*ORIGINAL PAPER*

# ON THE BOUNDS FOR THE SPECTRAL NORMS OF *R*-CIRCULANT MATRICES WITH A TYPE OF CATALAN TRIANGLE NUMBERS

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***Abstract.*** *In this paper, we give lower and upper bounds for the spectral norms of the r-circulant matrices whose entries are*

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$$

*and*

$$kB_{n,k} = \frac{k^2}{n}\binom{2n}{n-k},$$

*respectively, where* $n \in \mathbb{N}, k = 0,1,2,...,n-1, \ k \le n$. *Then we present some bounds for the spectral norms of Kronecker and Hadamard products of these matrices.*

***Keywords:*** *Catalan numbers, binomial coefficients, spectral norm, r-circulant matrix.*
***2010 for the AMS Sub. Classification:*** *Primary 15A60; Secondary 11B83, 11B65.*

## 1. INTRODUCTION

The $n \times n$ $r$-circulant matrix $C_r = Circ_r\left(c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}\right)$ associated with the numbers $c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}$ is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \ge i \\ rc_{n+j-i}, & j < i \end{cases}$$

that is

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$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}.$$

Particularly, for $r = 1$,

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

is called a circulant matrix. For more informations and details about the circulant matrices, we refer the interested reader to [1].

Circulant and *r*-circulant matrices have been studied in several papers. For instance, in [2], Solak gave some bounds for the spectral norms of circulant matrices with the Fibonacci and Lucas number entries. Afterwards, Shen and Cen [3] generalized Solak's results. In [4], same authors studied spectral norms of the matrices $A = C_r\left(F_{k,0}, F_{k,1}, ..., F_{k,n-1}\right)$ and $B = C_r\left(L_{k,0}, L_{k,1}, ..., L_{k,n-1}\right)$. Later, many researchers studied spectral norms of circulant, *r*-circulant and geometric circulant matrices with special numbers. For more details, we refer the interested reader to [5-13].

Inspired by the above papers, in this paper, we give lower and upper bounds for the spectral norms of the *r*-circulant matrices whose entries are

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$$

and

$$kB_{n,k} = \frac{k^2}{n}\binom{2n}{n-k},$$

respectively, where $n \in \mathbb{N}, k = 0,1,2,...,n-1, k \le n$. Then we present some bounds for the spectral norms of Kronecker and Hadamard products of these matrices.

## 2. PRELIMINARIES

In [14], Shapiro presented the following triangle:

| $n/k$ | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | 2 | 1 | | | | | |
| 3 | 5 | 4 | 1 | | | | |
| 4 | 14 | 14 | 6 | 1 | | | |
| 5 | 42 | 48 | 27 | 8 | 1 | | |
| 6 | 132 | 165 | 110 | 44 | 10 | 1 | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

where the entries are defined by

$$B_{n,k}=\frac{k}{n}\binom{2n}{n-k}, \quad n,k\in\mathbb{N},\ k\le n.$$

The above triangle is called Catalan triangle because the numbers in the first column are Catalan numbers. The well known Catalan numbers are defined by the following recursive formula:

$$C_0=1,\ \ C_n=\sum_{k=0}^{n-1}C_kC_{n-1-k}. \qquad (n\ge 1)$$

The $n$th term of the Catalan sequence is given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}. \qquad (n\ge 1)$$

Shapiro also gave some summation formulas involving the $B_{n,k}$ numbers. An interesting one of them is

$$\sum_{k=1}^{n}\left(B_{n,k}\right)^2=C_{2n-1}. \tag{1}$$

Afterwards, Gutiérrez et al. [15] gave some new identities involving the $B_{n,k}$ and Catalan numbers. They obtained a nice proof for the following formula:

$$\sum_{k=1}^{n}\left(kB_{n,k}\right)^2=\left(3n-2\right)C_{2(n-1)}. \tag{2}$$

Now we give some definitions and lemmas related to our work.

Let $A=\left(a_{ij}\right)$ be any $m\times n$ matrix. Then the Euclidean (Frobenius) norm of matrix $A$ is

$$\|A\|_E=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2}$$

and also the spectral norm of matrix $A$ is

$$\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i\left(A^H A\right)}$$

where $\lambda_i\left(A^H A\right)$ is eigenvalue of $A^H A$ and $A^H$ is conjugate transpose of matrix $A$.

The following inequality between Euclidean norm and spectral norm is well known:

$$\frac{1}{\sqrt{n}}\|A\|_E\le\|A\|_2\le\|A\|_E. \tag{3}$$

**Lemma 2.1.** [16] Let $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$ be any $m\times n$ matrices. Then

$$\|A\circ B\|_2\le\|A\|_2\|B\|_2 \tag{4}$$

where $A\circ B$ is the Hadamard product of $A$ and $B$.

**Lemma 2.2.** [17] Let $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$ be any $m\times n$ matrices. Then

$$\|A\circ B\|_2\le r_1(A)c_1(B) \tag{5}$$

where

$$r_1(A)=\max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}\left|a_{ij}\right|^2}\quad\text{and}\quad c_1(B)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{m}\left|b_{ij}\right|^2}.$$

**Lemma 2.3.** [17] Let $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$ be any $m\times n$ matrices. Then

$$\|A\otimes B\|_2=\|A\|_2\|B\|_2 \tag{6}$$

where $A\otimes B$ is the Kronecker product of $A$ and $B$.

## 3. MAIN RESULTS

**Theorem 3.1.** Let $X = C_r\left(B_{n,0}, B_{n,1}, B_{n,2}, ..., B_{n,n-2}, B_{n,n-1}\right)$ be an $n \times n$ $r$-circulant matrix where $r \in \mathbb{C}$.

(i) If $|r| \geq 1$, we have

$$\sqrt{C_{2n-1} - 1} \leq \| X \|_2 \leq |r|\left(C_{2n-1} - 1\right),$$

(ii) If $|r| < 1$, we have

$$|r|\sqrt{C_{2n-1} - 1} \leq \| X \|_2 \leq \sqrt{(n-1)\left(C_{2n-1} - 1\right)}.$$

*Proof*. From the definition of $X = C_r\left(B_{n,0}, B_{n,1}, B_{n,2}, ..., B_{n,n-2}, B_{n,n-1}\right)$, we have the following matrix:

$$X = \begin{bmatrix} B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n-2} & B_{n,n-1} \\ rB_{n,n-1} & B_{n,0} & B_{n,1} & \cdots & B_{n,n-3} & B_{n,n-2} \\ rB_{n,n-2} & rB_{n,n-1} & B_{n,0} & \cdots & B_{n,n-4} & B_{n,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rB_{n,2} & rB_{n,3} & rB_{n,4} & \cdots & B_{n,0} & B_{n,1} \\ rB_{n,1} & rB_{n,2} & rB_{n,3} & \cdots & rB_{n,n-1} & B_{n,0} \end{bmatrix}.$$

Thus we obtain

$$\| X \|_E^2 = \sum_{k=0}^{n-1}(n-k)\left(B_{n,k}\right)^2 + \sum_{k=1}^{n-1} k|r|^2\left(B_{n,k}\right)^2.$$

(i) For $|r| \geq 1$, by the aid of Eq.(1), we obtain

$$\begin{aligned} \| X \|_E^2 &\geq \sum_{k=0}^{n-1}(n-k)\left(B_{n,k}\right)^2 + \sum_{k=1}^{n-1} k\left(B_{n,k}\right)^2 \\ &= n\sum_{k=0}^{n-1}\left(B_{n,k}\right)^2 \\ &= n\left(C_{2n-1} - 1\right) \end{aligned}$$

that is

$$\frac{1}{\sqrt{n}}\| X \|_E \geq \sqrt{C_{2n-1} - 1}.$$

So we have

$$\| X \|_2 \geq \sqrt{C_{2n-1} - 1}.$$

Now, we want to find an upper bound for the spectral norm of the matrix *X*. Let the matrices *D* and *E* be as

$$D = \begin{bmatrix} B_{n,0} & 1 & 1 & \cdots & 1 & 1 \\ rB_{n,n-1} & B_{n,0} & 1 & \cdots & 1 & 1 \\ rB_{n,n-2} & rB_{n,n-1} & B_{n,0} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rB_{n,2} & rB_{n,3} & rB_{n,4} & \cdots & B_{n,0} & 1 \\ rB_{n,1} & rB_{n,2} & rB_{n,3} & \cdots & rB_{n,n-1} & B_{n,0} \end{bmatrix}$$

and

$$E = \begin{bmatrix} B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n-2} & B_{n,n-1} \\ 1 & B_{n,0} & B_{n,1} & \cdots & B_{n,n-3} & B_{n,n-2} \\ 1 & 1 & B_{n,0} & \cdots & B_{n,n-4} & B_{n,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & B_{n,0} & B_{n,1} \\ 1 & 1 & 1 & \cdots & 1 & B_{n,0} \end{bmatrix}.$$

It is easily seen that $X = D \circ E$. Then we get

$$\begin{aligned} r_1(D) &= \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |d_{nj}|^2} \\ &= \sqrt{|r|^2 \sum_{k=1}^{n-1} (B_{n,k})^2 + (B_{n,0})^2} \\ &= |r| \sqrt{C_{2n-1} - 1} \end{aligned}$$

and

$$\begin{aligned} c_1(E) &= \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |e_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |e_{in}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} (B_{n,k})^2} \\ &= \sqrt{C_{2n-1} - 1}. \end{aligned}$$

By using Lemma 2.2, we have

$$\|X\|_2 \le r_1(D)\,c_1(E) = |r|\left(C_{2n-1} - 1\right).$$

Thus,

$$\sqrt{C_{2n-1} - 1} \le \|X\|_2 \le |r|\left(C_{2n-1} - 1\right).$$

(ii) For $|r| < 1$, by using Eq.(1), we get

$$\|X\|_E^2 \ge \sum_{k=0}^{n-1} (n-k)|r|^2 \left(B_{n,k}\right)^2 + \sum_{k=1}^{n-1} k|r|^2 \left(B_{n,k}\right)^2$$
$$= n|r|^2 \sum_{k=0}^{n-1} \left(B_{n,k}\right)^2$$

that is

$$\frac{1}{\sqrt{n}}\|X\|_E \ge |r|\sqrt{C_{2n-1} - 1}\,.$$

So we obtain

$$|r|\sqrt{C_{2n-1} - 1} \le \|X\|_2\,.$$

Let the matrices $F$ and $G$ be as

$$F = \begin{bmatrix} B_{n,0} & 1 & 1 & \cdots & 1 & 1 \\ r & B_{n,0} & 1 & \cdots & 1 & 1 \\ r & r & B_{n,0} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \cdots & B_{n,0} & 1 \\ r & r & r & \cdots & r & B_{n,0} \end{bmatrix}$$

and

$$G = \begin{bmatrix} B_{n,0} & B_{n,1} & B_{n,2} & \cdots & B_{n,n-2} & B_{n,n-1} \\ B_{n,n-1} & B_{n,0} & B_{n,1} & \cdots & B_{n,n-3} & B_{n,n-2} \\ B_{n,n-2} & B_{n,n-1} & B_{n,0} & \cdots & B_{n,n-4} & B_{n,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n,2} & B_{n,3} & B_{n,4} & \cdots & B_{n,0} & B_{n,1} \\ B_{n,1} & B_{n,2} & B_{n,3} & \cdots & B_{n,n-1} & B_{n,0} \end{bmatrix}$$

such that $X = F \circ G$. Then we get

$$r_1(F)=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|f_{ij}\right|^2}=\sqrt{n-1}$$

and

$$c_1(G)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|g_{ij}\right|^2}=\sqrt{C_{2n-1}-1}.$$

Thus we have

$$|r|\sqrt{C_{2n-1}-1}\le\|X\|_2\le\sqrt{(n-1)(C_{2n-1}-1)}.$$

**Theorem 3.2.** Let $Y=C_r\left(0,B_{n,1},2B_{n,2},...,(n-2)B_{n,n-2},(n-1)B_{n,n-1}\right)$ be an $n\times n$ $r$-circulant matrix where $r\in\mathbb{C}$.

(i) If $|r|\ge1$, we have

$$\sqrt{(3n-2)C_{2(n-1)}-n^2}\le\|Y\|_2\le|r|\left[(3n-2)C_{2(n-1)}-n^2\right],$$

(ii) If $|r|<1$, we have

$$|r|\sqrt{(3n-2)C_{2(n-1)}-n^2}\le\|Y\|_2\le\sqrt{(n-1)(3n-2)C_{2(n-1)}-(n-1)n^2}.$$

*Proof.* From the definition of $Y=C_r\left(0,B_{n,1},2B_{n,2},...,(n-2)B_{n,n-2},(n-1)B_{n,n-1}\right)$, we have the following matrix:

$$Y=\begin{bmatrix} 0 & B_{n,1} & 2B_{n,2} & \cdots & (n-2)B_{n,n-2} & (n-1)B_{n,n-1}\\ r(n-1)B_{n,n-1} & 0 & B_{n,1} & \cdots & (n-3)B_{n,n-3} & (n-2)B_{n,n-2}\\ r(n-2)B_{n,n-2} & r(n-1)B_{n,n-1} & 0 & \cdots & (n-4)B_{n,n-4} & (n-3)B_{n,n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 2rB_{n,2} & 3rB_{n,3} & 4rB_{n,4} & \cdots & 0 & B_{n,1}\\ rB_{n,1} & 2rB_{n,2} & 3rB_{n,3} & \cdots & r(n-1)B_{n,n-1} & 0 \end{bmatrix}.$$

Thus we obtain

$$\begin{aligned}\|Y\|_E^2&=\sum_{k=0}^{n-1}(n-k)\left(kB_{n,k}\right)^2+\sum_{k=1}^{n-1}k|r|^2\left(kB_{n,k}\right)^2\\&=\sum_{k=0}^{n-1}k^2(n-k)\left(B_{n,k}\right)^2+\sum_{k=1}^{n-1}k^3|r|^2\left(B_{n,k}\right)^2.\end{aligned}$$

(i) For $|r|\ge1$, by the aid of Eq.(2), we obtain

$$\begin{aligned}\|Y\|_E^2 &\geq \sum_{k=0}^{n-1} k^2 (n-k)\left(B_{n,k}\right)^2 + \sum_{k=1}^{n-1} k^3 \left(B_{n,k}\right)^2 \\ &= n\sum_{k=0}^{n-1}\left(kB_{n,k}\right)^2 \\ &= n(3n-2)C_{2(n-1)} - n^3\end{aligned}$$

that is

$$\frac{1}{\sqrt{n}}\|Y\|_E \geq \sqrt{(3n-2)C_{2(n-1)} - n^2}\ .$$

So we have

$$\sqrt{(3n-2)C_{2(n-1)} - n^2} \leq \|Y\|_2 .$$

Alternatively, let the matrices $H$ and $K$ be as

$$H = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ r(n-1)B_{n,n-1} & 0 & 1 & \cdots & 1 & 1 \\ r(n-2)B_{n,n-2} & r(n-1)B_{n,n-1} & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2rB_{n,2} & 3rB_{n,3} & 4rB_{n,4} & \cdots & 0 & 1 \\ rB_{n,1} & 2rB_{n,2} & 3rB_{n,3} & \cdots & r(n-1)B_{n,n-1} & 0 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 0 & B_{n,1} & 2B_{n,2} & \cdots & (n-2)B_{n,n-2} & (n-1)B_{n,n-1} \\ 1 & 0 & B_{n,1} & \cdots & (n-3)B_{n,n-3} & (n-2)B_{n,n-2} \\ 1 & 1 & 0 & \cdots & (n-4)B_{n,n-4} & (n-3)B_{n,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & B_{n,1} \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

such that $Y = H \circ K$ . Then we obtain

$$r_1(H) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|h_{ij}\right|^2} = \sqrt{|r|^2\sum_{k=1}^{n-1}\left(kB_{n,k}\right)^2} = |r|\sqrt{(3n-2)C_{2(n-1)} - n^2}$$

and

$$c_1(K)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|k_{ij}\right|^2}=\sqrt{\sum_{k=1}^{n-1}\left(kB_{n,k}\right)^2}=\sqrt{(3n-2)C_{2(n-1)}-n^2}\ .$$

Thus we get

$$\sqrt{(3n-2)C_{2(n-1)}-n^2}\le\|Y\|_2\le|r|\left[(3n-2)C_{2(n-1)}-n^2\right]$$

as desired.

(ii) For $|r|<1$, by using Eq.(2), we get

$$\begin{aligned}\|Y\|_E^2 &\ge\sum_{k=0}^{n-1}\left(k|r|\right)^2(n-k)\left(B_{n,k}\right)^2+\sum_{k=1}^{n-1}k^3|r|^2\left(B_{n,k}\right)^2\\ &=n|r|^2\sum_{k=1}^{n-1}\left(kB_{n,k}\right)^2\end{aligned}$$

that is

$$\frac{1}{\sqrt{n}}\|Y\|_E\ge|r|\sqrt{(3n-2)C_{2(n-1)}-n^2}\ .$$

So we get the lower bound as

$$|r|\sqrt{(3n-2)C_{2(n-1)}-n^2}\le\|Y\|_2\,.$$

On the other hand, let the matrices $M$ and $N$ be as

$$M=\begin{bmatrix}0&1&1&\cdots&1&1\\ r&0&1&\cdots&1&1\\ r&r&0&\cdots&1&1\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ r&r&r&\cdots&0&1\\ r&r&r&\cdots&r&0\end{bmatrix}$$

and

$$N=\begin{bmatrix} B_{n,0} & B_{n,1} & 2B_{n,2} & \cdots & (n-2)B_{n,n-2} & (n-1)B_{n,n-1} \\ (n-1)B_{n,n-1} & B_{n,0} & B_{n,1} & \cdots & (n-3)B_{n,n-3} & (n-2)B_{n,n-2} \\ (n-2)B_{n,n-2} & (n-1)B_{n,n-1} & B_{n,0} & \cdots & (n-4)B_{n,n-4} & (n-3)B_{n,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2B_{n,2} & 3B_{n,3} & 4B_{n,4} & \cdots & B_{n,0} & B_{n,1} \\ B_{n,1} & 2B_{n,2} & 3B_{n,3} & \cdots & (n-1)B_{n,n-1} & B_{n,0} \end{bmatrix}$$

such that $Y=M\circ N$. Then we have

$$r_1(M)=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|m_{ij}\right|^2}=\sqrt{n-1}$$

and

$$c_1(N)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|n_{ij}\right|^2}=\sqrt{(3n-2)C_{2(n-1)}-n^2}\ .$$

Thus we have

$$|r|\sqrt{(3n-2)C_{2(n-1)}-n^2}\le\|Y\|_2\le\sqrt{(n-1)(3n-2)C_{2(n-1)}-(n-1)n^2}\ .$$

**Corollary 3.3.** Let $X=C_r\left(B_{n,0},B_{n,1},...,B_{n,n-1}\right)$ and $Y=C_r\left(0,B_{n,1},2B_{n,2},...,(n-1)B_{n,n-1}\right)$ be $n\times n$ $r$-circulant matrices where $r\in\mathbb{C}$.

(i) If $|r|\ge 1$, we have

$$\|X\circ Y\|_2\le|r|^2\left(C_{2n-1}-1\right)\left[(3n-2)C_{2(n-1)}-n^2\right]$$

(ii) If $|r|<1$, we have

$$\|X\circ Y\|_2\le(n-1)\sqrt{\left(C_{2n-1}-1\right)\left[(3n-2)C_{2(n-1)}-n^2\right]}$$

*Proof*. The proof is obvious by Lemma 2.1, Theorem 3.1 and Theorem 3.2.

**Corollary 3.4.** Let $X=C_r\left(B_{n,0},B_{n,1},...,B_{n,n-1}\right)$ and $Y=C_r\left(0,B_{n,1},2B_{n,2},...,(n-1)B_{n,n-1}\right)$ be $n\times n$ $r$-circulant matrices where $r\in\mathbb{C}$.

(i) If $|r|\ge 1$, we have

$$\|X\otimes Y\|_2\ge\sqrt{\left(C_{2n-1}-1\right)\left[(3n-2)C_{2(n-1)}-n^2\right]}$$

and

$$\|X\otimes Y\|_2\le|r|^2\left(C_{2n-1}-1\right)\left[(3n-2)C_{2(n-1)}-n^2\right].$$

(ii) If $|r|<1$, we have

$$\left\| X \otimes Y \right\|_2 \geq |r|^2 \sqrt{\left(C_{2n-1}-1\right)\left[\left(3n-2\right)C_{2(n-1)}-n^2\right]}$$

and

$$\left\| X \otimes Y \right\|_2 \leq \left(n-1\right) \sqrt{\left(C_{2n-1}-1\right)\left[\left(3n-2\right)C_{2(n-1)}-n^2\right]}.$$

*Proof.* The proof is obvious by Lemma 2.3, Theorem 3.1 and Theorem 3.2.

## 4. CONCLUSIONS

In this paper, we have obtained some bounds for the spectral norms of the *r*-circulant $X = C_r\left(B_{n,0}, B_{n,1}, B_{n,2}, ..., B_{n,n-2}, B_{n,n-1}\right)$ and $Y = C_r\left(0, B_{n,1}, 2B_{n,2}, ..., \left(n-2\right)B_{n,n-2}, \left(n-1\right)B_{n,n-1}\right)$ matrices. Also we have presented some bounds for the spectral norms of Kronecker and Hadamard products of these matrices. Naturally the following questions come to one's mind:

- What are the eigenvalues and eigenvectors of these matrices?
- What are the determinants of these matrices?
- Are these matrices invertible? If true, then what are the inverses of these matrices?
- Are there closed-form expressions for the spectral norms of these matrices?

We leave these questions for future work.

## REFERENCES

[1] Davis, P. J., *Circulant Matrices*, A Wiley-Interscience Publication, 1979.
[2] Solak, S., *Appl. Math. Comput*., **160**, 125, 2005.
[3] Shen, S. Q., Cen, J. M., *Appl. Math. Comput*., **216**, 2891, 2010.
[4] Shen, S. Q., Cen, J. M., *Int. J. Contemp. Math. Sci*., **5**, 569, 2010.
[5] Bahşi, M., Solak, S., *J. Math. Inequal*., **8**, 693, 2014.
[6] Bahşi, M., *TWMS J. Pure Appl. Math*., **6**, 84, 2015.
[7] Bahşi, M., *J. Math. Inequal*., **10**, 445, 2016.
[8] Tuğlu, N., Kızılateş, C., *J. Inequal. Appl*., **2015**, Article ID: 253, 1, 2015.
[9] Tuğlu, N., Kızılateş, C., *Gazi Univ. J. Sci*., **28**, 447, 2015.
[10] Kızılateş, C., Tuğlu, N., *J. Inequal. Appl*., **2016**, Article ID: 312, 1, 2016.
[11] Kızılateş, C., Tuğlu, N., *Gazi Univ. J. Sci*., **31**, 555, 2018.
[12] Köme, C., Yazlık, Y., *J. Inequal. Appl*., **2017**, Article ID: 192, 1, 2017.
[13] Yazlık, Y., Taşkara, N., *J. Inequal. Appl*., **2013**, Article ID: 394, 1, 2013.
[14] Shapiro, L. W., *Discrete Math*., **14**, 83, 1976.
[15] Gutiérrez, J. M., Hernández, M. A., Miana, P. J., Romero, N., *J. Math. Anal. Appl*., **341**, 52, 2008.
[16] Mathias, R., *Linear Algebra Appl*., **139**, 269, 1990.
[17] Horn, R. A., Johnson, C. R., *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.