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APPLICATIONS OF THE FERMI-WALKER DERIVATIVE

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Abstract. *The present paper presents the Fermi-Walker derivative on the spherical indicatrices of a space curve according to the quasi frame. Firstly, we mention about the fundamentals of the Fermi-Walker derivative and parallelism. Then, we give some geometric properties of these concepts according to the quasi frame. Finally, we give the Fermi-Walker derivative on the spherical indicatrices of a space curve according to the quasi frame which is our main subject. Moreover, we give an Appendix in which we present a method about how to solve a homogenous normal linear system of differential equations.*

Keywords: *Spherical Indicatrix, Quasi Frame, Fermi-Walker Derivative.*

1. INTRODUCTION

A frame which is under the effect of linear and rotational acceleration may be described by the Frenet-Serret frame. With the help of the Fermi-Walker transport, the Frenet-Serret curve analysis has been extended from nonnull to null trajectories in a generic space-time. Fermi-Walker transported frames are important in several investigations. Especially, one can show that they are very useful in understanding the properties of timelike circular orbits in stationary axisymmetric space-times and of Fermi-Walker transport along them and it helps visualise the geometry of this family of orbits [1].

On the other hand, Fermi-Walker derivative is essential to understand the geodesics also. Let ∇ be the connection of $\mathbb{R}^n$ and T be the tangent vector field of a curve α in $\mathbb{R}^n$, then if $\nabla_T T = 0$, it means that T is parallel along the curve α. This gives us being geodesic in $\mathbb{R}^n$. Similarly, let M be a hypersurface in $\mathbb{R}^n$, $\overline{\nabla}$ be the connection of M and T be the tangent vector field of a curve α on M, then if T is parallel along the curve α, $\overline{\nabla}_T T = 0$ holds. This gives us being geodesic on the hypersurface M. All lines in $\mathbb{R}^n$ are geodesics. The answer of the question of if all curves in $\mathbb{R}^n$ are geodesics, can be given considering the connection obtained by the Fermi derivative. If α is a curve in $\mathbb{R}^n$ and $\widetilde{\nabla}$ is the Fermi derivative, then $\widetilde{\nabla}_T T = 0$ holds for every curve in $\mathbb{R}^n$, [2]. There are many applications of parallel vector fields in differential geometry. In [1] and [3], the Fermi-Walker derivative and parallelism are studied. For the applications of the Fermi-Walker derivative and parallelism in physics, one can see the references [4] and [5].

Motivated by the importance of Fermi-Walker derivative and parallelism mentioned above and considering the advantages of using the quasi frame instead of other frames, in this paper, we study the Fermi-Walker derivative and the quasi frame together. In Section 2, we give some basic concepts about space curves and the quasi frame and we present the fundamentals of the Fermi-Walker derivative and parallelism and some properties of them. In Section 3, we give some properties of the Fermi-Walker derivative according to the quasi

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frame. In Section 4, we give the Fermi-Walker derivative along the spherical indicatrices of a space curve parameterized with arc-length according to the quasi frame.

2. PRELIMINARIES

In this section, we present some basic concepts on differential geometry of space curves and the Fermi-Walker derivative. First of all, we mention about the quasi frame of a space curve.

Let $\alpha = \alpha(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The quasi frame of α consists of the vectors $\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}$ which are given by

$$\boldsymbol{t_q} = \boldsymbol{t} = \alpha'(s),$$

$$\boldsymbol{n_q} = \frac{\boldsymbol{t} \times \vec{k}}{\|\boldsymbol{t} \times \vec{k}\|},$$

$$\boldsymbol{b_q} = \boldsymbol{t} \times \boldsymbol{n_q},$$

where $\vec{k}$ is the projection vector which can be chosen as $\vec{k} = (1,0,0)$ or $\vec{k} = (0,1,0)$ or $\vec{k} = (0,0,1)$. In this paper, we choose the projection vector $\vec{k} = (0,0,1)$. $\boldsymbol{n_q}$ and $\boldsymbol{b_q}$ are called the quasi normal vector and the quasi binormal vector, respectively, [6].

Let θ be the angle between the principal normal $\boldsymbol{n} = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and the quasi normal $\boldsymbol{n_q}$. The quasi formulas are given by

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix},$$

where k_i are called the quasi curvatures $(1 \leq i \leq 3)$. The quasi curvatures are given by

$$k_1 = \kappa \cos\theta = \langle \boldsymbol{t_q}', \boldsymbol{n_q} \rangle,$$

$$k_2 = -\kappa \sin\theta = \langle \boldsymbol{t_q}', \boldsymbol{b_q} \rangle,$$

$$k_3 = \theta' + \tau = -\langle \boldsymbol{n_q}, \boldsymbol{b_q}' \rangle,$$

where κ is the Frenet curvature and τ is the Frenet torsion of the curve α, [7].

Let (M, g) be a n-dimensional Riemannian manifold and ∇ be an affine connection defined on M. If for every $X, Y, Z \in \chi(M)$ the conditions

$$i)\ \nabla_X Y - \nabla_Y X = [X, Y],$$

$$ii)\ Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

hold, then ∇ is said to be a torsion-free Riemannian connection on M or Levi-Civita connection of M.

Let G be a Lie group, $\alpha: I \subset \mathbb{R} \to G$ be a curve on G, V be the speed vector field of the curve α and W be a vector field along the curve α. Then,

$$\nabla_V W = W - \frac{1}{2}[W, V],$$

where ∇ is Levi-Civita connection of G.

Let $\alpha: I \subset \mathbb{R} \to M$ be a unit-speed curve on a hypersurface M, X be a differentiable vector field which is tangent to the hypersurface M along the curve α and perpendicular to the curve α everywhere. Then, the derivative $\frac{\delta X}{\delta s}$ which is defined by

$$\frac{\delta X}{\delta s} = \nabla_{\boldsymbol{T}} X - \langle \nabla_{\boldsymbol{T}} X, \boldsymbol{T} \rangle \boldsymbol{T},$$

is called the Fermi derivative of the vector field X along the curve α. Here, $\boldsymbol{T}$ is the tangent vector field of the curve α and ∇ is Levi-Civita connection of M, [8],[9].

Definition 2.1. Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length and X be a vector field along the curve α. Then, the Fermi-Walker derivative of X along the curve α is defined as

$$\widetilde{\nabla}_{\boldsymbol{t}} X = \nabla_{\boldsymbol{t}} X - \langle \boldsymbol{t}, X \rangle A + \langle A, X \rangle \boldsymbol{t}.$$

Here, $\boldsymbol{t}$ is the tangent vector field of the curve α and $A = \nabla_{\boldsymbol{t}} \boldsymbol{t}$, [2].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length and X be a vector field along the curve α. Then, the Fermi-Walker derivative of X along the curve α can be expressed as

$$\widetilde{\nabla}_{\boldsymbol{t}} X = \nabla_{\boldsymbol{t}} X - \kappa(\boldsymbol{b} \times X),$$

where $\boldsymbol{b}$ is the binormal vector field and κ is the Frenet curvature of the curve α. As a result, the necessary and sufficient condition for the Fermi-Walker derivative to coincide with the Euclidean derivative is $X = \lambda \boldsymbol{b}$, where $\lambda \in \mathbb{R}$.

Definition 2.2. Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length and X be a vector field along the curve α. If $\widetilde{\nabla}_{\boldsymbol{t}} X = 0$, then X is called Fermi-Walker parallel along the curve α, [2].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be the Frenet frame of α. The vector ϖ which has the following properties,

$$\widetilde{\nabla}_{\boldsymbol{t}} \boldsymbol{t} = \varpi \times \boldsymbol{t},$$

$$\widetilde{\nabla}_{\boldsymbol{t}} \boldsymbol{n} = \varpi \times \boldsymbol{n},$$

$$\widetilde{\nabla}_{t} \boldsymbol{b} = \varpi \times \boldsymbol{b},$$

is $\varpi = \tau \boldsymbol{t}$ where τ is the Frenet torsion of the curve α. The vector ϖ is called the Darboux vector in the sense of Fermi-Walker according to the Frenet frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$.

Definition 2.3. Let $\alpha: I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve parameterized with arc-length and $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$ be a orthogonal frame along the curve α. If the Fermi-Walker derivatives of the vector fields $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ vanish, then $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$ is called a non-rotating frame, [2].

3. THE FERMI-WALKER DERIVATIVE AND THE QUASI FRAME

The Fermi-Walker derivative is used in physics and has many applications in this area. Sometimes, Levi-Civita parallelism is not useful and we have to use Fermi-Walker parallelism. For example, an object is freely falling and there is an accelerated observer. Then, in order to define constant directions, we have to use Fermi-Walker parallelism [10]. Considering the advantages of using quasi frame instead of other frames, we study the Fermi-Walker derivative and the quasi frame together.

In this section, we give some properties of the Fermi-Walker derivative according to the quasi frame. Let $\alpha: I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve parameterized with arc-length and X be a vector field along the curve α. Then, the Fermi-Walker derivative of X along the curve α which is defined at Definition 2.1 can be written according to the quasi frame as

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X - \langle \boldsymbol{t_q}, X \rangle A + \langle A, X \rangle \boldsymbol{t_q},$$

where $A = \nabla_{\boldsymbol{t_q}} \boldsymbol{t_q}$.

Theorem 3.1. Let $\alpha: I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve parameterized with arc-length and X be a vector field along the curve α. Then, the Fermi-Walker derivative of X along the curve α can be expressed as

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X - k_1 (\boldsymbol{b_q} \times X) + k_2 (\boldsymbol{n_q} \times X).$$

Proof. Using the definition of the Fermi-Walker derivative and the quasi formulas, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X - \langle \boldsymbol{t_q}, X \rangle (k_1 \boldsymbol{n_q} + k_2 \boldsymbol{b_q}) + \langle k_1 \boldsymbol{n_q} + k_2 \boldsymbol{b_q}, X \rangle \boldsymbol{t_q}.$$

In the light of the properties of the inner product, we write

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X - k_1 (\langle \boldsymbol{t_q}, X \rangle \boldsymbol{n_q} - \langle X, \boldsymbol{n_q} \rangle \boldsymbol{t_q}) - k_2 (\langle \boldsymbol{t_q}, X \rangle \boldsymbol{b_q} - \langle X, \boldsymbol{b_q} \rangle \boldsymbol{t_q}).$$

With the help of the property

$$(u \times v) \times w = \langle w, u \rangle v - \langle w, v \rangle u$$

of the cross-product, we obtain

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X - k_1 (\boldsymbol{b_q} \times X) + k_2 (\boldsymbol{n_q} \times X).$$

Corollary 3.2. The Fermi-Walker derivative coincides with the Euclidean derivative if and only if $X = \lambda (k_2 \boldsymbol{n_q} - k_1 \boldsymbol{b_q})$, where $\lambda \in \mathbb{R}$.

Proof.

$$\widetilde{\nabla}_{\boldsymbol{t_q}} X = \nabla_{\boldsymbol{t_q}} X \Leftrightarrow -k_1(\boldsymbol{b_q} \times X) + k_2(\boldsymbol{n_q} \times X) = 0$$

$$\Leftrightarrow (k_2 \boldsymbol{n_q} - k_1 \boldsymbol{b_q}) \times X = 0$$

$$\Leftrightarrow X = \lambda(k_2 \boldsymbol{n_q} - k_1 \boldsymbol{b_q}).$$

4. THE FERMI-WALKER DERIVATIVE AND THE SPHERICAL INDICATRICES OF A SPACE CURVE

In this section, we give the Fermi-Walker derivative along the spherical indicatrices of a space curve parameterized with arc-length according to the quasi frame. Firstly, we recall the definition of the spherical indicatrices of a space curve according to the quasi frame of the curve.

Definition 4.1. Let $\alpha = \alpha(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The following space curves lie on a unit sphere

$$\beta(s) = \boldsymbol{t_q}(s),$$

$$\gamma(s) = \boldsymbol{n_q}(s),$$

$$\delta(s) = \boldsymbol{b_q}(s)$$

and they are called the spherical indicatrix of the tangent, the quasi normal and the quasi binormal to the curve, respectively, [11].

4.1. THE FERMI-WALKER DERIVATIVE AND THE TANGENT INDICATRIX

In this subsection, we give the Fermi-Walker derivative along the tangent indicatrix of a space curve parameterized with arc-length according to quasi frame. Let s be the arc-length parameter of a space curve $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ and s_β be the arc-length parameter of the tangent indicatrix $\beta = \boldsymbol{t_q}(s)$ of the curve α, i.e. $\beta(s_\beta) = \boldsymbol{t_q}(s)$. Firstly, we give a theorem which is about the relationship between the Frenet frame of the tangent indicatrix of a space curve and the quasi frame of this curve.

Theorem 4.1.1. Let $\{\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$ be the quasi frame of the curve $\alpha = \alpha(s)$ parameterized with arc-length and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be the Frenet frame of the curve $\beta(s) = \boldsymbol{t_q}(s)$. The quasi curvatures of the curve α are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve β are denoted by κ and τ, respectively. Then, the Frenet elements of β can be given in terms of the quasi elements of α as follows [11]:

$$\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_2}{\sqrt{k_1^2+k_2^2}} \\ \frac{A_1}{\sqrt{U_1}} & \frac{B_1}{\sqrt{U_1}} & \frac{C_1}{\sqrt{U_1}} \\ \frac{K_1}{\sqrt{V_1}} & \frac{L_1}{\sqrt{V_1}} & \frac{M_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix},$$

$$\kappa = (1 + \frac{K_1}{(k_1^2+k_2^2)^3})^{\frac{1}{2}},$$

$$\tau = \frac{W_1}{V_1}.$$

Theorem 4.1.2. The Fermi-Walker derivative along the tangent indicatrix β of a space curve α parameterized with arc-length can be expressed as

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)} X = \frac{1}{\sqrt{k_1^2+k_2^2}} \left[\nabla_{\boldsymbol{t_q}} X + (X \times W) + \frac{k_1 k_2' - k_1' k_2}{k_1^2+k_2^2} (X \times \boldsymbol{t_q}) \right],$$

where $W = k_3 \boldsymbol{t_q} - k_2 \boldsymbol{n_q} + k_1 \boldsymbol{b_q}$ is the Darboux vector of the curve α according to the quasi frame and $k_1(s), k_2(s), k_3(s)$ are the quasi curvatures of the curve α at s.

Proof. By the definition of the Fermi-Walker derivative, we can write

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)} X = \nabla_{\boldsymbol{t_q}(s_\beta)} X - \langle \boldsymbol{t_q}(s_\beta), X \rangle \nabla_{\boldsymbol{t_q}(s_\beta)} \boldsymbol{t_q}(s_\beta) + \langle \nabla_{\boldsymbol{t_q}(s_\beta)} \boldsymbol{t_q}(s_\beta), X \rangle\, \boldsymbol{t_q}(s_\beta).$$

With the help of the property

$$(u \times v) \times w = \langle w, u \rangle v - \langle w, v \rangle u$$

of the cross-product, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)} X = \nabla_{\boldsymbol{t_q}(s_\beta)} X + X \times \left(\boldsymbol{t_q}(s_\beta) \times \nabla_{\boldsymbol{t_q}(s_\beta)} \boldsymbol{t_q}(s_\beta) \right).$$

Using the following equalities:

$$\boldsymbol{t_q}(s_\beta) = \frac{k_1 n_q + k_2 b_q}{\sqrt{k_1^2+k_2^2}},$$

$$\frac{ds_\beta}{ds} = \sqrt{k_1^2 + k_2^2}\,,$$

$$\nabla_{\boldsymbol{t_q}(s_\beta)} \boldsymbol{t_q}(s_\beta) = -\boldsymbol{t_q} - \frac{k_2 K}{(k_1^2+k_2^2)^2} \boldsymbol{n_q} + \frac{k_1 K}{(k_1^2+k_2^2)^2} \boldsymbol{b_q},$$

where $K = k_1 k_2' - k_1' k_2 + k_1^2 k_3 + k_2^2 k_3$, we obtain

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)} X = \nabla_{\boldsymbol{t_q}(s_\beta)} X + X \times (\frac{W}{\sqrt{k_1^2+k_2^2}} + \frac{k_1 k_2' - k_1' k_2}{(k_1^2+k_2^2)^{\frac{3}{2}}} \boldsymbol{t_q})\ .$$

Then, in the light of the properties of the cross-product, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)}X = \frac{1}{\sqrt{k_1^2+k_2^2}}\left[\nabla_{\boldsymbol{t_q}}X + (X \times W) + \frac{k_1k_2'-k_1'k_2}{k_1^2+k_2^2}(X \times \boldsymbol{t_q})\right].$$

Theorem 4.1.3. Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length, β be the tangent indicatrix of the curve α and $X = \lambda_1\boldsymbol{t_q} + \lambda_2\boldsymbol{n_q} + \lambda_3\boldsymbol{b_q}$ be a vector field along the curve β, where λ_1, λ_2 and λ_3 are continuously differentiable functions of the parameter s. Then, the vector field X is Fermi-Walker parallel along the curve β if and only if

$$\lambda_1(s) = constant,$$

$$\lambda_2(s) = c_1\cos\left(\int_1^s f(s)ds\right) - c_2\sin\left(\int_1^s f(s)ds\right),$$

$$\lambda_3(s) = c_2\cos\left(\int_1^s f(s)ds\right) + c_1\sin\left(\int_1^s f(s)ds\right),$$

where c_1, c_2 are arbitrary real constants and $f = \frac{k_1k_2'-k_1'k_2}{k_1^2+k_2^2}$.

Proof. We use Theorem 4.1.2. One can easily get the following equalities,

$$X \times W = (\lambda_2k_1 + \lambda_3k_2)\boldsymbol{t_q} + (\lambda_3k_3 - \lambda_1k_1)\boldsymbol{n_q} - (\lambda_1k_2 + \lambda_2k_3)\boldsymbol{b_q},$$

$$X \times \boldsymbol{t_q} = \lambda_3\boldsymbol{n_q} - \lambda_2\boldsymbol{b_q},$$

$$\nabla_{\boldsymbol{t_q}}X = (\lambda_1' - \lambda_2k_1 - \lambda_3k_2)\boldsymbol{t_q} + (\lambda_2' + \lambda_1k_1 - \lambda_3k_3)\boldsymbol{n_q} + (\lambda_3' + \lambda_1k_2 + \lambda_2k_3)\boldsymbol{b_q}.$$

Using Theorem 4.1.2, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)}X = \frac{1}{\sqrt{k_1^2+k_2^2}}\left[\lambda_1'\boldsymbol{t_q} + (\lambda_2' + f(s)\lambda_3)\boldsymbol{n_q} + (\lambda_3' - f(s)\lambda_2)\boldsymbol{b_q}\right].$$

By the definition of Fermi-Walker parallelism, the vector field X is Fermi-Walker parallel along the curve β if and only if $\widetilde{\nabla}_{\boldsymbol{t_q}(s_\beta)}X = 0$. This requires a system of differential equations as follows:

$$\begin{bmatrix}\lambda_1' \\ \lambda_2' \\ \lambda_3'\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -f(s) \\ 0 & f(s) & 0\end{bmatrix}\begin{bmatrix}\lambda_1 \\ \lambda_2 \\ \lambda_3\end{bmatrix}. \tag{4.1.1}$$

For the details to obtain the solution of this system, see Appendix.

4.2. THE FERMI-WALKER DERIVATIVE AND THE QUASI NORMAL INDICATRIX

In this subsection, we give the Fermi-Walker derivative along the quasi normal indicatrix of a space curve parameterized with arc-length according to quasi frame. Let s be the arc-length parameter of a space curve $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ and s_γ be the arc-length parameter of the quasi normal indicatrix $\gamma = \boldsymbol{n_q}(s)$ of the curve α, i.e. $\gamma(s_\gamma) = \boldsymbol{n_q}(s)$. Firstly, we give a theorem which is about the relationship between the Frenet frame of the quasi normal indicatrix of a space curve and the quasi frame of this curve.

Theorem 4.2.1. Let $\{\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$ be the quasi frame of the curve $\alpha = \alpha(s)$ parameterized with arc-length and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be the Frenet frame of the curve $\gamma(s) = \boldsymbol{n_q}(s)$. The quasi curvatures of the curve α are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve γ are denoted by κ and τ, respectively. Then, the Frenet elements of γ can be given in terms of the quasi elements of α as follows [11]:

$$\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{\sqrt{k_1^2+k_3^2}} & 0 & \frac{k_3}{\sqrt{k_1^2+k_3^2}} \\ \frac{A_2}{\sqrt{U_2}} & \frac{B_2}{\sqrt{U_2}} & \frac{C_2}{\sqrt{U_2}} \\ \frac{K_2}{\sqrt{V_2}} & \frac{L_2}{\sqrt{V_2}} & \frac{M_2}{\sqrt{V_2}} \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix},$$

$$\kappa = (1 + \frac{K_2}{(k_1^2+k_3^2)^3})^{\frac{1}{2}},$$

$$\tau = \frac{W_2}{V_2} \ .$$

Theorem 4.2.2. The Fermi-Walker derivative along the quasi normal indicatrix γ of a space curve α parameterized with arc-length can be expressed as

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)} X = \frac{1}{\sqrt{k_1^2+k_3^2}} \left[\nabla_{\boldsymbol{t_q}} X + (X \times W) + \frac{k_1 k_3' - k_1' k_3}{k_1^2+k_3^2} (X \times \boldsymbol{n_q}) \right],$$

where $W = k_3 \boldsymbol{t_q} - k_2 \boldsymbol{n_q} + k_1 \boldsymbol{b_q}$ is the Darboux vector of the curve α according to the quasi frame and $k_1(s), k_2(s), k_3(s)$ are the quasi curvatures of the curve α at s.

Proof. By the definition of the Fermi-Walker derivative, we can write

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)} X = \nabla_{\boldsymbol{t_q}(s_\gamma)} X - \langle \boldsymbol{t_q}(s_\gamma), X \rangle \nabla_{\boldsymbol{t_q}(s_\gamma)} \boldsymbol{t_q}(s_\gamma) + \langle \nabla_{\boldsymbol{t_q}(s_\gamma)} \boldsymbol{t_q}(s_\gamma), X \rangle \, \boldsymbol{t_q}(s_\gamma).$$

With the help of the property

$$(u \times v) \times w = \langle w, u \rangle v - \langle w, v \rangle u$$

of the cross-product, we get

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)}X = \nabla_{\boldsymbol{t_q}(s_\gamma)}X + X \times \left(\boldsymbol{t_q}(s_\gamma) \times \nabla_{\boldsymbol{t_q}(s_\gamma)}\boldsymbol{t_q}(s_\gamma)\right).$$

Using the following equalities,

$$\boldsymbol{t_q}(s_\gamma) = \frac{-k_1 n_q + k_3 b_q}{\sqrt{k_1^2+k_3^2}},$$

$$\frac{ds_\gamma}{ds} = \sqrt{k_1^2 + k_3^2},$$

$$\nabla_{\boldsymbol{t_q}(s_\gamma)}\boldsymbol{t_q}(s_\gamma) = \frac{k_3 L}{(k_1^2+k_3^2)^2}\boldsymbol{t_q} - \boldsymbol{n_q} + \frac{k_1 L}{(k_1^2+k_3^2)^2}\boldsymbol{b_q},$$

where $L = k_1 k_3' - k_1' k_3 - k_1^2 k_2 - k_3^2 k_2$, we obtain

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)}X = \nabla_{\boldsymbol{t_q}(s_\gamma)}X + X \times \left(\frac{W}{\sqrt{k_1^2+k_3^2}} + \frac{k_1 k_3' - k_1' k_3}{(k_1^2+k_3^2)^{\frac{3}{2}}}\boldsymbol{n_q}\right).$$

Then, in the light of the properties of the cross-product, we get

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)}X = \frac{1}{\sqrt{k_1^2+k_3^2}}\left[\nabla_{\boldsymbol{t_q}}X + (X \times W) + \frac{k_1 k_3' - k_1' k_3}{k_1^2+k_3^2}(X \times \boldsymbol{n_q})\right].$$

Theorem 4.2.3. Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve parameterized with arc-length, γ be the quasi normal indicatrix of the curve α and $X = \hat{\lambda}_1\boldsymbol{t_q} + \hat{\lambda}_2\boldsymbol{n_q} + \hat{\lambda}_3\boldsymbol{b_q}$ be a vector field along the curve γ, where $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ are continuously differentiable functions of the parameter *s*. Then, the vector field *X* is Fermi-Walker parallel along the curve γ if and only if

$$\hat{\lambda}_1(s) = c_2\cos\left(\int_1^s g(s)ds\right) + c_1\sin\left(\int_1^s g(s)ds\right),$$

$$\hat{\lambda}_2(s) = constant,$$

$$\hat{\lambda}_3(s) = c_1\cos\left(\int_1^s g(s)ds\right) - c_2\sin\left(\int_1^s g(s)ds\right),$$

where c_1, c_2 are arbitrary real constants and $g = \frac{k_1 k_3' - k_1' k_3}{k_1^2+k_3^2}$.

Proof. We use Theorem 4.2.2. One can easily get the following equalities,

$$X \times W = (\hat{\lambda}_2 k_1 + \hat{\lambda}_3 k_2)\boldsymbol{t_q} + (\hat{\lambda}_3 k_3 - \hat{\lambda}_1 k_1)\boldsymbol{n_q} - (\hat{\lambda}_1 k_2 + \hat{\lambda}_2 k_3)\boldsymbol{b_q},$$

$$X \times \boldsymbol{n_q} = -\hat{\lambda}_3\boldsymbol{t_q} + \hat{\lambda}_1\boldsymbol{b_q},$$

$$\nabla_{\boldsymbol{t_q}}X = (\hat{\lambda}_1' - \hat{\lambda}_2 k_1 - \hat{\lambda}_3 k_2)\boldsymbol{t_q} + (\hat{\lambda}_2' + \hat{\lambda}_1 k_1 - \hat{\lambda}_3 k_3)\boldsymbol{n_q} + (\hat{\lambda}_3' + \hat{\lambda}_1 k_2 + \hat{\lambda}_2 k_3)\boldsymbol{b_q}.$$

Using Theorem 4.2.2, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)}X = \frac{1}{\sqrt{k_1^2+k_3^2}}\left[\left(\hat{\lambda}_1' - g(s)\hat{\lambda}_3\right)\boldsymbol{t_q} + \hat{\lambda}_2'\boldsymbol{n_q} + (\hat{\lambda}_3' + g(s)\hat{\lambda}_1)\boldsymbol{b_q}\right].$$

By the definition of Fermi-Walker parallelism, the vector field X is Fermi-Walker parallel along the curve γ if and only if $\widetilde{\nabla}_{\boldsymbol{t_q}(s_\gamma)}X = 0$. This requires a system of differential equations as follows:

(4.2.1)
$$\begin{bmatrix}\hat{\lambda}_1' \\ \hat{\lambda}_2' \\ \hat{\lambda}_3'\end{bmatrix} = \begin{bmatrix} 0 & 0 & g(s) \\ 0 & 0 & 0 \\ -g(s) & 0 & 0 \end{bmatrix}\begin{bmatrix}\hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3\end{bmatrix}.$$

For the details to obtain the solution of this system, see Appendix.

4.3. THE FERMI-WALKER DERIVATIVE AND THE QUASI BINORMAL INDICATRIX

In this subsection, we give the Fermi-Walker derivative along the quasi binormal indicatrix of a space curve parameterized with arc-length according to quasi frame. Let s be the arc-length parameter of a space curve $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ and s_δ be the arc-length parameter of the quasi binormal indicatrix $\delta = \boldsymbol{b_q}(s)$ of the curve α, i.e. $\delta(s_\delta) = \boldsymbol{b_q}(s)$. Firstly, we give a theorem which is about the relationship between the Frenet frame of the quasi binormal indicatrix of a space curve and the quasi frame of this curve.

Theorem 4.3.1. Let $\{\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$ be the quasi frame of the curve $\alpha = \alpha(s)$ parameterized with arc-length and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be the Frenet frame of the curve $\delta(s) = \boldsymbol{b_q}(s)$. The quasi curvatures of the curve α are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve δ are denoted by κ and τ, respectively. Then, the Frenet elements of δ can be given in terms of the quasi elements of α as follows:

$$\begin{bmatrix}\boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b}\end{bmatrix} = \begin{bmatrix} \frac{-k_2}{\sqrt{k_2^2+k_3^2}} & 0 & \frac{-k_3}{\sqrt{k_2^2+k_3^2}} \\ \frac{A_3}{\sqrt{U_3}} & \frac{B_3}{\sqrt{U_3}} & \frac{C_3}{\sqrt{U_3}} \\ \frac{K_3}{\sqrt{V_3}} & \frac{L_3}{\sqrt{V_3}} & \frac{M_3}{\sqrt{V_3}} \end{bmatrix}\begin{bmatrix}\boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q}\end{bmatrix},$$

$$\kappa = \left(1 + \frac{K_3}{(k_2^2+k_3^2)^3}\right)^{\frac{1}{2}},$$

$$\tau = \frac{W_3}{V_3}.$$

Theorem 4.3.2. The Fermi-Walker derivative along the quasi binormal indicatrix δ of a space curve α parameterized with arc-length can be expressed as

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\delta)}X = \frac{1}{\sqrt{k_2^2+k_3^2}}\left[\nabla_{\boldsymbol{t_q}}X + (X\times W) + \frac{k_2k_3'-k_2'k_3}{k_2^2+k_3^2}(X\times \boldsymbol{b_q})\right],$$

where $W = k_3\boldsymbol{t_q} - k_2\boldsymbol{n_q} + k_1\boldsymbol{b_q}$ is the Darboux vector of the curve α according to the quasi frame and $k_1(s), k_2(s), k_3(s)$ are the quasi curvatures of the curve α at s.

Proof. By the definition of the Fermi-Walker derivative, we can write

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\delta)}X = \nabla_{\boldsymbol{t_q}(s_\delta)}X - \langle \boldsymbol{t_q}(s_\delta), X\rangle \nabla_{\boldsymbol{t_q}(s_\delta)}\boldsymbol{t_q}(s_\delta) + \langle \nabla_{\boldsymbol{t_q}(s_\delta)}\boldsymbol{t_q}(s_\delta), X\rangle\, \boldsymbol{t_q}(s_\delta).$$

With the help of the property

$$(u\times v)\times w = \langle w,u\rangle v - \langle w,v\rangle u$$

of the cross-product, we get

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\delta)}X = \nabla_{\boldsymbol{t_q}(s_\delta)}X + X\times\left(\boldsymbol{t_q}(s_\delta)\times \nabla_{\boldsymbol{t_q}(s_\delta)}\boldsymbol{t_q}(s_\delta)\right).$$

Using the following equalities:

$$\boldsymbol{t_q}(s_\delta) = \frac{-k_2\boldsymbol{t_q}-k_3\boldsymbol{n_q}}{\sqrt{k_2^2+k_3^2}},$$

$$\frac{ds_\delta}{ds} = \sqrt{k_2^2+k_3^2},$$

$$\nabla_{\boldsymbol{t_q}(s_\delta)}\boldsymbol{t_q}(s_\delta) = \frac{k_3M}{(k_2^2+k_3^2)^2}\boldsymbol{t_q} - \frac{k_2M}{(k_2^2+k_3^2)^2}\boldsymbol{n_q} - \boldsymbol{b_q},$$

where $M = k_2k_3' - k_2'k_3 + k_1k_2^2 + k_1k_3^2$, we obtain

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\delta)}X = \nabla_{\boldsymbol{t_q}(s_\delta)}X + X\times\left(\frac{W}{\sqrt{k_2^2+k_3^2}} + \frac{k_2k_3'-k_2'k_3}{(k_2^2+k_3^2)^{\frac{3}{2}}}\boldsymbol{b_q}\right).$$

Then, in the light of the properties of the cross-product, we get

$$\tilde{\nabla}_{\boldsymbol{t_q}(s_\delta)}X = \frac{1}{\sqrt{k_2^2+k_3^2}}\left[\nabla_{\boldsymbol{t_q}}X + (X\times W) + \frac{k_2k_3'-k_2'k_3}{k_2^2+k_3^2}(X\times \boldsymbol{b_q})\right].$$

Theorem 4.3.3. Let $\alpha: I\subset\mathbb{R}\to\mathbb{R}^3$ be a curve parameterized with arc-length, δ be the quasi binormal indicatrix of the curve α and $X = \bar{\lambda}_1\boldsymbol{t_q} + \bar{\lambda}_2\boldsymbol{n_q} + \bar{\lambda}_3\boldsymbol{b_q}$ be a vector field along the curve δ, where $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are continuously differentiable functions of the parameter s. Then, the vector field X is Fermi-Walker parallel along the curve δ if and only if

$$\bar{\lambda}_1(s) = c_1\cos\left(\int_1^s h(s)ds\right) - c_2\sin\left(\int_1^s h(s)ds\right),$$

$$\bar{\lambda}_2(s) = c_2 \cos\left(\int_1^s h(s)ds\right) + c_1 \sin\left(\int_1^s h(s)ds\right),$$

$$\bar{\lambda}_3(s) = constant,$$

where c_1, c_2 are arbitrary real constants and $h = \frac{k_2 k_3' - k_2' k_3}{k_2^2 + k_3^2}$.

Proof. We use Theorem 4.3.2. One can easily get the following equalities:

$$X \times W = \left(\bar{\lambda}_2 k_1 + \bar{\lambda}_3 k_2\right)\boldsymbol{t_q} + \left(\bar{\lambda}_3 k_3 - \bar{\lambda}_1 k_1\right)\boldsymbol{n_q} - \left(\bar{\lambda}_1 k_2 + \bar{\lambda}_2 k_3\right)\boldsymbol{b_q},$$

$$X \times \boldsymbol{b_q} = \bar{\lambda}_2 \boldsymbol{t_q} - \bar{\lambda}_1 \boldsymbol{n_q},$$

$$\nabla_{\boldsymbol{t_q}} X = \left(\bar{\lambda}_1' - \bar{\lambda}_2 k_1 - \bar{\lambda}_3 k_2\right)\boldsymbol{t_q} + \left(\bar{\lambda}_2' + \bar{\lambda}_1 k_1 - \bar{\lambda}_3 k_3\right)\boldsymbol{n_q} + \left(\bar{\lambda}_3' + \bar{\lambda}_1 k_2 + \bar{\lambda}_2 k_3\right)\boldsymbol{b_q}.$$

Using Theorem 4.3.2, we get

$$\widetilde{\nabla}_{\boldsymbol{t_q}(s_\delta)} X = \frac{1}{\sqrt{k_2^2 + k_3^2}}\left[\left(\bar{\lambda}_1' + h(s)\bar{\lambda}_2\right)\boldsymbol{t_q} + (\bar{\lambda}_2' - h(s)\bar{\lambda}_1)\boldsymbol{n_q} + \bar{\lambda}_3' \boldsymbol{b_q}\right].$$

By the definition of Fermi-Walker parallelism, the vector field X is Fermi-Walker parallel along the curve δ if and only if $\widetilde{\nabla}_{\boldsymbol{t_q}(s_\delta)} X = 0$. This requires a system of differential equations as follows:

(4.2.1)
$$\begin{bmatrix} \bar{\lambda}_1' \\ \bar{\lambda}_2' \\ \bar{\lambda}_3' \end{bmatrix} = \begin{bmatrix} 0 & -h(s) & 0 \\ h(s) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \\ \bar{\lambda}_3 \end{bmatrix}.$$

For the details to obtain the solution of this system, see Appendix.

5. APPENDIX

In this section, we present a method about how to solve a homogenous normal linear system of differential equations. A normal linear system can be written as follows:

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + F_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + F_2(t)$$

(5.1)
$$\vdots \qquad \qquad \vdots$$

$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + F_n(t)$$

where $a_{ij}(t)$, $F_i(t)$ are given functions and $x_i(t)$ are unknown functions, $1 \le i, j \le n$. If for all, $(i = 1, \ldots, n)$ $F_i(t) = 0$, then the system (5.1) is called a homogenous normal linear system. We can write the homogenous normal linear system in matrix form as follows:

(5.2)
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t)\, a_{12}(t) \ldots a_{1n}(t) \\ a_{21}(t)\, a_{22}(t) \ldots a_{2n}(t) \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}(t)\, a_{n2}(t) \cdots a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Let $a_{ij}(t)$ be continuous functions defined at an interval I and $X_1, X_2, \ldots, X_n$ be linearly independent n solutions of the system (5.2) at I. Then, each solution of the system can be expressed in only one way for a certain choice of the real constants $c_1, c_2, \ldots, c_n$ as follows:

$$X = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_n X_n(t).$$

Here, $\{X_1, X_2, \ldots, X_n\}$ is called a fundamental set of solutions of the system (5.2) and the solution $X = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_n X_n(t)$ is called the general solution of the system (5.2) [11].

Now, let us consider the homogenous normal linear system (4.1.1). The vector functions

$$\lambda_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 \\ \cos\left(\int_1^s f(s)ds\right) \\ \sin\left(\int_1^s f(s)ds\right) \end{bmatrix}, \lambda_3 = \begin{bmatrix} 0 \\ -\sin\left(\int_1^s f(s)ds\right) \\ \cos\left(\int_1^s f(s)ds\right) \end{bmatrix}$$

are linearly independent solutions of this system. So, the general solution of this system is

$$\lambda = c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3.$$

Let us show that the vector function λ is really solution of this system. We can write

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2\cos\left(\int_1^s f(s)ds\right) - c_3\sin\left(\int_1^s f(s)ds\right) \\ c_2\sin\left(\int_1^s f(s)ds\right) + c_3\cos\left(\int_1^s f(s)ds\right) \end{bmatrix}.$$

Calculating the derivatives, we get

$$\lambda' = \begin{bmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{bmatrix} = \begin{bmatrix} 0 \\ -f(s)(c_2\sin\left(\int_1^s f(s)ds\right) + c_3\cos\left(\int_1^s f(s)ds\right)) \\ f(s)(c_2\cos\left(\int_1^s f(s)ds\right) - c_3\sin\left(\int_1^s f(s)ds\right)) \end{bmatrix},$$

so this gives the system (4.1.1) and that is what we want to show. In other words, the solution of the system (4.1.1) is

$$\lambda_1(s) = constant,$$

$$\lambda_2(s) = c_1\cos\left(\int_1^s f(s)ds\right) - c_2\sin\left(\int_1^s f(s)ds\right),$$

$$\lambda_3(s) = c_2\cos\left(\int_1^s f(s)ds\right) + c_1\sin\left(\int_1^s f(s)ds\right),$$

where c_1, c_2 are arbitrary real constants.

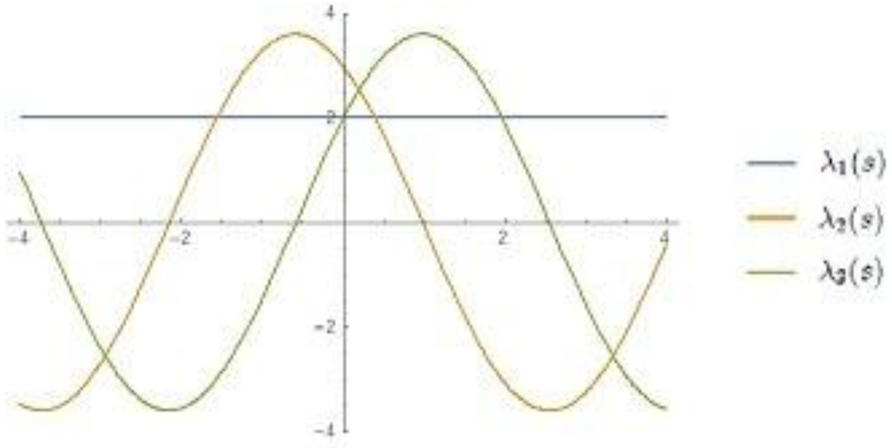

Figure 1. The graph is a special solution of the system (4.1.1).

If we consider the homogenous normal linear system (4.2.1), similarly we get

$$\hat{\lambda}_1(s) = c_2 \cos\left(\int_1^s g(s)ds\right) + c_1 \sin\left(\int_1^s g(s)ds\right),$$

$$\hat{\lambda}_2(s) = constant,$$

$$\hat{\lambda}_3(s) = c_1 \cos\left(\int_1^s g(s)ds\right) - c_2 \sin\left(\int_1^s g(s)ds\right),$$

where c_1, c_2 are arbitrary real constants.

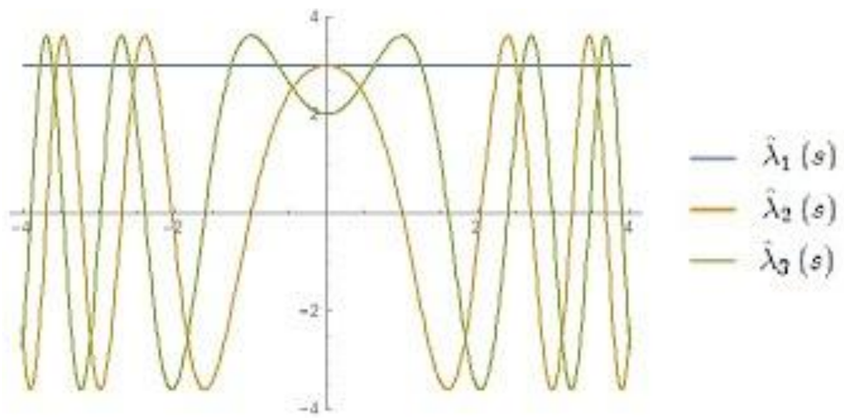

Figure 2. The graph is a special solution of the system (4.2.1).

If we consider the homogenous normal linear system (4.3.1), similarly we get

$$\bar{\lambda}_1(s) = c_1 \cos\left(\int_1^s h(s)ds\right) - c_2 \sin\left(\int_1^s h(s)ds\right),$$

$$\bar{\lambda}_2(s) = c_2 \cos\left(\int_1^s h(s)ds\right) + c_1 \sin\left(\int_1^s h(s)ds\right),$$

$$\bar{\lambda}_3(s) = constant,$$

where c_1, c_2 are arbitrary real constants.

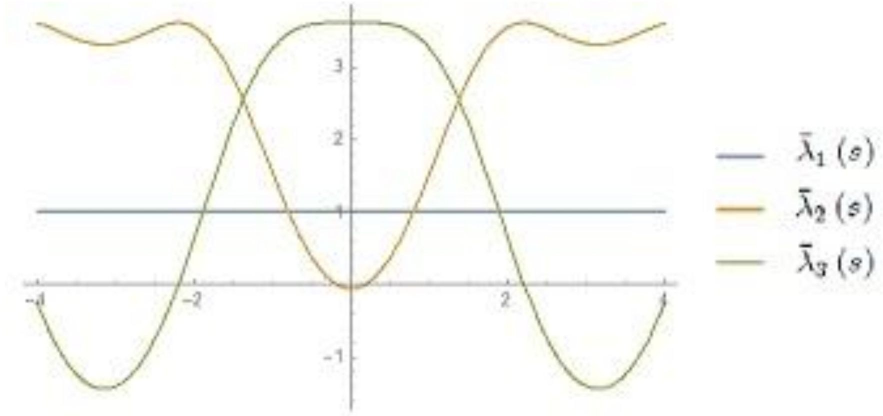

Figure 3. The graph is a special solution of the system (4.3.1).

6. CONCLUSION

The Fermi-Walker derivative and parallelism have many applications in differential geometry and in physics so this subject is very important in these areas. There are studies about the Fermi-Walker derivative, for example one can see [10]. They used the Frenet frame of the curves in that study. In this work, we consider this subject using the quasi frame of the curves. There are two main advantages of the quasi frame over the Frenet frame [7]: 1) It is well defined even if the curve has vanishing second derivative, 2) it avoids the unnecessary twist around the tangent. Moreover, the computation of the quasi frame is easier than the rotation minimizing frames, for example one of them is Bishop frame. In this study, we give the Fermi-Walker derivative on the spherical indicatrices of a space curve according to the quasi frame. Additionally, we give an Appendix in which we present a method about how to solve a homogenous normal linear system of differential equations.

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