$$\lim_{i,j \to \infty} \| x_{ij} - x_{st}, z \| = 0 \qquad \text{for every } z \in X.$$

If every Cauchy sequence in X converges to some number $L \in X$, then X is said to be complete with respect to the 2–norm. Any complete 2–normed space is said to be a 2–Banach space.

Definition 1.4 [8] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be statistically convergent to $L \in X$, if for each $\epsilon > 0$, for every nonzero $z \in X$, the double natural density of the set $\{i, j \in \mathbb{N}: \| x_{ij} - L, z \| \geq \epsilon\}$ equal zero.

Definition 1.5 [9] Let X be a linear metric space. A function $g: X \to \mathbb{R}$ is said to be paranorm, if for all $x, y \in X$,

(i). $g(x) \geq 0$ *for all* $x \in X$,

(ii). $g(-x) = g(x)$,

(iii). $g(x + y) \leq g(x) + g(y)$ for all $x, y \in X$,

If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and x_n is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$, then $g(\lambda_n x - \lambda x) \to 0$ as $n \to \infty$.) [10].

A paranorm g for which $g(x) = 0$ implies that $x = 0$ is called total paranorm and the pair (X, g) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm ([11] - Theorem 10.4.2, p. 183).

Let ℓ_∞ and c denotes the Banach spaces of bounded and convergent sequences $x = (x_k)$, respectively with the usual norm $\| x \| = sup_k |x_k|$. Let σ be a one–to–one mapping from the set of positive integers into itself having no finite orbits for all positive integers and

T be an operator on ℓ_∞ defined by $T_x = T(x_k) = (x_{\sigma(k)})$ for all $x = (x_k) \in \ell_\infty$. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or σ–mean if and only if:

(i). $\varphi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k.

(ii). $\varphi(e) = 1$ where $e = \{1,1,1, \ldots\}$.

(iii). $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in \ell_\infty$.

By V_σ (see, [12]) we denote the set of bounded sequences of all whose invariant means are equal. That is

$$V_\sigma = \left\{ x = (x_k) \colon \sum_{m=1}^{\infty} t_{m,k}(x) = L \quad \text{uniformly in } k, L = \sigma - \lim x \right\}, \quad (1.1)$$

where $m \geq 1, k > 0$ and

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \cdots + x_{\sigma^m(k)}}{m+1} \quad such\ that\ t_{-1,k} = 0\,, \quad (1.2)$$

where $\sigma^m(k)$ denote m^{th}–iterate of σ at k. Invariant means have recently been studied by Ahmad and Mursaleen [13], Raimi [14] and many others. Later on, the concept of invariant mean for double sequences was defined in [15].

Definition 1.6 A sequence $x \in \ell_\infty$ is of σ–bounded variation if and only if: $\sum |\varphi_{mk}(x)|$ converges uniformly in k, $\lim_{m\to\infty} t_{m,k}(x)$ which must exist, should take the same value for all k.

By BV_σ we denote the space of all sequences of σ–bounded variation which was defined by Mursaleen [16] as follow:

$$BV_\sigma = \left\{ x \in \ell_\infty \colon \sum_{m=1}^{\infty} |\varphi_{mk}(x)| < \infty, \text{uniformly in } k \right\}, \quad (1.3)$$

where

$$\varphi_{mk}(x) = t_{m,k}(x) - t_{m-1,k}(x), \quad and\ t_{-1,k} = \quad (1.4)$$

having the following properties, for any sequences x, y and scalar λ,

$$\varphi_{mk}(x + y) = \varphi_{mk}(x) + \varphi_{mk}(y).$$

$$\varphi_{mk}(\lambda x) = \lambda \varphi_{mk}(x).$$

Vakeel et al. [17] developed the same space we just mentioned above to a double sequences. Later with the help of the concept of I–convergence, Vakeel et al. [18] have defined many different spaces related to the space BV_σ that is being studied. For more details see [3, 19-21].

Definition 1.7 [22] Let X be a non–empty set. A family of sets $I \subseteq 2^X$ is said to be an ideal in X if :

(i). $\emptyset \in I$,

(ii). I is additive, that is $A, B \in I \Rightarrow A \cup B \in I$, (iii). I is hereditary, that is $A \in I, B \subseteq A \Rightarrow B \in I$.

- An ideal $I \subseteq 2^X$ is said to be non–trivial if $I \neq 2^X$.
- A non–trivial ideal $I \subseteq 2^X$ is said to be admissible if $I \supseteq \{\{x\}: x \in X\}$.
- A non–trivial ideal $I \subseteq 2^X$ is said to be maximal if there cannot exist any non–trivial ideal $J \neq I$ containing I as a subset.

Definition 1.8 [22] A non–empty family of sets $\mathcal{F} \subseteq 2^{\mathrm{X}}$ is said to be a filter on X if and only if

(i). $\emptyset \notin \mathcal{F}$,

(ii). For $A, \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,

(iii). For each $A \in \mathcal{F}$ with $A \subseteq B$ implies $B \in \mathcal{F}$.

Remark 1.1 [22] For each ideal I there is a filter $\mathcal{F}(I)$ which corresponds to I (filter associate with ideal I), that is

$$\mathcal{F}(I) = \{K \subseteq X: K^c \in I\}, \text{ where } K^c = X \setminus K. \tag{1.5}$$

Definition 1.9 [2] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be $I-$convergent to a number $L \in \mathbb{R}$ if, for every $\epsilon > 0$, the set

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N}: \| \left(x_{ij}\right) - L, z \| \geq \epsilon\right\} \in I, \text{ for every } z \in X. \tag{1.6}$$

and we write $I - \lim_{i,j} \| x_{ij}, z \| = L$.

Definition 1.10 [2] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be I–null if, for every $\epsilon > 0$, the set

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N}: \| \left(x_{ij}\right), z \| \geq \epsilon\right\} \in I, \text{ for every } z \in X. \tag{1.7}$$

and we write $I - \lim_{i,j} \| x_{ij}, z \| = 0$.

Definition 1.11 [2] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be I–Cauchy if, for each $\epsilon > 0$, there exists a numbers $s = s(\epsilon)$ and $t = t(\epsilon)$ such that the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: \| (x_{ij}) - (x_{st}), z \| \geq \epsilon\} \in I, \text{ for every } z \in X.$$

Definition 1.12 [20] A double sequence (x_{ij}) in a 2–normed space $(X, \|\cdot,\cdot\|)$ is said to be I–bounded if there exists $M > 0$, such that, the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: \| x_{ij}, z \| \geq M\} \in I, \text{ for every } z, \in X.$$

Definition 1.13 [20] A double sequence space E is said to be solid or normal, if $(\alpha_{ij} x_{ij}) \in E$ whenever $(x_{ij}) \in E$ and for any double sequence of scalars (α_{ij}) with $|(\alpha_{ij})| < 1$, for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.14 [20] A double sequence space E is said to be symmetric, if $(x_{\pi(i,j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i,j)$ is a permutation on $\mathbb{N} \times \mathbb{N}$.

Definition 1.15 [20] A double sequence space E is said to be sequence algebra, if $(x_{ij}) * (y_{ij}) = (x_{ij}.y_{ij}) \in E$ whenever $x_{ij}, y_{ij} \in E$.

Definition 1.16 [20] A double sequence space E is said to be convergent free, if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies that $y_{ij} = 0$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.17 [20] Let $K = \{(i_s j_t) \in \mathbb{N} \times \mathbb{N}: i_1 < i_2 < \cdots \text{ and } j_1 < j_2 < \cdots\} \subseteq N \times N$ and let E be a double sequence space. A K–step space of E is a double sequence space

$$\lambda_K^E = \{x = (x_{i_s j_t}) \in {}_2\,\omega : x_{st} \in E\}$$

A canonical pre–image of a double sequence $(x_{i_s j_t}) \in \lambda_K^E$ is a double sequence $y_{ij} \in {}_2\,\omega$ defined by

$$y_{ij} = \begin{cases} x_{ij} & \text{if } i,j \in K \\ 0 & \text{otherwise.} \end{cases}$$

A canonical pre–image of a step space is a set of canonical pre–images of all elements in λ_K^E, i.e., y is in the canonical pre–image of λ_K^E iff y is a canonical pre–image of some element $x \in \lambda_K^E$.

Definition 1.18 [20] A double sequence space E is said to be monotone, if it is contains the canonical pre–images of it is step space.

The idea of modulus function was introduced by Nakano [23] in 1953. It is defined as a function $f: [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i). $f(t) = 0$ if and only if $t = 0$,

(ii). $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,

(iii). f is increasing,

(iv). f is continuous from the right at zero.

Ruckle [24] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = x_k : \textstyle\sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [25, 26] proved that, for any modulus f

$$X(f) \subset \ell_1 \text{ and } X(f)^{\alpha} = \ell_{\infty},$$

The spaces $X(f)$ is a Banach space with respect to the norm.

$$\| x \| = \textstyle\sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

After then Kolk [27, 28] gave an extension of $\chi(f)$ by considering a sequence of modulus functions called the moduli $F = (f_k)$ and defined the sequence space:

$$X(F) = \{x = (x_k) : (f_k(|x_k|) \in X\}.$$

From the above four properties of modulus function it can be clearly seen that $f(x)$ must be continuous everywhere on $[0,\infty)$. For a sequence of moduli, we have further two properties:

$$\sup_k f_k(t) < \infty \text{ for all } t > 0,$$

$$\lim_{t\to 0} f_k(t) = 0 \text{ uniformly in } X \text{ and for } k \geq 1.$$

Example 1.2 Let f be a function from $[0,\infty)$ to $[0,\infty)$. If we take $f(x) = \frac{x}{x+1}$, then the function f is a bounded modulus function and if we take $f(x) = x^p$, $0 < p < 1$, *then* f is an unbounded modulus function.

The following popular inequalities will be used throughout the article. Let $p = (p_{ij})$ be the bounded double sequence of positive real numbers. For any complex λ, with $0 < p_{ij} \leq H = \sup_{i,j}(p_{ij}) < \infty$, we have

$$|\lambda|^{p_{ij}} \leq \max\{1, |\lambda|^H\}.$$

Let $D = \max\{1, 2^{H-1}\}$, then for the factorable double sequences (a_{ij}) and (b_{ij}) in the complex plane, we have

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq D\{|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}\}, \tag{1.8}$$

for all i and j. Also $|a_{ij}|^{p_{ij}} \leq max\{1, |a_{ij}|^H\}$ for all $a \in \mathbb{C}$. We used the following lemmas to establish some results of this article.

Lemma 1.1 [29] Every solid space is monotone.

Lemma 1.2 Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

Lemma 1.3 If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$.

2. MAIN RESULTS

Let I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$, let $F = (f_{ij})$ be a double sequence of moduli, let $(X, \|\cdot,\cdot\|)$ be a 2–normed space, let $p = (p_{ij})$ be a factorable double sequence of strictly positive real numbers, let ${}_2\omega(2X)$ be the space of all double sequences defined over the 2–normed space $(X, \|\cdot,\cdot\|)$, then for each $\epsilon > 0$, we define the following new spaces of double sequences:

$$ {}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|] = \{x = (x_{ij}) \in_2 \omega(2X) : \{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \geq \epsilon\} \in I, \text{ for some } L \in X \text{ and for every } z \in X\}, \tag{2.1}$$

$$ {}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]) = \{x = (x_{ij}) \in_2 \omega(2X) : \{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} \geq \epsilon\} \in I, \text{for every } z \in X \}, \tag{2.2}$$

$$ {}_2({}_\infty BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]) = \{x = (x_{ij}) \in_2 \omega(2X) : \{(i,j) \in \mathbb{N}\times\mathbb{N} : \exists\ K > 0 \ s.t \quad u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} \geq K\} \in I, \text{for } z \in X \ \}, \tag{2.3}$$

$$_2(_\infty BV_\sigma[F,u,p,\|\cdot,\cdot\|]) = \{x=(x_{ij}) \in_2 \omega(2X) : \sup_{i,j} u_{ij}\left[F\left(\| \varphi_{mkij}(x), z \|\right)\right]^{p_{ij}} < \infty\}. \quad (2.4)$$

We also denote

$$_2\mathcal{M}^I_{BV_\sigma}[F,u,p,\|\cdot,\cdot\|] =_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|] \cap_2 (_\infty BV_\sigma[F,u,p,\|\cdot,\cdot\|]),$$

and

$$_2(_0\mathcal{M}^I_{BV_\sigma}[F,u,p,\|\cdot,\cdot\|]) =_2 (_0BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]) \cap_2 (_\infty BV_\sigma[F,u,p,\|\cdot,\cdot\|]).$$

Theorem 2.1 For any sequence of moduli F the spaces of double sequence

$_2BV^I_\sigma[F,u,p,\|\cdot,\cdot\|],_2 (_0BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$, $_2\mathcal{M}^I_{BV_\sigma}[F,u,p,\|\cdot,\cdot\|]$ and $_2(_0\mathcal{M}^I_{BV_\sigma}[F,u,p,\|\cdot,\cdot\|])$

are linear spaces.

Proof. We shall proof the result for the space $_2BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$, for other spaces the proof will follow similarly. Let $x=(x_{ij})$, $y=(y_{ij}) \in_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$ be two arbitrary elements and let α, β are scalars. Now, since $x=(x_{ij})$, $y=(y_{ij}) \in_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$, then, there exists $L_1, L_2 \in \mathbb{C}$ such that the sets

$$A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| \varphi_{mkij}(x) - L_1, z \|)]^{p_{ij}} < \frac{\epsilon}{2}, \text{ for every } z\,X\} \in I, \quad (2.5)$$

$$A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| \varphi_{mkij}(y) - L_2, z \|)]^{p_{ij}} < \frac{\epsilon}{2}, \text{ for every } z \in X\} \in I, \quad (2.6)$$

Since F is sequence moduli, we have,

$$\begin{aligned} &u_{ij}[F(\| (\alpha\varphi_{mkij}(x) + \beta\varphi_{mkij}(y)) - (\alpha L_1 + \beta L_2) \|)]^{p_{i,j}} \\ &\le u_{ij}[F(\| |\alpha|(\varphi_{mkij}(x) - L_1) + |\beta|(\varphi_{mkij}(y) - L_2) \|)]^{p_{ij}} \\ &\le u_{ij}[F\left(\| \left(\varphi_{mkij}(x) - L_1\right) + \left(\varphi_{mkij}(y) - L_2\right) \|\right)]^{p_{ij}} \quad (2.7) \end{aligned}$$

Therefore, by (2.5), (2.6) and (2.7), we have,

$$\begin{aligned} &\{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| (\alpha\varphi_{mkij}(x) + \beta\varphi_{mkij}(y)) - (\alpha L_1 + \beta L_2) \ge \epsilon \|)]^{p_{ij}}\} \\ &\subseteq [A_1 \cup A_2] \in I \end{aligned}$$

implies that

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : u_{ij}[F(\| (\alpha\varphi_{mkij}(x) + \beta\varphi_{mkij}(y)) - (\alpha L_1 + \beta L_2) \ge \epsilon \|)]^{p_{ij}}\} \in I.$$

But $x=(x_{ij})$, $y=(y_{ij}) \in_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]$ are arbitrary elements. Therefore $\alpha(x_{ij}) + \beta(y_{i,j}) \in_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]$ and for $x=(x_{ij})$, $y=(y_{ij}) \in_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]$. Hence, $_2BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$ is linear spaces.

Theorem 2.2 The inclusions

$$_2(_0BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]) \subset_2 BV^I_\sigma[F,u,p,\|\cdot,\cdot\|] \subset_2 (_\infty BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]) \text{ hold.}$$

Proof. For this let us consider $x=(x_{ij}) \in_2 (_0BV^I_\sigma[F,u,p,\|\cdot,\cdot\|])$. It is obvious that it must belong to $_2BV^I_\sigma[F,u,p,\|\cdot,\cdot\|]$. Consider

$$u_{i,j}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \le u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} + u_{ij}[F(\| L, z \|)]^{p_{ij}}.$$

Taking the limit on both sides we get

$$I - \lim_{i,j} u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} = 0.$$

Hence $x = (x_{ij}) \in_2 BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$.

Now it remains to show that ${}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|] \subset_2 ({}_\infty BV_\sigma^I[F, u, p, \|\cdot,\cdot\|])$. For this let us consider $x = (x_{ij}) \in_2 BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$, then

$$I - \lim_{i,j} u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} = 0.$$

Consider

$$u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} \le u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} + u_{ij}[F(\| L, z \|)]^{p_{ij}}.$$

Taking the supremum on both sides we get

$$\sup_{i,j} u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} < \infty.$$

Hence $x = (x_{ij}) \in_2 ({}_\infty BV_\sigma^I[F, u, p, \|\cdot,\cdot\|])$.

Lemma 2.1 Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$, one has

$$f(x) \le 2f(1)\delta^{-1}x. \tag{2.8}$$

Theorem 2.3 Let $F = (f_{ij})$ be a sequence of moduli and $0 \le \inf p_{ij} = h \le \sup p_{ij} = H < \infty$. Then the following statements hold:

$${}_2BV_\sigma^I[u, p, \|\cdot,\cdot\|] \subset {}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|].$$

$${}_2({}_0BV_\sigma^I[u, p, \|\cdot,\cdot\|]) \subset {}_2({}_0BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]).$$

Proof. For some $\delta > 0$, choose $\delta_0 > 0$ such that $\max\{\delta_0^h, \delta_0^H\} < \delta$. By the continuity of $F = (f_{ij})$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, we can choose some $\epsilon \in (0,1)$ such that for every t with $0 < t \le \epsilon$ we have

$$F(t) < \delta_0 \quad \forall (i,j) \in \mathbb{N} \times \mathbb{N}. \tag{2.9}$$

Let $x = (x_{ij}) \in_2 BV_\sigma^I[u, p, \|\cdot,\cdot\|]$, then for some $L > 0$, $\delta > 0$ and for every $z \in X$, $i = 1,2,3, \dots, (n-1)$, we have

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : u_{ij}[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \ge \epsilon\} \in I.$$

Therefore for $(i,j) \notin A$, we have

$$u_{ij}[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} < \epsilon^H,$$

$$u_{ij}[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} < \epsilon.$$

So by inequality (2.9), we have

$$F[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \leq \delta_0,$$

$$[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} < max\{\delta_0^h, \delta_0^H\} < \delta,$$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \geq \delta\} \in I.$$

This implies that $x = \in_2 BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$. Hence ${}_2BV_\sigma^I[u, p, \|\cdot,\cdot\|] \subset_2 BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$. The other part can be proved similarly. The inclusion is strict as for the reverse inclusion we need the condition given in the next theorem.

Theorem 2.4 Let $F = (f_{ij})$ be a sequence of moduli if $\lim_t sup(f_{ij}(t)/t) = A > 0$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, then

$${}_2BV_\sigma^I[u, p, \|\cdot,\cdot\|] = {}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|],$$

$${}_2({}_0BV_\sigma^I[u, p, \|\cdot,\cdot\|]) = {}_2({}_0BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]).$$

Proof. (i) To prove ${}_2BV_\sigma^I[u, p, \|\cdot,\cdot\|] = {}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$, it is sufficient to show that ${}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|] \subset {}_2BV_\sigma^I[u, p, \|\cdot,\cdot\|]$. Let $x \in_2 BV_\sigma^I[F, u, p, \|\cdot,\cdot\|]$, then, by definition, we get

$$u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} < \delta. \tag{2.10}$$

By the given condition $\lim_t \sup(f_{ij}(t)/t) = A > 0$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, we have $f_{ij} \geq At$ for all (i,j), that is,

$$F(\| \varphi_{mkij}(x) - L, z \|) \geq A(\| \varphi_{mkij}(x) - L, z \|).$$

$$u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \geq A^H[u_{ij}(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}}. \tag{2.11}$$

from inequalities (2.10) and (2.11), we get

$$\delta > u_{ij}[F(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \geq A^H[u_{ij}(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}}. \tag{2.12}$$

which consequently implies that

$$u_{ij}[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} < \delta.$$

That is,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: u_{ij}[(\| \varphi_{mkij}(x) - L, z \|)]^{p_{ij}} \geq \delta\} \in I. \tag{2.13}$$

This implies that $x \in_2 BV_\sigma^I[u, p, \|\cdot,\cdot\|]$. Hence ${}_2BV_\sigma^I[F, u, p, \|\cdot,\cdot\|] \subseteq_2 BV_\sigma^I[u, p, \|\cdot,\cdot\|]$. Hence, from the previous theorem and inclusion (2.13), we get the required result. The other part can be proved similarly.

Corollary Let $F^1 = (f_{ij}^1)$ *and* $F^2 = (f_{ij}^2)$ be a sequence of moduli if $lim_t sup(f_{ij}(t)/t) < \infty$ *for all* $(i,j) \in \mathbb{N} \times \mathbb{N}$, then

$$_2BV_\sigma^I[F^1, u, p, \|\cdot,\cdot\|] = {}_2BV_\sigma^I[F^2, u, p, \|\cdot,\cdot\|],$$

$$_2({}_0BV_\sigma^I[F^1, u, p, \|\cdot,\cdot\|]) = {}_2({}_0BV_\sigma^I[F^2, u, p, \|\cdot,\cdot\|]).$$

Theorem 2.5 Let $F^1 = (f_{ij}^1)$ and $F^2 = (f_{ij}^2)$ be sequence of moduli, then

$$Z[F^2, u, p, \|\cdot,\cdot\|]) \subset Z[F^1F^2, u, p, \|\cdot,\cdot\|].$$

$$Z[F^1, u, p, \|\cdot,\cdot\|] \cap Z[F^2, u, p, \|\cdot,\cdot\|] \subset Z[(F^1 + F^2), u, p, \|\cdot,\cdot\|],$$

$$\text{for } Z =_2 BV_\sigma^I,\ {}_2({}_0BV_\sigma^I),\ {}_2({}_0\mathcal{M}_{BV_\sigma}^I),\ {}_2\mathcal{M}_{BV_\sigma}^I.$$

Proof. For some $\epsilon > 0$, we choose $\epsilon_o > 0$ such that $\max\epsilon_0^H < \epsilon$. Now as F^1 is a sequence of moduli, we can choose $\delta \in (0,1)$ such that, for every $t \in (0,\delta)$ we get $F^1(t) < \epsilon_o$. Let $x = (x_{ij}) \in_2 BV_\sigma^I[F^2, u, p, \|\cdot,\cdot\|]$. Then by definition, we have

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N}: u_{ij}[(F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} \geq \delta^H\} \in I. \quad (2.14)$$

For $(i,j) \notin A$, we get

$$u_{ij}[(F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} < \delta^H \quad implies \quad u_{ij}[(F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} < \delta.$$

Now, by the continuity of F^1, we have

$$F^1[u_{ij}[F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}}] < \epsilon,$$

which further implies that

$$F^1[u_{ij}[F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}}] < max\epsilon^H < \epsilon,$$

which implies that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: u_{ij}[F^1F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} \geq \epsilon\} \in I.$$

Therefore

$$x \in_2 BV_\sigma^I[F^1F^2, u, p, \|\cdot,\cdot\|].$$

The rest result can be proved similarly.

Let $x \in_2 BV_\sigma^I[F^1, u, p, \|\cdot,\cdot\|] \cap_2 BV_\sigma^I[F^2, u, p, \|\cdot,\cdot\|]$. Then by the definition of both the spaces, we get

$$u_{ij}[F^1(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} < \epsilon,$$

and

$$u_{ij}[F^2(\| \varphi_{kmij}(x) - L, z \|)]^{p_{ij}} < \epsilon,$$

using the fact that $(F^1+F^2)(x) \leq KF^1(x)+KF^2(x)$, we have

$$\begin{aligned} u_{ij}[(F^1+F^2)(\| \varphi_{kmij}(x)-L,z \|)]^{p_{ij}} &\leq K u_{ij}[F^1(\| \varphi_{kmij}(x)-L,z \|)]^{p_{ij}} \\ &+K u_{ij}[F^2(\| \varphi_{kmij}(x)-L,z \|)]^{p_{ij}} \\ &\leq K(\epsilon+\epsilon)=2K(\epsilon)=\epsilon' \ (say). \end{aligned}$$

So we get

$$\{(i,j)\in \mathbb{N}\times\mathbb{N}: u_{ij}[(F^1+F^2)(\| \varphi_{kmij}(x)-L,z \|)]^{p_{ij}} \geq \epsilon\} \in I. \qquad (2.15)$$

Therefore we have $x \in {}_2BV_\sigma^I[(F^1+F^2),u,p,\|\cdot,\cdot\|]$. Hence we have

$${}_2BV_\sigma^I[F^1,u,p,\|\cdot,\cdot\|] \cap {}_2BV_\sigma^I[F^2,u,p,\|\cdot,\cdot\|] \subseteq {}_2BV_\sigma^I[(F^1+F^2),u,p,\|\cdot,\cdot\|]$$

The rest results can be proved similarly.

Theorem 2.6 For any sequence of moduli F and a factorable double sequence of strictly positive real numbers $p=(p_{ij})$, the spaces ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and ${}_2({}_0\mathcal{M}_{\mathrm{BV}_\sigma}^{\mathrm{I}}[\mathrm{F},\mathrm{u},\mathrm{p},\|\cdot,\cdot\|])$ are solid and monotone.

Proof. Let us consider ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and for ${}_2({}_0\mathcal{M}_{BV_\sigma}^I[F,u,p,\|\cdot,\cdot\|])$ the proof will be similar. Let $x=(x_{ij}) \in {}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ be an arbitrary element, then

$$I-\lim\ u_{ij}[F(\| \varphi_{mkij}(x),z \|)]^{p_{ij}}=0.$$

Let α_{ij} be a double sequence of scalars with $|\alpha_{ij}|\leq 1$ for $i,j\in\mathbb{N}$. Now, F is a sequence of moduli. Therefore

$$\begin{aligned} u_{ij}[F(\| \alpha_{ij}\varphi_{mkij}(x),z \|)]^{p_{ij}} &= u_{ij}[F(\| \alpha_{ij}\varphi_{mkij}(x),z \|)]^{p_{ij}} \\ &\leq u_{ij}[|\alpha_{ij}|^{p_{ij}}F(\| \varphi_{mkij}(x),z \|)]^{p_{ij}} \end{aligned}$$

$$\Rightarrow u_{ij}[F(\| \alpha_{ij}\varphi_{mkij}(x),z \|)]^{p_{ij}} \leq u_{ij}[F(\| \varphi_{mkij}(x),z \|)]^{p_{ij}}, \qquad \text{for all } i,j\in\mathbb{N}$$

$$\Rightarrow \qquad I-\lim u_{ij}[F(\| \alpha_{ij}\varphi_{mkij}(x),z \|)]^{p_{ij}}=0.$$

Thus we have $\alpha_{ij}x_{ij} \in {}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$. Hence ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ is solid. Therefore by lemma 1.1 the space ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ is monotone. The rest results can be proved similarly.

Theorem 2.7 For any sequence of moduli F and a factorable double sequence of strictly positive real numbers $p=p_{ij}$, the spaces ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ are not convergence free.

Proof. To show this, let $I=I_f$ and $F(x)=x$ for all $x\in[0,\infty)$. Now consider the double sequences (x_{ij}) and (y_{ij}) which is defined as follows:

$$x_{ij}=\frac{1}{i+j} \qquad \text{and} \qquad y_{ij}=i+j, \quad \text{for all } i,j\in\mathbb{N}.$$

Then we have (x_{ij}) belong to both ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$, but (y_{ij}) does not belong to ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$. Hence, the spaces ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ and ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ are not convergence free.

Theorem 2.8 The spaces ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ and ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ are sequence algebra.

Proof. Let $x=(x_{ij}), y=(y_{ij}) \in_2 ({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ be any two arbitrary elements, such that

$$I-\lim_{i,j} u_{ij}[F(\| \varphi_{mkij}(x), z \|)]^{p_{ij}} = 0,$$

$$I-\lim_{i,j} u_{ij}[F(\| \varphi_{mkij}(y), z \|)]^{p_{ij}} = 0.$$

$$u_{ij}[F(\| \varphi_{mkij}(xy), z \|)]^{p_{ij}} \leq u_{ij}[F(\| \varphi_{mkij}(x)\varphi_{mkij}(y), z \|)]^{p_{ij}}$$

Therefore,

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: u_{ij}[F(\| \varphi_{mkij}(xy), z \|)]^{p_{ij}} \geq \epsilon\} \in I.$$

$$\Rightarrow \qquad I-\lim_{i,j} u_{ij}[F(\| \varphi_{mkij}(x)\varphi_{mkij}(y), z \|)]^{p_{ij}} = 0.$$

Therefore we have $(x_{ij}y_{ij}) \in_2 ({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$. Hence ${}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|])$ is sequence algebra. The rest results can be proved similarly.

Theorem 2.9 For any sequence of moduli F and a factorable double sequence of strictly positive real numbers $p=(p_{ij})$, the spaces ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ and ${}_2\mathcal{M}_{BV_\sigma}^I[F,u,P,\|\cdot,\cdot\|]$ are neither solid nor monotone in general.

Proof. Here we give counter example for establishment of this result. Let $X =_2 BV_\sigma^I$ and ${}_2\mathcal{M}_{BV_\sigma}^I$. Let us consider $I=I_f$ and $F(x)=x$, for all $x=(x_{ij}) \in [0,\infty)$. Consider, the K–step space $X_K[F,u,p,\|\cdot,\cdot\|]$ of $X[F,u,p,\|\cdot,\cdot\|]$ defined as follows:

Let $x=(x_{ij}) \in X[F,u,p,\|\cdot,\cdot\|]$ and $y=(y_{ij}) \in X_K[F,u,p,\|\cdot,\cdot\|]$ be such that

$$y_{ij} = \begin{cases} x_{ij}, & if\ \ i,j\ \ is\ \ even \\ 0, & otherwise\,. \end{cases}$$

Consider the double sequence $(x_{i,j})$ defined by $(x_{ij})=1$ for all $i,j\in\mathbb{N}$. Then $x=(x_{ij}) \in_2 BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ and ${}_2\mathcal{M}_{BV_\sigma}^I[F,u,p,\|\cdot,\cdot\|]$, but K–step space pre–image does not belong to ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ and ${}_2\mathcal{M}_{BV_\sigma}^I[F,u,p,\|\cdot,\cdot\|]$. Thus ${}_2BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]$ and ${}_2\mathcal{M}_{BV_\sigma}^I[F,u,p,\|\cdot,\cdot\|]$ are not monotone and hence they are not solid.

6. CONCLUSIONS

In this present paper we have defined and introduced some spaces of I–convergent double sequences of σ–bounded variation, that is

$$BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]),\ {}_2({}_0BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]),\ {}_2({}_\infty BV_\sigma^I[F,u,p,\|\cdot,\cdot\|]),\ {}_2({}_\infty BV_\sigma[F,u,p,\|\cdot,\cdot\|])$$

over 2–normed spaces by using different operators. In addition, we studied some basic topological and algebraic properties of these spaces. These definitions and results provide new tools to deal with the convergence problems of double sequences occurring in many branches of science and engineering.

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