ORIGINAL PAPER

DUAL SPHERICAL CURVES OF NATURAL LIFT CURVE AND TANGENT BUNDLES OF UNIT 2-SPHERE

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Abstract. *The paper is concerned with the ruled surfaces generated by the main curve and its natural lift curve in dual space. Then it gives a detailed analysis of ruled surfaces for dual spherical curves of natural lift curve. Striction curves and distribution parameters of these ruled surfaces are calculated. The developability condition is examined by using obtained results. Finally, some examples are given to support the main results.*

Keywords: *Dual spherical curves, natural lift, ruled surface, tangent bundle, distribution parameter.*

1. INTRODUCTION

The theories of surfaces and curves play important roles in differential geometry. Especially, in Riemannian geometry, the theories of curves and surfaces are given in [2].The concept of natural lift curve was first encountered in J.A. Thorpe's book, see [3]. In that book, natural lift curve is defined as the curve which is generated by the endpoints of tangent vectors of main curve. In literature, there are a lot of studies about the natural lift curve, see[4-11].

Ruled surfaces have a lot of applications in geometry, computer modelling systems, kinematics, physics, etc. As known, ruled surface is obtained by moving a line along the curve. For detailed information for the characteristic properties and definition of ruled surface, see [15] In literature, there are a lot of studies about the concepts of the ruled surfaces and integral invariants of closed ruled surfaces, see [16-20].

Dual numbers were introduced by W. K. Clifford in 1873. Then E. Study constructed the correspondence between the geometry of lines and unit dual sphere by the theorem which is called his name. E. Study mapping says that there exists one-to-one correspondence between the oriented lines in Euclidean space and the points on the unit dual sphere, DS^2.

Firstly, the relationship between DS^2, the tangent bundle of unit 2 sphere, TS^2 and ruled surfaces in IR^3 is given in detailed analysis, [14]. Then the correspondence between ruled surfaces in IR^3 and the tangent bundle of unit 2-sphere is given by constructing isomorphism between TS^2 and DS^2 in [12]. The same authors give the correspondence between tangent bundle of pseudo-sphere and ruled surfaces in Minkowski space in [13].

However, there is a little research about ruled surfaces generated by dual spherical curves of natural lift curve. The remainder of this paper is divided into six sections. Section 2 is about the basic definitions and theorems about the tangent bundle of unit 2-sphere, natural

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lift curve, dual spherical curves of natural lift curve and ruled surfaces. Section 3 covers some basic properties and definitions about dual vectors. Moreover, the isomorphism between the subset $T\overline{M}$ of the tangent bundle of unit 2-sphere and unit dual sphere, DS^2. Section 4 is about ruled surfaces and dual spherical curves of the main curve and its natural lift curve. In Section 5, the ruled surfaces generated by dual spherical curves of the natural lift curve are given in detail. The striction curves and distribution parameters are calculated. Then an example is given to support the main results. Finally, in Section 6, we discuss the obtained results and give the fields of application.

2. PRELIMINARIES

In this section, we will recall basic definitions and theorems about tangent bundle of unit 2-sphere, natural lift curve, dual spherical curves, properties of ruled surfaces (i.e. striction curve, distribution parameter):

Definition 1: Let S^2 be the unit 2-sphere in IR^3. The tangent bundle of S^2 is given in the following equation:

$$TS^2 = \left\{\Gamma(s) = (\gamma(s), \nu(s)) \in IR^3 \times IR^3 : \left|\gamma(s)\right| = 1, \left\langle \gamma(s), \nu(s) \right\rangle = 1\right\} \quad (1)$$

Definition 2: Let $\Gamma : I \to \overline{M}$ be a curve. Here $\overline{M}$ represents the subset of unit 2-sphere S^2. Γ is called an integral curve of X

$$\frac{d(\Gamma(s))}{ds} = X(\Gamma(s)). \quad (2)$$

Here X is the smooth tangent vector field on $\overline{M}$.
Assume that $T\overline{M}$ is a subset of TS^2. $T\overline{M}$ is defined as

$$T\overline{M} = \{(\overline{\gamma}(s), \overline{\nu}(s)) \in IR^3 \times IR^3 : \left|\overline{\gamma}(s)\right| = 1, \left\langle \overline{\gamma}(s), \overline{\nu}(s) \right\rangle = 0\} \ . \quad (3)$$

Here $\overline{\gamma}(s)$ and $\overline{\nu}(s)$ are derivatives of $\gamma(s)$ and $\nu(s)$, respectively.

Definition 3: For the curve $\Gamma(s)$, $\overline{\Gamma}(s)$ is called as the natural lift of $\Gamma(s)$, which produces in the following equation:

$$\overline{\Gamma}(s) = (\overline{\gamma}(s), \overline{\nu}(s)) = (\dot{\gamma}(s)_{\gamma(s)}, \dot{\nu}(s)_{\nu(s)}). \quad (4)$$

Accordingly, we can write

$$\frac{d(\overline{\Gamma}(s))}{ds} = \frac{d(\dot{\Gamma}(s)_{\Gamma(s)})}{ds} = D_{\dot{\Gamma}(s)}\dot{\Gamma}(s). \quad (5)$$

Here D refers Levi-Civita connection on IR^3. We have

$$T\overline{M} = \bigcup T_P \overline{M}, \qquad p \in \overline{M}, \tag{6}$$

where $T_P\overline{M}$ is the tangent space of $\overline{M}$ at p and $\chi(\overline{M})$ is the space of vector fields on $\overline{M}$. Let $\{T(s), N(s), B(s)\}$ and $\{\overline{T}(s), \overline{N}(s), \overline{B}(s)\}$ be Frenet frames of $\Gamma(s)$ and $\overline{\Gamma}(s)$, respectively.

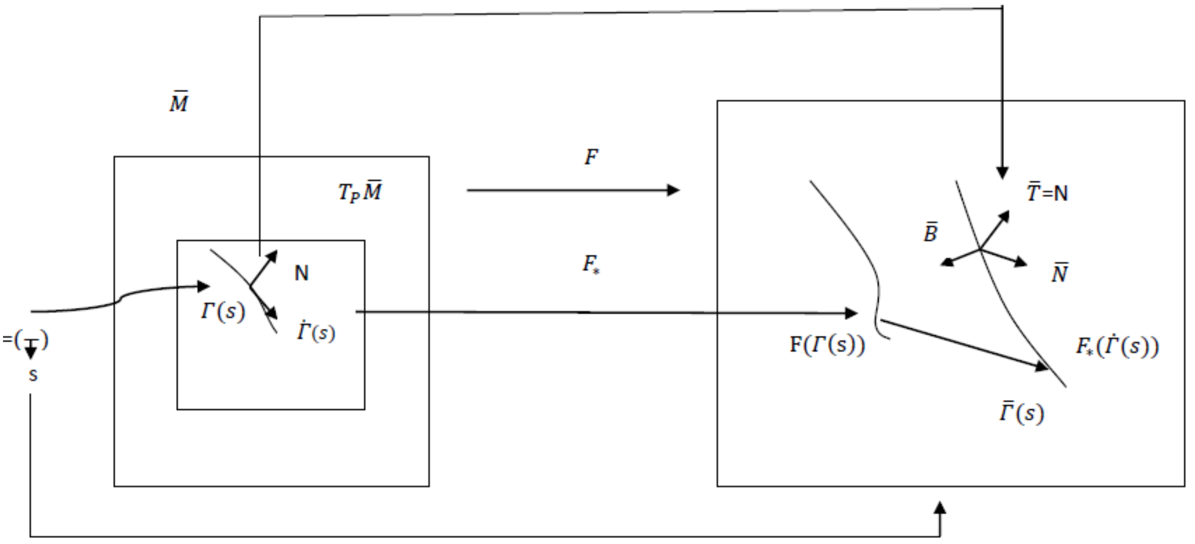
The natural lift curve is the curve which is drawn by the end points of the tangents of the main curve. The following figure is about the representation of natural lift curve. In this figure,

$$F : \overline{M} \to T\overline{M}$$

and

$$F_{*p} : T_p\overline{M} \to T_{F(p)}(T\overline{M})$$

are functions among $\overline{M}$, $T\overline{M}$, $T_P\overline{M}$ and $T_{F(p)}(T\overline{M})$.



$$\overline{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s)) = (\dot{\gamma}(s)_{\gamma(s)}, \dot{v}(s)_{v(s)})$$

Figure 1. The natural lift curve of the main curve $\Gamma(s)$.

The real and dual parts of Frenet operators for the curve is given as follows:

$$\dot{t}(s) = k_1(s)n(s),$$

$$\dot{n}(s) = -k_1(s)t(s) + k_2(s)b(s),$$

$$\dot{b}(s) = -k_2(s)n(s),$$

$$\dot{t}^*(s) = k_1(s)n^*(s) + k_1^*(s)n(s),$$

$$\dot{n}^*(s) = -k_1(s)t^*(s) - k_1^*(s)t(s) + k_2(s)b^*(s) + k_2^*(s)b(s),$$

$$\dot{b}^*(s) = -k_2(s)n^*(s) - k_2^*(s)n(s).$$

Here $\kappa(s) = k_1(s) + \varepsilon k_1^*(s)$ and $\tau(s) = k_2(s) + \varepsilon k_2^*(s)$ are curvature and torsion of the main curve, respectively.

The real and dual parts of Frenet operators for the natural lift curve is given as follows:

$$\dot{\bar{t}}(s) = -k_1(s)t(s) + k_2(s)b(s),$$

$$\dot{\bar{n}}(s) = (\sin(\Psi))t(s) - (k_1(s)\cos(\Psi) + k_2(s)\sin(\Psi))n(s) + (\cos(\Psi))b(s),$$

$$\dot{\bar{b}}(s) = (\cos(\Psi))t(s) + (k_1(s)\sin(\Psi) - k_2(s)\cos(\Psi))n(s) - (\sin(\Psi))b(s),$$

$$\dot{\bar{t}}*(s) = -k_1(s)t^*(s) - k_1^*(s)t(s) + k_2(s)b^*(s) + k_2^*(s)b(s),$$

$$\dot{\bar{n}}*(s) = (\Psi^* \sin(\Psi))' t(s) + (\Psi^*(k_1(s)\sin(\Psi) - k_2(s)\cos(\Psi)) + (k_1^*(s)\cos(\Psi) - k_2^*(s)\sin(\Psi)))n(s)$$
$$+(\Psi^* \cos(\Psi))' b(s) + (\sin(\Psi))t^*(s) + (-k_1(s)\cos(\Psi) - k_2(s)\sin(\Psi))n^*(s) + \cos(\Psi)b^*(s),$$

$$\dot{\bar{b}}*(s) = (\Psi^* \cos(\Psi))' t(s) + ((k_1^*(s)\sin(\Psi) - k_2^*(s)\cos(\Psi)) + \Psi^*(k_1^*(s)\sin(\Psi) + k_2^*(s)\cos(\Psi)))n(s)$$
$$+(\Psi^* \sin(\Psi))' b(s) + (\cos(\Psi))t^*(s) + (k_1(s)\sin(\Psi) - k_2(s)\cos(\Psi))n^*(s) + (\sin(\Psi))b^*(s).$$

Here $\Theta = \Psi + \varepsilon \Psi^*$ is dual angle between Darboux vector and binormal vector.

Given one-parameter family lines $\{\beta(s), \varsigma(s)\}$, the ruled surface generated by the family $\{\beta(s), \varsigma(s)\}$ is

$$\Phi(s,u) = \beta(s) + u.\varsigma(s), \quad s \in I, \quad u \in IR. \tag{7}$$

The striction curve of the ruled surface is defined as

$$\alpha(s) = \beta(s) - \frac{\langle \dot{\beta}(s), \dot{u}(s) \rangle}{\langle \dot{u}(s), \dot{u}(s) \rangle}.u(s). \tag{8}$$

The sitriction curve $\alpha(s)$ coincides the base curve $\beta(s)$ if the multiplication $\dot{\beta}(s)$ and $\dot{u}(s)$ is equal to zero.

Definition 4: For the curve $\Gamma(s) = (\gamma(s), \nu(s))$, the distribution parameter of the ruled surface is defined as

$$P_u = \frac{\det(\dot{\beta}(s), u(s), \dot{u}(s))}{\langle \dot{u}(s), \dot{u}(s) \rangle}. \tag{9}$$

3. RULED SURFACES GENERATED BY THE NATURAL LIFT CURVES AND UNIT DUAL SPHERE

This section presents some basic properties and definitions about dual vectors. Moreover, the isomorphism between the subset $T\overline{M}$ of the tangent bundle of unit 2-sphere and unit dual sphere, DS^2.

The set of dual numbers is defined as

$$ID = \{A = a + \varepsilon a^* : (a, a^*) \in IR \times IR, \varepsilon^2 = 0\}. \quad (10)$$

The combination of a and a^* is called as dual vectors in IR^3. These vevctors are real and dual part of A, respectively. If a and a^* are vectors in IR^3, then $A = a + \varepsilon a^*$ is called as dual vector. Let $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ be dual vectors. The addition, inner product and vector product are presented as follows:

The addition is

$$A + B = (a + b) + \varepsilon(a^* + b^*)$$

and the inner product is

$$A.B = \langle a, b\rangle + \varepsilon(\langle a^*, b\rangle + \langle a, b^*\rangle).$$

The vector product is also given as

$$A \times B = a \times b + \varepsilon(a^* \times b + a \times b^*).$$

The norm of $A = a + \varepsilon a^*$ is given as

$$\|A\| = \sqrt{a.a} + \frac{a.a^*}{\sqrt{a.a}}.$$

The norm of $A = a + \varepsilon a^*$ exists only for $a \neq 0$. If the norm $\|A\|$ is equal to 1, the dual vector is called unit dual vector.

The unit dual sphere which consists of all unit dual vectors is defined as

$$DS^2 = \left\{A = a + \varepsilon a^* : \|A\| = 1\right\}. \quad (11)$$

For detailed information for dual vectors, see [1]. The correspondence between unit dual sphere and the subset of tangent bundle of unit 2-sphere is givenby using Eq.(3) and Eq.(11):

$$T\overline{M} \subseteq TS^2 \to DS^2,$$

$$\overline{\Gamma}(s) = (\overline{\gamma}(s), \overline{v}(s)) \mapsto \overline{\Gamma}(s) = \overline{\gamma}(s) + \varepsilon\overline{v}(s).$$

Theorem 1: There exists one –to-one correspondence between the oriented lines in IR^3 and the points on DS^2, see [1].

Theorem 2: Let $\overline{\Gamma}(s)=(\overline{\gamma}(s),\overline{v}(s))$ be natural lift curve of the curve $\Gamma(s)=(\gamma(s),\nu(s))$ on DS^2. In IR^3, the ruled surface generated by the natural lift curve is presented as

$$\overline{\Phi}(s,u)=(\overline{\gamma}(s)\times\overline{v}(s))+u.\overline{\gamma}(s), \tag{12}$$

where $\overline{\beta}(t)=\overline{\gamma}(t)\times\overline{\vartheta}(t)$ is the base curve of the ruled surface $\overline{\Phi}(s,u)$.

Consequently, the isomorphism between the subset $T\overline{M}$ of the tangent bundle of unit 2-sphere and unit dual sphere, DS^2 is given in the following equation:

$$T\overline{M}\subseteq TS^2\to DS^2\to IR^3,$$

$$\overline{\Gamma}(s)=(\overline{\gamma}(s),\overline{v}(s))\mapsto\overline{\Gamma}(s)=\overline{\gamma}(s)+\varepsilon\overline{v}(s)\mapsto\overline{\Phi}(s,u)=(\overline{\gamma}(s)\times\overline{v}(s))+u.\overline{\gamma}(s).$$

4. RULED SURFACES AND DUAL SPHERICAL CURVES

In this section, we will give spherical indicatries of the main curve and ruled surfaces generated by the spherical indicatries of this curve. Then we will give spherical indicatries of natural lift curve and ruled surfaces generated by the spherical indicatries of the natural lift curve.

Let $\Gamma(s)=(\varphi(s),\upsilon(s))$ be a curve on $\overline{M}$.

Dual spherical indicatries of the main curve	Ruled surfaces generated by the dual spherical curves of the main curve
$\beta_T(s)=t(s)\times t^*(s)$,	$\Phi_T(s,u)=t(s)\times t^*(s)+u.t(s)$,
$\beta_N(s)=n(s)\times n^*(s)$,	$\Phi_N(s,u)=n(s)\times n^*(s)+u.n(s)$,
$\beta_B(s)=b(s)\times b^*(s)$.	$\Phi_B(s,u)=b(s)\times b^*(s)+u.b(s)$.

Let $\overline{\Gamma}(s)=(\overline{\gamma}(s),\overline{\upsilon}(s))$ be the natural lift curve on $T\overline{M}$.

Dual spherical indicatries of the natural lift curve	Ruled surfaces generated by the dual spherical curves of the natural lift curve
$\overline{\beta}_{\overline{T}}(s)=\overline{t}(s)\times\overline{t}^*(s)$,	$\overline{\Phi}_{\overline{T}}(s,u)=\overline{t}(s)\times\overline{t}^*(s)+u.\overline{t}(s)$,
$\overline{\beta}_{\overline{N}}(s)=\overline{n}(s)\times\overline{n}^*(s)$,	$\overline{\Phi}_{\overline{N}}(s,u)=\overline{n}(s)\times\overline{n}^*(s)+u.\overline{n}(s)$,
$\overline{\beta}_{\overline{B}}(s)=\overline{b}(s)\times\overline{b}^*(s)$.	$\overline{\Phi}_{\overline{B}}(s,u)=\overline{b}(s)\times\overline{b}^*(s)+u.\overline{b}(s)$.

5. RULED SURFACES AND DUAL SPHERICAL CURVES OF NATURAL LIFT CURVES

This section deals with the dual spherical curves of natural lift curves and ruled surfaces of generated by these curves. Then distribution parameters, striction curves of these ruled surfaces are calculated.

Assume that $\Gamma(s)=(\varphi(s),\upsilon(s))$ is a curve on $\overline{M}$. Ruled surface generated by the curve $\Gamma(t)$ is given in the following equation:

$$\Phi(s,u)=\varphi(s)\times\upsilon(s)+u.\varphi(s), \quad (1)$$

where $\beta(t)=\varphi(t)\times\upsilon(t)$ is defined as the base curve of the ruled surface of $\Gamma(t)$.

Ruled surfaces generated by the tangent, principal normal and binormal vectors of the curve $\Gamma(s)=(\varphi(s),\upsilon(s))$ are given as follows:

$$\Phi_T(s,u)=t(s)\times t^*(s)+u.t(s),$$

$$\Phi_N(s,u)=n(s)\times n^*(s)+u.n(s),$$

$$\Phi_B(s,u)=b(s)\times b^*(s)+u.b(s).$$

Let $\overline{\Gamma}(s)=(\overline{\gamma}(s),\overline{\upsilon}(s))$ be the natural lift curve on $T\overline{M}$. Ruled surface generated by the natural lift curve $\overline{\Gamma}(s)$ is presented as the following equation:

$$\overline{\Phi}(s,u)=\overline{\gamma}(s)\times\overline{\upsilon}(s)+u.\overline{\gamma}(s), \quad (2)$$

where $\overline{\beta}(t)=\overline{\gamma}(t)\times\overline{\vartheta}(t)$ is defined as the base curve of the ruled surface of $\overline{\Gamma}(t)$.

Ruled surfaces generated by the tangent, principal normal and binormal vectors of the curve $\overline{\Gamma}(s)=(\overline{\gamma}(s),\overline{\upsilon}(s))$ are given as follows:

$$\overline{\Phi}_{\overline{T}}(s,u)=\overline{t}(s)\times\overline{t}^*(s)+u.\overline{t}(s),$$

$$\overline{\Phi}_{\overline{N}}(s,u)=\overline{n}(s)\times\overline{n}^*(s)+u.\overline{n}(s),$$

$$\overline{\Phi}_{\overline{B}}(s,u)=\overline{b}(s)\times\overline{b}^*(s)+u.\overline{b}(s).$$

Corollary 1. Assume that $\Phi(s,u)$ and $\overline{\Phi}(s,u)$ are ruled surfaces. Ruled surfaces generated by tangent indicatrices of the main curve and its natural lift curve are given:

$$\Phi_T(s,u)=t(s)\times t^*(s)+u.t(s),$$

$$\overline{\Phi}_{\overline{T}}(s,u)=\overline{t}(s)\times\overline{t}^*(s)+u.\overline{t}(s).$$

Here $\beta_T(s)=t(s)\times t^*(s)$ and $\overline{\beta}_{\overline{T}}(s)=\bar{t}(s)\times \bar{t}^*(s)$ are base curves of $\Phi_T(s,u)$ and $\overline{\Phi}_{\overline{T}}(s,u)$, respectively. The striction curves of $\Phi_T(s,u)$ and $\overline{\Phi}_{\overline{T}}(s,u)$ are presented as follows:

$$\alpha_T(s)=t(s)\times t^*(s)-\lambda.t(s),$$

$$\overline{\alpha}_{\overline{T}}(s)=\bar{t}(s)\times \bar{t}^*(s)-\eta.\bar{t}(s).$$

After some calculations, we obtain the constant coefficients λ and η as follows:

$$\lambda=\frac{\left\langle \dot{t}(s)\times \dot{t}^*(s),\dot{t}(s)\right\rangle}{\left\langle \dot{t}(s),\dot{t}(s)\right\rangle},$$

$$\lambda=\frac{\left\langle k_1(s).n(s)\times(k_1(s).n^*(s)+k_1^*(s).n(s)),k_1(s).n(s)\right\rangle}{\left\langle k_1(s).n(s),k_1(s).n(s)\right\rangle}.t(s),$$

$$\lambda=0$$

and

$$\eta=\frac{\left\langle \dot{\bar{t}}(s)\times \dot{\bar{t}}*(s),\dot{\bar{t}}(s)\right\rangle}{\left\langle \dot{\bar{t}}(s),\dot{\bar{t}}(s)\right\rangle},$$

$$\eta=\frac{\left\langle (k_1(s)t(s)+k_2(s)b(s))\times(k_1(s).t^*(s)+k_1^*(s).t(s)+k_2(s).b^*(s)+k_2^*(s).b(s)),(k_1(s)t(s)+k_2(s)b(s))\right\rangle}{\left\langle (k_1(s)t(s)+k_2(s)b(s)),(k_1(s)t(s)+k_2(s)b(s))\right\rangle},$$

$$\eta=0.$$

So, the base curves $\beta_T(s)$ and $\overline{\beta}_{\overline{T}}(s)$ of the ruled surfaces $\Phi(s,u)$ and $\overline{\Phi}(s,u)$ coincide the striction curves $\alpha_T(s)$ and $\overline{\alpha}_{\overline{T}}(s)$. If the ruled surface $\Phi(s,u)$ is developable, then $\overline{\Phi}(s,u)$ is developable.

The distribution parameters of the ruled surfaces are presented as follows:

$$P_T=\frac{\det(\frac{d\beta_T}{ds_T},T(s),\dot{T}(s))}{\left\|\dot{T}(s)\right\|^2},$$

$$\overline{P}_{\overline{T}} = \frac{\det(\frac{d\overline{\beta}_T}{d\overline{s}_T}, \overline{T}(s), \dot{\overline{T}}(s))}{\left\| \dot{\overline{T}}(s) \right\|^2},$$

So, we obtain $P_T = 0$ and $\overline{P}_{\overline{T}} = \dfrac{k_1^*(s)k_1(s) + k_2^*(s)k_2(s)}{k_1^2(s) + k_2^2(s)}$, respectively.

Corollary 2. Assume that $\Phi(s,u)$ and $\overline{\Phi}(s,u)$ are ruled surfaces. The ruled surfaces generated by principal normal indicatrices of the main curve and its natural lift curve are given:

$$\Phi_N(s,u) = n(s) \times n^*(s) + u.n(s),$$

$$\overline{\Phi}_{\overline{N}}(s,u) = \overline{n}(s) \times \overline{n}^*(s) + u.\overline{n}(s).$$

Here $\beta_N(s) = n(s) \times n^*(s)$ and $\overline{\beta}_{\overline{N}}(s) = \overline{n}(s) \times \overline{n}^*(s)$ are base curves of $\Phi_N(s,u)$ and $\Phi_{\overline{N}}(s,u)$, respectively. The striction curves of $\Phi_N(s,u)$ and $\Phi_{\overline{N}}(s,u)$ are given as follows:

$$\alpha_N(s) = n(s) \times n^*(s) - \mu.n(s),$$

$$\overline{\alpha}_N(s) = \overline{n}(s) \times \overline{n}^*(s) - \rho.\overline{n}(s).$$

After some calculations, we obtain the constant coefficients μ and ρ as follows:

$$\mu = \frac{\left\langle \dot{n}(s) \times \dot{n}^*(s), \dot{n}(s) \right\rangle}{\left\langle \dot{n}(s), \dot{n}(s) \right\rangle},$$

$$\mu = \frac{\left\langle (k_1(s).t(s) + k_2(s).b(s)) \times (k_1(s).t^*(s) + k_1^*(s).t(s) + k_2(s).b^*(s) + k_2^*(s).b(s)), (k_1(s).t(s) + k_2(s).b(s)) \right\rangle}{\left\langle (k_1(s).t(s) + k_2(s).b(s)), (k_1(s).t(s) + k_2(s).b(s)) \right\rangle},$$

$$\mu = 0$$

and

$$\rho = \frac{\left\langle \dot{\overline{n}}(s) \times \dot{\overline{n}}*(s), \dot{\overline{n}}(s) \right\rangle}{\left\langle \dot{\overline{n}}(s)(s), \dot{\overline{n}}(s) \right\rangle},$$

$$\rho = 0.$$

The distribution parameters of the ruled surfaces are given as:

$$P_N = \frac{\det(\frac{d\beta_N}{ds_N}, N(s), \dot{N}(s))}{\left\| \dot{N}(s) \right\|^2},$$

$$\overline{P}_{\overline{N}} = \frac{\det(\frac{d\overline{\beta}_N}{d\overline{s}_N}, \overline{N}(s), \dot{\overline{N}}(s))}{\left\| \dot{\overline{N}}(s) \right\|^2},$$

$$P_N = \frac{k_2(s)}{k_1^2(s) + k_2^2(s)}$$

and

$$\overline{P}_{\overline{N}} = \frac{k_1(s).k_1^*(s)\cos^2\Psi + (k_1(s).k_2^*(s) + k_2(s).k_1^*(s))\sin\Psi.\cos\Psi + k_2(s).k_2^*(s)\sin^2\Psi}{1 + (k_1\cos\Psi + k_2\sin\psi)^2}.$$

Corollary 3: Assume that $\Phi(s,u)$ and $\overline{\Phi}(s,u)$ are ruled surfaces. The binormal indicatrices of the main curve and its natural lift curve are given:

$$\Phi_B(s,u) = b(s) \times b^*(s) + u.b(s),$$

$$\overline{\Phi}_{\overline{B}}(s,u) = \overline{b}(s) \times \overline{b}^*(s) + u.\overline{b}(s).$$

Here $\beta_B(s) = b(s) \times b^*(s)$ and $\overline{\beta}_{\overline{B}}(s) = \overline{b}(s) \times \overline{b}^*(s)$ are base curves of $\Phi_B(s,u)$ and $\Phi_{\overline{B}}(s,u)$, respectively. The striction curves of $\Phi_B(s,u)$ and $\Phi_{\overline{B}}(s,u)$ are given as follows:

$$\alpha_B(s) = b(s) \times b^*(s) - \nu .b(s),$$

$$\overline{\alpha}_{\overline{B}}(s) = \overline{b}(s) \times \overline{b}^*(s) - \omega.\overline{b}(s).$$

After some calculations, we obtain the constant coefficients ν and w as follows:

$$\nu = \frac{\left\langle \dot{b}(s) \times \dot{b}^*(s), \dot{b}(s) \right\rangle}{\left\langle \dot{b}(s), \dot{b}(s) \right\rangle},$$

$$v=\frac{\langle(-k_2(s).n(s))\times(-k_2^*(s).n(s)-k_2(s).n^*(s)),(-k_2(s).n(s))\rangle}{\langle(-k_2(s).n(s)),(-k_2(s).n(s))\rangle},$$

$$v=0$$

and

$$w=\frac{\left\langle \dot{\bar{b}}(s)\times\dot{\bar{b}}^*(s),\dot{\bar{b}}(s)\right\rangle}{\left\langle \dot{\bar{b}}(s),\dot{\bar{b}}(s)\right\rangle},$$

$$w=\frac{A.B}{(1+A^2)}.$$

Here $A=k_1(s)\sin(\Psi)-k_2(s)\cos(\Psi)$ and $B=k_1^*(s)\sin(\Psi)-k_2^*(s)\cos(\Psi)$ are taken. The distribution parameters of the ruled surfaces are given as:

$$P_B=\frac{\det(\frac{d\beta_B}{ds_B},B(s),\dot{B}(s))}{\left\|\dot{B}(s)\right\|^2},$$

$$\bar{P}_{\bar{B}}=\frac{\det(\frac{d\bar{\beta}_B}{d\bar{s}_B},\bar{B}(s),\dot{\bar{B}}(s))}{\left\|\dot{\bar{B}}(s)\right\|^2},$$

$$P_B=\frac{1}{k_2(s)}$$

and

$$\bar{P}_{\bar{B}}=\frac{(\cos(\Psi))\langle\Psi,t(s)\rangle(\sin^2\Psi+A^2)+\langle\Psi,n(s)\rangle.A-\langle\Psi,b(s)\rangle(\sin(\Psi))(A^2+\cos^2\Psi)-B.(\cos(\Psi)).(\sin(\Psi))}{1+(A)^2}.$$

Example: Let $\gamma(s)=(\cos s,\sin s,0)$ be the curve on unit 2-sphere and $\nu(s)=(\sin s,-\cos s,5s)$ is the vector in IR^3. Because of $|\gamma(s)|=1$ and $\langle\gamma(s),\nu(s)\rangle=0$, $\Gamma(s)=(\gamma(s),\nu(s))$ is in TS^2. Assume that $\bar{\gamma}(s)=(-\sin s,\cos s,0)$ is the curve on the subset $\bar{M}$ of unit 2 sphere and $\bar{\nu}(s)=(\cos s,\sin s,5)$ is the vector in IR^3. Because of $|\bar{\gamma}(s)|=1$ and $\langle\bar{\gamma}(s),\bar{\nu}(s)\rangle=0$, $\bar{\Gamma}(s)=(\bar{\gamma}(s),\bar{\nu}(s))$ is in $T\bar{M}$ .The ruled surface generated by $\bar{\Gamma}(s)=(\bar{\gamma}(s),\bar{\nu}(s))$ is given as

$$\bar{\Phi}(s,u)=(5\cos s-u.\sin s,5\sin s+u.\cos s,-1).$$

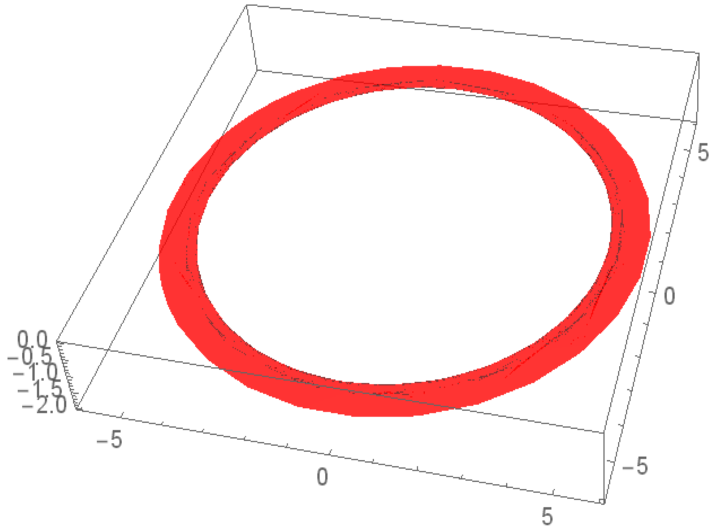
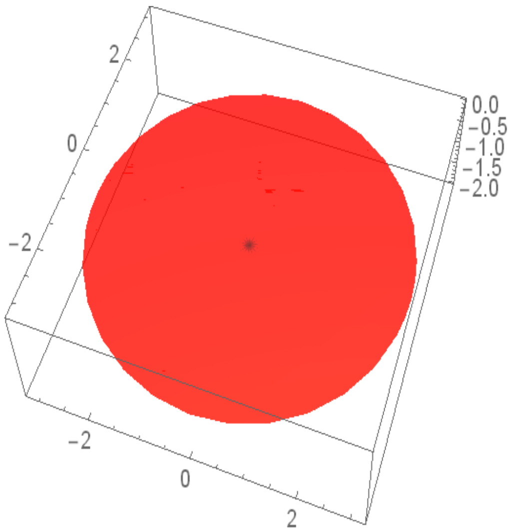
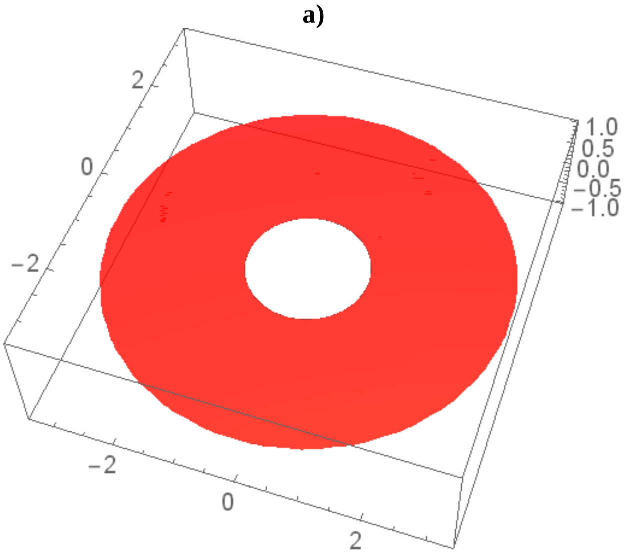


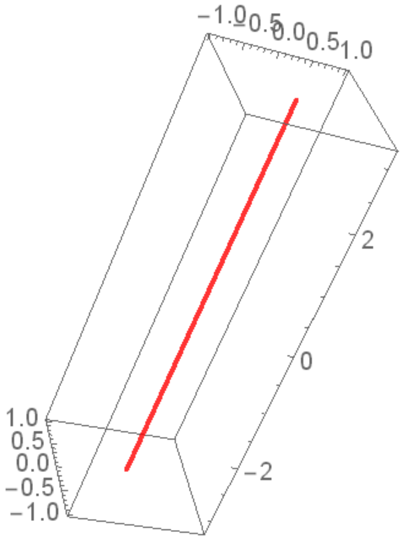
Figure 2. The ruled surface generated by the natural lift curve $\overline{\Gamma}(s) = (\overline{\gamma}(s), \overline{\nu}(s))$.



a)



b)



c)

Figure 3. Ruled surfaces generated by: a) tangent, b) principal normal and c) binormal indicatries.

The ruled surface generated by tangent, principal normal and binormal indicatries of the natural lift curve is presented as follows:

$$\overline{\Phi}_{\overline{T}}(s,u) = (-u.\cos s, -u.\sin s, -1),$$

$$\overline{\Phi}_{\overline{N}}(s,u) = (\sin s - u.\cos s, -\cos s - u.\sin s, 0),$$

$$\overline{\Phi}_{\overline{B}}(s,u) = (0,0,u).$$

6. CONCLUSION

Study mapping plays vital role to construct correspondence between Euclidean space and dual space. In this study, ruled surfaces generated by dual spherical curves of the natural lift curve are given by using isomorphism between the subset of tangent bundle of unit 2-sphere and unit dual sphere. Then distribution parameters and striction curves of ruled surfaces are calculated. Moreover, calculations are applied to an example. Obtained results have significiant role to model in physics, mathematics and computer modeling systems.

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