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EXTENSION OF VARIATIONAL INEQUALITIES WITH GENERALIZED WEAKLY RELAXED α-SEMI-PSEUDOMONOTONE MAPPING

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Abstract. *In this paper, we have introduced a new kind of variational inequality and offered its solution by proposing generalized weakly relaxed α-pseudomonotonicity and generalized weakly relaxed α-semi-pseudomonotonicity of set-valued map. The KKM theorem has been instrumental in proving the existence results. We also present the solvability of our proposed problem with weakly relaxed α-semi-pseudomonotone set-valued maps in arbitrary Banach spaces using Kakutani-Fan-Glicksberg fixed point theorem.*

Keywords: *generalized weakly relaxed α-pseudomonotone, generalized weakly relaxed α-semi-pseudomonotone, f-hemicontinuity, Fan-KKM theorem, f-coercivity.*

1. INTRODUCTION

The role of Monotonicity in variational inequality is the same as that of convexity in optimization theory, and hence it becomes imperative to study monotonicity. In variational inequality problems, optimization problems etc. monotonicity of map has very important place.

In past years, many authors [1-10] observed some vital generalizations of monotone maps such as quasi monotone maps, pseudomonotone maps, strict pseudomonotone maps etc. Earlier, Chen [2], Verma [8] and Fang & Huang [11] studied and introduced some new generalizations of previous existing monotone maps in different spaces and gave some results for their problems by using their generalized monotone maps.

Later Kassay & Kolumban [12] gave some results for variational inequalities with semi-pseudomonotone maps. Results of Chang, Lee & Chen [1], Chen [2], Fang & Huang [11], Kassay & Kolumban [12] and Lee & Lee [13] spiritualized & prompted us and we introduce Generalized weakly relaxed α-pseudomonotonicity and Generalized weakly relaxed α-semi-pseudomonotonicity of set-valued maps. We have obtained existence of solutions of variational inequalities with generalized weakly relaxed α-pseudomonotone set-valued maps in reflexive Banach space by using KKM technique. By using Kakutani-Fan-Glicksberg fixed point theorem we also have proved some results for variational inequality with generalized weakly relaxed α-semi-pseudomonotone set-valued maps in arbitrary Banach spaces.

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2. GENERALIZED WEAKLY RELAXED α-PSEUDOMONOTONE SET-VALUED VARIATIONAL INEQUALITIES

Throughout this paper we consider E as real reflexive Banach space with its topological dual E^*.

2.1. DEFINITION

Let K be a nonempty subset of E. A set valued map $T: K \to 2^{E^*}$ is said to be generalized weakly relaxed α-pseudomonotone if there exists a function $\alpha: E \to \mathbb{R}$ with $\alpha(sw) = k(s).\alpha(w)$ and $s \in (0,1)$, where k is a function from $(0,1)$ to $(0,1)$ with $\lim_{s\to 0} \frac{k(s)}{s} = 0$, such that for every $x, y \in K$, any fixed $z \in K$ and for every $u \in T(x)$, we have

$$\langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0$$

Implies

$$\langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \alpha(x-y),$$

where $f: K \times K \to E$, $P: E^* \to E^*$ and $h: K \times K \to \mathbb{R}$ are maps.

2.2. DEFINITION

Let K be a nonempty convex subset of $E, T: K \to 2^{E^*}$, $f: K \times K \to E$, $P: E^* \to E^*$ are maps. T is said to be *f*-hemicontinuous if for any fixed $x, y \in K$ the map

$$s \mapsto \langle P(T(x+(y-x))), f((tx+(1-t)z), y)\rangle, 0 < s < 1$$

is upper semicontinuous at 0^+.

2.3. DEFINITION

Let K be a nonempty subset of E, $T: K \to 2^{E^*}$, $P: E^* \to E^*$ and $f: K \times K \to E$ are three maps and $h: K \times K \to \mathbb{R} \cup \{\infty\}$ be a proper function. T is said to be *f*-coercive with respect to h if there exists an $x_0 \in K$ such that

$$\frac{\langle P(u_r) - P(u_0), f(x_0, (tx_r + (1-t)z))\rangle + h(x_r, x_0)}{\| f((tx_r + (1-t)z), x_0)\|} \to \infty$$

whenever $\|x\| \to \infty$ for all $u \in T(x)$ and $u_0 \in T(x_0)$.

2.4. THEOREM

Let K be a convex subset of E, $T: K \to 2^{E^*}$ be an *f*-hemicontinuous and generalized weakly relaxed α-pseudomonotone set valued map, and $h: K \times K \to \mathbb{R} \cup \{\infty\}$ be a proper function. Assume that

(i) for fixed $v \in E^*, x \mapsto \langle P(v), f(., x)\rangle + h(., x)$ is convex, and

(ii) $f((tx+(1-t)z), x) = \bar{0}$, and $h(x,x) = 0$ for $x \in K$

Then the following variational inequalities (1) and (2) are equivalent.

Find $x \in K$ and $u \in T(x)$ such that for each $y \in K$

$$\langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0 \tag{1}$$

Find $x \in K$ and $v \in T(y)$ such that for each $y \in K$

$$\langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \alpha(x-y) \tag{2}$$

Proof: By generalized weakly relaxed α-pseudomonotonicity of T, (1) implies (2).

Conversely, Let y be a point of K such that for $x \in K$ with $h(x,y) < \infty$ there exists $v \in T(y)$ satisfying $\langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \alpha(x-y)$.

Put $y_s = (1-s)x + sy$, $s \in (0,1)$, then $y_s \in K$. It follows that

$$\langle P(v_s), f((tx+(1-t)z), y_s)\rangle + h(x, y_s) \geq \alpha(x-y_s), \text{ for some } v_s \in T(y_s).$$

Since $\alpha(x-y_s) = \alpha(s(x-y)) = k(s).\alpha(x-y)$, by the condition (i) and (ii) we have

$$s(\langle P(v_s), f((tx+(1-t)z), y)\rangle + h(x,y)) \geq k(s)\alpha(x-y)$$

Hence

$$\langle P(v_s), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \frac{k(s)}{s}\alpha(x-y)$$

Since T is *f*-hemicontinuous and generalized weakly relaxed α-pseudomonotone, by letting $s \to 0$ we have

$$\langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0.$$

for some $u \in T(x)$ and $\forall y \in K$ with $h(x,y) < \infty$. When $h(x,y) = +\infty$, the inequality $\langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0$ is trivial. Therefore $x \in K$ is a solution of (1).

2.5. DEFINITION

Let K be a nonempty subset of a vector space E. A set valued map $T: K \to 2^E$ is said to be KKM-map if for any finite subset N of K, we have

$$co(N) \subset \bigcup_{x \in N} T(x)$$

2.6. THEOREM (Fan-KKM Theorem [14])

Let E be Topological vector space, $K \subset E$ an arbitrary subset, and $T: K \to 2^E$ a KKM-map. If all the sets *T(x)* are closed in E and at least one of them is compact, then $\bigcap\{T(x): x \in K\}$ is nonempty.

2.7. THEOREM

Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E. Let $T: K \to 2^{E^*}$ be an *f*-hemicontinuous and generalized weakly relaxed α-pseudomonotone set-valued map with nonempty compact values, $P: E^* \to E^*$ be a map and $h: K \times K \to \mathbb{R} \bigcup \{\infty\}$ be a proper function. Assume that
(i) for fixed $v \in E^*, x \mapsto \langle P(v), f(.,x)\rangle + h(.,x)$ is convex, weakly lower semicontinuous,
(ii) $f((tx+(1-t)z), y) + f((ty+(1-t)z), x) = \bar{0}$, and $h(x,y)+h(y,x)=0$ for $x, y \in K$, and
(iii) $\alpha: E \to \mathbb{R}$ is weakly lower semicontinuous.
Then problem (1) is solvable.

Proof: Define a set-valued map $A: K \to 2^E$ as

$$A(y) = \{x \in K : \langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0 \text{ for some } u \in T(x)\} \text{ for } y \in K .$$

Then A is KKM-map. In fact, suppose $\exists\ \{y_1, y_2, ..., y_n\}$ in K and $s_i > 0$ with $\sum_{i=1}^{n} s_i = 1$ such that

$$y = \sum_{i=1}^{n} s_i y_i \notin \bigcup_{i=1}^{n} A(y_i)$$

Then $\forall\ v \in T(y)$

$$\langle P(v), f(y_i, y)\rangle + h(y_i, y) < 0, \quad \text{i=1,2,...,n.}$$

It follows that

$$0 = \langle P(v), f(y,y)\rangle + h(y,y)$$

$$= \left\langle P(v), f(\sum_{i=1}^{n} s_i y_i, y)\right\rangle + h(\sum_{i=1}^{n} s_i y_i, y)$$

$$\le \sum_{i=1}^{n} s_i (\langle P(v), f(y_i, y)\rangle + h(y_i, y))$$

$$< 0$$

which is contradiction.

Define another set-valued map $B: K \to 2^E$ by

$$B(y) = \{x \in K: \text{ for some } v \in T(y),\ \langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \ge \alpha(x-y)\} \text{ for } y \in K$$

Then by the relaxed α-pseudomonotonicity of T it follows that B is also a KKM-map. On the other hand, let $\{x_\xi\}$ be a net in $B(y)$ converging weakly to x. Then for some $v_\xi \in T(y)$,

$$\langle P(v_\xi), f((tx_\xi + (1-t)z), y)\rangle + h(x_\xi, y) \ge \alpha(x_\xi - y) \quad \text{for } \xi \in I .$$

Hence *T(y)* is compact, so that we can assume that $\{v_\xi\}$ converges to some $v \in T(y)$. Since $x \mapsto \langle P(v), f(x,.)\rangle + h(x,.)$ is weakly upper semicontinuous, and α is weakly lower semicontinuous, we have

$$\langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \ge \overline{\lim_{\xi}}(\langle P(v_\xi), f((tx_\xi + (1-t)z), y)\rangle + h(x_\xi, y))$$

$$\ge \underline{\lim}_{\xi}(\langle P(v_\xi), f((tx_\xi + (1-t)z), y)\rangle + h(x_\xi, y))$$

$$= \underline{\lim}_{\xi}\ \alpha(x_\xi - y)$$

$$= \alpha(x-y)$$

It follows that $x \in B(y)$ and $B(y)$ is weakly closed $\forall\, y \in K$. Since K is bounded closed and convex, K is weakly compact, So $B(y)$ is weakly compact in K $\forall\, y \in K$. It follows from Theorem 2.4 and Fan-KKM theorem that

$$\bigcap_{y\in K} A(y) = \bigcap_{y\in K} B(y) \ne \phi$$

Hence $\exists$ an $x \in K$ such that $\forall\ y \in K,\ \exists\ u \in T(x)$ satisfying

$$\langle P(u), f((tx+(1-t)z), y)\rangle + h(x, y) \geq 0.$$

2.8. THEOREM

Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E. Let $T: K \to 2^{E^*}$ be an *f*-hemicontinuous and generalized relaxed α -pseudomonotone set-valued map, $P: E^* \to E^*$ be a map and $h: K \times K \to \mathbb{R} \bigcup \{\infty\}$ be a proper function. Assume that

(i) for fixed $v \in E^*, x \mapsto \langle P(v), f(.,x)\rangle + h(.,x)$ is convex, weakly lower semicontinuous,

(ii) $f((tx+(1-t)z), y) + f((ty+(1-t)z), x) = \overline{0}$, and $h(x, y) + h(y, x) = 0$ for $x, y \in K$, and

(iii) $\alpha: E \to \mathbb{R}$ is weakly lower semicontinuous.

If T is *f*-coercive with respect to *h*, then problem (1) is also solvable.

Proof: Let $C_r = \{y \in E : \|y\| \leq r\}$ and consider the following problem;

Find $x_r \in K \bigcap C_r$ such that $\forall$ y $\in K \bigcap C_r$, $\exists\ u_r \in T(x_r)$ satisfying

$$\langle P(u_r), f((tx_r + (1-t)z), y)\rangle + h(x_r, y) \geq 0 \tag{3}$$

By Theorem 2.7, problem (3) has a solution $x_r \in K \bigcap C_r$. Choose $r > \|x_0\|$ with x_0 in the coercivity condition of definition 2.3. Since $x_r \in K \bigcap C_r$, we have $u_r \in T(x_r)$ satisfying

$$\langle P(u_r), f((tx_r + (1-t)z), x_0)\rangle + h(x_r, x_0) \geq 0 \tag{4}$$

Moreover

$$\langle P(u_r), f((tx_r + (1-t)z), x_0)\rangle + h(x_r, x_0)$$

$$= -\langle P(u_r) - P(u_0), f(x_0, (tx_r + (1-t)z))\rangle + h(x_r, x_0) + \langle P(u_0), f((tx_r + (1-t)z), x_0)\rangle$$

$$\leq -\langle P(u_r) - P(u_0), f(x_0, (tx_r + (1-t)z))\rangle + h(x_r, x_0) + \|P(u_0)\| \|f((tx_r + (1-t)z), x_0)\|$$

$$\leq \|f((tx_r + (1-t)z), x_0)\| \left(-\frac{\langle P(u_r) - P(u_0), f(x_0, (tx_r + (1-t)z))\rangle + h(x_r, x_0)}{\|f((tx_r + (1-t)z), x_0)\|} + \|P(u_0)\|\right)$$

for $u_0 \in T(x_0)$.

Now if $\|x_r\| = r\ \forall\ r$, we can choose r large enough so that the above inequality and the *f*-coercivity of T with respect to h imply that

$$\langle P(u_r), f((tx_r + (1-t)z), x_0)\rangle + h(x_r, x_0) < 0,$$

which contradicts the inequality (4).

Hence $\exists\, r$ such that $\|x_r\| < r$. For any $y \in K$, we can choose $\delta \in (0,1)$ small enough such that $x_r + \delta(y - x_r) \in K \cap C_r$. It follows from (4) that

$$\langle P(u_r), f((tx_r + (1-t)z), x_r + \delta(y - x_r))\rangle + h(x_r, x_r + \delta(y - x_r)) \geq 0 \quad \text{for some } u_r \in T(x_r).$$

By the conditions (i) and (ii), we have

$$\langle P(u_r), f((tx_r + (1-t)z), y)\rangle + h(x_r, y) \geq 0.$$

Thus $x_r \in K$ is a solution of (1).

2.9 REMARK

Known results in Goeleven & Motreanu [5], Hadjisavvas & Schaible [6] and corresponding results in Kang, Huang & Lee [7], Verma [8], Fang & Huang [11], Lee & Lee [13] and Siddiqi, Ansari & Kazmi [15] are the special cases of my proposed results.

3. GENERALIZED WEAKLY RELAXED α-SEMI-PSEUDOMONOTONE SET-VALUED VARIATIONAL INEQUALITIES

For semi-monotone single-valued map $F : K \times K \to E^*$ Chen [2] considered the variational inequality (VI-1) given below

(VI-1) Find $x \in K$ such that

$$\langle F(x,x), y - x\rangle \geq 0 \qquad \forall\, y \in K.$$

where K is bounded closed convex subset of second dual of E i.e. E^*.

And also for semi-monotone single-valued map $F : K \times K \to E^*$ Fang & Huang [11] presented the generalized variational-like inequality (VI-2) given below

(VI-2) Find an $x \in K$ such that

$$\langle F(x,x), \eta(y,x)\rangle + h(y) - h(x) \geq 0 \qquad \forall\, y \in K,$$

where $\eta : K \times K \to E^{**}$ and h:K $\to \mathbb{R} \cup \{\infty\}$ are maps.

And further, In 2000, for semi-pseudomonotone set-valued maps $F : K \times K \to 2^E$, Kassay & Kolumban [12] proved some results for following variational inequalities (VI-3) and (VI-4).

(VI-3) Find $x \in K$ such that

$$\sup_{u \in F(x,x)} \langle u, y-x \rangle \geq 0 \qquad \forall\, y \in K$$

(VI-4) Find an element $x \in K$ such that

$$\sup_{u \in F(y,x)} \langle u, y-x \rangle \geq 0 \qquad \forall\, y \in K$$

They showed that (VI-3) and (VI-4) are eqauivalent and they obtained the existence results.

After that, for weakly relaxed α-semi-pseudomonotone set-valued map $T : K \to 2^{E^*}$ Lee & Lee [13] considered the variational inequalities (VI-5) and (VI-6) and after showing the equaivalence between them they have proved some results for (VI-5) and (VI-6) described below

(VI-5) Find $x \in K$ such that $\forall\, y \in K,\ \exists\, u \in T(x)$ satisfying

$$\langle u, \eta(y,x) \rangle + f(y,x) \geq 0\,.$$

(VI-6) Find $x \in K$ such that $\forall\, y \in K,\ \exists\, v \in T(y)$ satisfying

$$\langle v, \eta(y,x) \rangle + f(y,x) \geq \alpha(y-x)\,.$$

where $\eta : K \times K \to E$ and $f : K \times K \to \mathbb{R}$ are maps.

In this section, we consider the existence of solutions to the following variational inequality for a generalized weakly relaxed α-semi-pseudomonotone set-valued map $F : K \times K \to 2^{E^*}$, where K is a nonempty closed convex subset of E^{**}.

Find $x \in K$ such that $\forall\, y \in K,\ \exists\, u \in F(x,x)$ satisfying

$$\langle P(u), f((tx + (1-t)z), y) \rangle + h(x,y) \geq 0 \qquad (5)$$

3.1. DEFINITION

Let $f : K \times K \to E^{**}$, $P : E^{*} \to E^{*}$ and $h : K \times K \to \mathbb{R}$ are maps. Let K be a nonempty subset of E^{**}. A set-valued map $F : K \times K \to 2^{E^*}$ is said to be generalized weakly relaxed α-semi-pseudomonotone if the following conditions hold;

(a) for each fixed $p \in K,\ F(p,.) : K \to 2^{E^*}$ is generalized weakly relaxed α-pseudomonotone map, i.e., $\exists$ a function $\alpha : E^{**} \to \mathbb{R}$ with $\alpha(sw) = k(s).\alpha(w)$ for $w \in E^{**}$, where k is a function from $(0,1)$ to $(0,1)$ with $\lim\limits_{s \to 0} \dfrac{k(s)}{s} = 0$, such that for every $x, y \in K$, any fixed $z \in K$ and for every $u \in F(p,x)$, we have

$$\langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0$$

Implies

$$\langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \alpha(x-y) \quad \text{for some } v \in T(y),$$

(b) for each fixed $y \in K$, $F(.,y)$ is completely continuous, i.e., for any net $\{x_\xi\}$ in E^{**} such that $x_\xi \xrightarrow{*} x_0$, every net v_ξ in E^* with $v_\xi \in F(x_\xi, y)$ has a convergent subnet, of which limit belongs to $F(x_0, y)$ in the norm topology of E^*, where $\xrightarrow{*}$ denotes the weak* convergence in E^{**}.

3.2. THEOREM

Let E be a real Banach space and K a nonempty bounded closed convex subset of E^{**}. Let $F: K \times K \to 2^{E^*}$ be a generalized weakly relaxed α-semi-pseudomonotone set-valued map, $P: E^* \to E^*$ be a map and $h: K \times K \to \mathbb{R} \bigcup \{+\infty\}$ a proper function such that

(i) for fixed $v \in E^*, x \mapsto \langle P(v), f(.,x)\rangle + h(.,x)$ is linear, weakly lower semicontinuous,

(ii) $f((tx+(1-t)z), y) + f((ty+(1-t)z), x) = \bar{0}$, and $h(x,y) + h(y,x) = 0$ for $x, y \in K$,

(iii) $\alpha: E \to \mathbb{R}$ is convex, weakly lower semicontinuous, and

(iv) for each $x \in K$, $F(x,.): K \to 2^{E^*}$ is finite dimensional continuous.

Then problem (5) is solvable.

Proof: Let $G \subset E^{**}$ be a finite dimensional subspace with $K_G = K \bigcap G \neq \phi$. $\forall\, p \in K$, we consider the following problem;

Find $x_0 \in K_G$ such that $\forall\, y \in K_G$, $\exists\, u_0 \in F(p, x_0)$ satisfying

$$\langle P(u_0), f((tx_0+(1-t)z), y)\rangle + h(x_0, y) \geq 0 \qquad (6)$$

$\forall\, p \in K_G$, since $F(p,.)$ is generalized weakly relaxed α-pseudomonotone and continuous on a bounded closed convex subset K_G of G, by Theorem 2.7, the problem (6) has a solution x_p in K_G. If we define a set-valued map $T: K_G \to 2^{K_G}$ as follows;

$$T(p) = \{x \in K_G : \text{ for } y \in K_G \ \exists\, u \in F(p,x) \text{ such that } \langle P(u), f((tx+(1-t)z), y)\rangle + h(x,y) \geq 0\},$$

Then *T(p)* is nonempty, since $x_p \in T(p)$. By Theorem 2.4, *T(p)* is equal to the set

$$\{x \in K_G : \text{ for } y \in K_G,\ \exists\, v \in F(p,y) \text{ such that } \langle P(v), f((tx+(1-t)z), y)\rangle + h(x,y) \geq \alpha(x-y)\}.$$

By condition (i), (ii) and (iii), $T : K_G \to 2^{K_G}$ has a nonempty bounded closed and convex set-values, and T is upper semicontinuous by the complete continuity of $F(.,y)$. By Kakutani-Fan-Glicksberg fixed point theorem, T has a fixed point p_0 in K_G, i.e., $\forall\, y \in K_G,\ \exists\, u \in F(p_0, p_0)$ satisfying

$$\langle P(u), f((tp_0 + (1-t)z), y)\rangle + h(p_0, y) \geq 0 \tag{7}$$

Let $\mathfrak{A} = \{G : G$ is a finite dimensional subspace of E^{**} with $K_G \neq \phi\}$ and, for $G \in \mathfrak{A}$,

$H_G := \{x \in K : \langle P(v), f((tx + (1-t)z), y)\rangle + h(x, y) \geq \alpha(x - y)$ for $y \in K_G$, for some $v \in F(x, y)\}$.

By Theorem 2.5 and (7), we know that H_G is nonempty and bounded. Since the weak*closure $\overline{H_G}$ of H_G in E^{**}, for any $G_i \in \mathfrak{A}$ (i=1,2,…,n), we know that $H_{\bigcup_i G_i} \subset \bigcap_i H_{G_i}$ so $\{\overline{H_G} : G \in \mathfrak{A}\}$ has the finite intersection property. Therefore it follows that

$$\bigcup_{G \in \mathfrak{A}} \overline{H_G} \neq \phi .$$

Let $x \in \bigcup_{G \in \mathfrak{A}} \overline{H_G}$. We claim that $\forall\, y \in K$, $\exists\, u \in F(x, x)$ satisfying

$$\langle P(u), f((tx + (1-t)z), y)\rangle + h(x, y) \geq 0 .$$

Indeed $\forall\, p \in K$, let $G \in \mathfrak{A}$ be such that $p \in K_G$ and $x \in K_G$. There exists a net $\{x_\xi\}$ in H_G such that $x_\xi \to x$. It follows that

$$\langle P(v_\xi), f((tx_\xi + (1-t)z), y)\rangle + h(x_\xi, y) \geq \alpha(x_\xi - y) \quad \forall\, y \in K_G, \text{ for some } v_\xi \in F(x_\xi, y) .$$

Hence we have

$$\langle P(v), f((tx + (1-t)z), y)\rangle + h(x, y) \geq \alpha(x - y) \quad \forall\, y \in K, \text{ for some } v \in F(x, y) ,$$

by using the complete continuity of *F(.,y)* and the assumption on h, f and α. From Theorem 2.4 $\forall\, y \in K,\ \exists\, u \in F(x, x)$ satisfying

$$\langle P(u), f((tx + (1-t)z), y)\rangle + h(x, y) \geq 0 .$$

3.3. THEOREM

Let E be a real Banach space and K a nonempty unbounded closed convex subset of E^{**}. Let $F: K \times K \to 2^{E^*}$ be a generalized weakly relaxed α-semi-pseudomonotone set-valued map, $P: E^* \to E^*$ be a map and $h: K \times K \to \mathbb{R} \bigcup \{+\infty\}$ a proper function such that

(i) for fixed $v \in E^*, x \mapsto \langle P(v), f(., x) \rangle + h(., x)$ is linear, lower semicontinuous,

(ii) $f((tx+(1-t)z), y) + f((ty+(1-t)z), x) = \bar{0}$, and $h(x, y) + h(y, x) = 0$ for $x, y \in K$,

(iii) $\alpha: E \to \mathbb{R}$ is convex, weakly lower semicontinuous,

(iv) for each $x \in K,\ F(x,.): K \to 2^{E^*}$ is finite dimensional continuous, and

(v) $\exists$ an $x_0 \in K$ such that

$$\underline{\lim}_{\|x\| \to \infty} \langle P(u), f((tx+(1-t)z), x_0) \rangle + h(x, x_0) > 0$$

$\forall\ x \in K$ and $\forall\ u \in F(x, x)$. Then problem (5) is solvable.

Proof: Let C_r be the closed ball with radius r and center at 0 in E^{**}. By Theorem 3.2, there exists a solution $x_r \in C_r \bigcap K$ such that $\forall\ y \in C_r \bigcap K,\ \exists\ u_r \in F(x_r, x_r)$ satisfying

$$\langle P(u_r), f((tx_r + (1-t)z), y) \rangle + h(x_r, y) \geq 0 .$$

Let r be large enough so that $x_0 \in C_r$, therefore $\exists\ u_r \in F(x_r, x_r)$ such that

$$\langle P(u_r), f((tx_r + (1-t)z), x_0) \rangle + h(x_r, x_0) \geq 0 ,$$

By the condition (v), we know that $\{x_r\}$ is bounded. So we can suppose that $x_r \to x$ as $r \to \infty$. It follows from Theorem 2.4 that $\forall\ v_r \in F(x_r, y)$

$$\langle P(v_r), f((tx_r + (1-t)z), y) \rangle + h(x_r, y) \geq \alpha(x_r - y) \quad \forall\ y \in K .$$

Letting $r \to \infty,$ we have $\forall\ v \in F(x, y)$

$$\langle P(v), f((tx + (1-t)z), y) \rangle + h(x, y) \geq \alpha(x - y) \qquad \forall\ y \in K ,$$

Again by Theorem 2.4 we have for $y \in K,\ \exists\ u \in F(x, x)$ satisfying

$$\langle P(u), f((tx + (1-t)z), y) \rangle + h(x, y) \geq 0 .$$

3.4 REMARK

Results and theorems proved in Chen [2], Kang, Huang & Lee [7] and Fang & Huang [11] are special cases of my proposed results.

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