ORIGINAL PAPER

EXPRESSION OF RECIPROCAL SUM OF GAUSSIAN LUCAS SEQUENCES BY LAMBERT SERIES

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Abstract. *In this study, we first obtain infinite sums derived from the reciprocal of the Gaussian Lucas sequence using the Binet formula and some properties. Then, we express the infinite sums of these reciprocals in terms of the Lambert series.*

Keywords: *Binet's formula, Gaussian Lucas numbers, Lambert series, Reciprocal sum.*

1. INTRODUCTION

The Fibonacci sequence is perhaps one of the most well-known sequence and they have many interesting properties and important applications to diverse disciplines Fibonacci numbers and Lucas numbers. It is well known that generalized Fibonacci and Lucas numbers play an important role in many subjects such as algebra, geometry, and number theory. Their various elegant properties and wide applications have been studied by many authors, [1-4]. Horadam [5] has defined complex Fibonacci numbers and the generalization of the classical Fibonacci numbers to complex numbers. Then, Jordan [6] studied Gaussian Lucas numbers and extended some relationships, which are known about the common Fibonacci sequences to the complex Fibonacci sequences.

For any integer $n \geq 0$, the Gaussian Lucas numbers GL_n are defined by

$$GL_0 = 2 - i, \text{ and } GL_1 = 1 + 2i,$$

$$GL_n = GL_{n-1} + GL_{n-2}. \tag{1.1}$$

The n^{th} Gaussian Lucas number is given by the equality

$$GL_n = L_n + iL_{n-1},$$

where i is the imaginary unit which satisfies $i^2 = -1$.

There exists a Binet formula for the Gaussian Lucas numbers,

$$GF_n = \delta \cdot \alpha^n + \theta \cdot \beta^n, \tag{1.2}$$

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where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\delta = \frac{2-(1-\sqrt{5})i}{2}$ and $\theta = \frac{(\sqrt{5}-3)+(\sqrt{5}-1)i}{2}$.

Here, α and β satisfy the following equations

$$\alpha + \beta = 1, \alpha - \beta = \sqrt{5} \text{ and } \alpha\beta = -1. \quad (1.3)$$

Recent studies [7-9] show that there has been an increasing interest on reciprocal sums of the Fibonacci numbers. Indeed, the floor function has been used in all of the above studies. However, Horadam [10] obtained infinite sums of the reciprocals of Fibonacci numbers and some generalizations by the help of Lambert series.

Now, we recall the definition of Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}.$$

We speak of the Lambert series and generalized Lambert series respectively,

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \quad |x| < 1,$$

$$L(a,x) = \sum_{n=1}^{\infty} \frac{a \cdot x^n}{1-a \cdot x^n} \quad |x| < 1, |ax| < 1. \quad (1.4)$$

In this paper, we obtain infinite sums

$$\sum_{n=1}^{\infty} \frac{1}{GL_n}, \sum_{n=1}^{\infty} \frac{1}{GL_{2n}}, \; and \; \sum_{n=1}^{\infty} \frac{1}{GL_{2n-1}}$$

in terms of Lambert series.

2. MAIN RESULTS

Theorem 2.1.

i. If n is even, then

$$\sum_{n=1}^{\infty} \frac{1}{\mathrm{GL_n}} = \frac{\mathrm{i}\sqrt{\gamma}}{\gamma\sqrt{\lambda}} \left\{ \mathrm{L}\left(\frac{\mathrm{i}\sqrt{\lambda}}{\sqrt{\gamma}}, \beta \right) - \mathrm{L}\left(\frac{-\lambda}{\gamma}, \beta^2 \right) \right\},$$

ii. If n is odd, then

$$\sum_{n=1}^{\infty}\frac{1}{GL_n}=\frac{-\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta\right)-L\left(\frac{\lambda}{\gamma},\beta^2\right)\right\}.$$

Proof. Firstly, we let n be even. We consider

$$\frac{1}{GL_n}=\frac{1}{\delta\cdot\alpha^n+\theta\cdot\beta^n}$$
$$=\frac{1}{\delta\left(\alpha^n+\frac{\theta}{\delta}\cdot\beta^n\right)}$$
$$=\frac{1}{\delta}\cdot\frac{\beta^n}{(\alpha\beta)^n+\frac{\theta}{\delta}\cdot\beta^{2n}}. \tag{2.1}$$

Since n is even, then $(\alpha\beta)^n=(-1)^n=1$ and therefore,

$$\frac{1}{GF_n}=\frac{1}{\delta}\cdot\frac{\beta^n}{1+\frac{\theta}{\delta}\cdot\beta^{2n}}$$
$$=\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\cdot\frac{\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^n}{1-\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^n}-\frac{\frac{-\theta}{\delta}\cdot\beta^{2n}}{1+\frac{\theta}{\delta}\cdot\beta^{2n}}$$

Thus, we have

$$\sum_{n=1}^{\infty}\frac{1}{GF_n}=\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\sum_{n=1}^{\infty}\left(\frac{\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^n}{1-\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^n}\right)-\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\sum_{n=1}^{\infty}\left(\frac{\frac{-\theta}{\delta}\cdot\beta^{2n}}{1+\frac{\theta}{\delta}\cdot\beta^{2n}}\right).$$

By the help of Lambert series, we get

$$\sum_{n=1}^{\infty}\frac{1}{GL_n}=\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left\{L\left(\frac{i\sqrt{\theta}}{\sqrt{\delta}},\beta\right)-L\left(\frac{-\theta}{\delta},\beta^2\right)\right\},$$

where $\left|\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta\right|<1$ and $\left|\frac{-\theta}{\delta}\cdot\beta^2\right|<1$.

Secondly, we assume that n is odd. Then, $(\alpha\beta)^n=(-1)^n=-1$. Now, we impose this into the Eqn. (2.1) to get

$$\frac{1}{GL_n} = -\frac{1}{\delta} \cdot \frac{\beta^n}{1 - \frac{\theta}{\delta} \cdot \beta^{2n}}$$
$$= \frac{-\sqrt{\delta}}{\delta\sqrt{\theta}} \left(\frac{\frac{\sqrt{\theta}}{\sqrt{\theta}} \cdot \beta^n}{1 - \frac{\sqrt{\theta}}{\sqrt{\delta}} \cdot \beta^n} - \frac{\frac{\theta}{\delta} \cdot \beta^{2n}}{1 - \frac{\theta}{\theta} \cdot \beta^{2n}} \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{GL_n} = \frac{-\sqrt{\delta}}{\delta\sqrt{\theta}} \sum_{n=1}^{\infty} \left(\frac{\frac{\sqrt{\theta}}{\sqrt{\delta}} \cdot \beta^n}{1 - \frac{\sqrt{\theta}}{\sqrt{\delta}} \cdot \beta^n} \right) + \frac{\sqrt{\delta}}{\delta\sqrt{\theta}} \sum_{n=1}^{\infty} \left(\frac{\frac{\theta}{\delta} \cdot \beta^{2n}}{1 - \frac{\theta}{\delta} \cdot \beta^{2n}} \right)$$

and we have

$$\sum_{n=1}^{\infty} \frac{1}{GL_n} = \frac{-\sqrt{\delta}}{\delta\sqrt{\theta}} \left\{ L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}}, \beta \right) - L\left(\frac{\theta}{\delta}, \beta^2 \right) \right\},$$

where $\left| \frac{\sqrt{\theta}}{\sqrt{\delta}} \cdot \beta \right| < 1$ and $\left| \frac{\theta}{\delta} \cdot \beta^2 \right| < 1$.

Theorem 2.2. The following equality holds:

$$\sum_{n=1}^{\infty} \frac{1}{GL_{2n}} = \frac{i\sqrt{\delta}}{\delta\sqrt{\theta}} \left\{ L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}}, \beta^2 \right) - L\left(\frac{-\theta}{\delta}, \beta^4 \right) \right\}.$$

Proof. In the Eqn. (1.2), we consider $2n$ instead of n, then

$$\frac{1}{GL_{2n}} = \frac{1}{\delta \cdot \alpha^{2n} + \theta \cdot \beta^{2n}}$$
$$= \frac{1}{\delta\left(\alpha^{2n} + \frac{\theta}{\delta}\beta^{2n}\right)}$$
$$= \frac{1}{\delta} \cdot \frac{\beta^{2n}}{(\alpha\beta)^{2n} + \frac{\theta}{\delta} \cdot \beta^{4n}}.$$

Since $(\alpha\beta)^{2n} = 1$, we have

$$\frac{1}{GL_{2n}} = \frac{1}{\delta} \cdot \frac{\beta^{2n}}{1 + \frac{\theta}{\delta} \cdot \beta^{4n}}$$

$$= \frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left(\frac{\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^{2n}}{1-\frac{i\sqrt{\theta}}{\delta}\cdot\beta^{2n}}\right)-\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left(\frac{\frac{-\theta}{\delta}\cdot\beta^{4n}}{1+\frac{\theta}{\delta}\cdot\beta^{4n}}\right).$$

Therefore, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n}}=\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left(\sum_{n=1}^{\infty}\frac{\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^{2n}}{1-\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^{2n}}\right)-\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left(\sum_{n=1}^{\infty}\frac{\frac{-\theta}{\delta}\cdot\beta^{4n}}{1+\frac{\theta}{\delta}\cdot\beta^{4n}}\right),$$

which can be written in terms of Lambert series as:

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n}}=\frac{i\sqrt{\delta}}{\delta\sqrt{\theta}}\left\{L\left(\frac{i\sqrt{\theta}}{\sqrt{\delta}},\beta^2\right)-L\left(\frac{-\theta}{\delta},\beta^4\right)\right\},$$

where $\left|\frac{i\sqrt{\theta}}{\sqrt{\delta}}\cdot\beta^2\right|<1$ and $\left|-\frac{\theta}{\delta}\cdot\beta^4\right|<1$.

Lemma 2.3. We assume α,β,θ and δ are defined as in the Eqn. (1.2). Then,

$$\sum_{n=1}^{\infty}\frac{\beta^{2n-1}}{1-\frac{\theta}{\delta}\beta^{4n-2}}=\sum_{n=1}^{\infty}\frac{\beta^{n}}{1-\frac{\theta}{\delta}\beta^{2n}}-\sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\frac{\theta}{\delta}\beta^{4n}}.$$

Theorem 2.4. The following equality holds:

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n-1}}=-\frac{1}{\delta}\frac{\sqrt{\delta}}{\sqrt{\theta}}\left\{L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}},\beta\right)+L\left(\frac{\theta}{\delta},\beta^4\right)-L\left(\frac{\theta}{\delta},\beta^2\right)-L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}},\beta^2\right)\right\}.$$

Proof. In the Eqn. (1.2) we consider $2n-1$ instead of n, then

$$\frac{1}{GL_{2n-1}}=\frac{1}{\delta\cdot\alpha^{2n-1}+\theta\cdot\beta^{2n-1}}$$

$$=\frac{1}{\delta\left(\alpha^{2n-1}+\frac{\theta}{\delta}\beta^{2n-1}\right)}$$

$$=\frac{1}{\delta}\cdot\frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1}+\frac{\theta}{\delta}\cdot\beta^{4n-2}}.$$

Since $(\alpha\beta)^{2n-1}=-1$, we have

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n-1}}=-\frac{1}{\delta}\cdot\sum_{n=1}^{\infty}\frac{\beta^{2n-1}}{1-\frac{\theta}{\delta}\beta^{4n-2}}$$

and from Lemma 2.3 we get

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n-1}}=-\frac{1}{\delta}\cdot\left(\sum_{n=1}^{\infty}\frac{\beta^{n}}{1-\frac{\theta}{\delta}\beta^{2n}}-\sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\frac{\theta}{\delta}\beta^{4n}}\right)$$

$$=-\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot\left(\sum_{n=1}^{\infty}\frac{\frac{\sqrt{\theta}}{\sqrt{\delta}}\beta^{n}}{1-\frac{\sqrt{\theta}}{\sqrt{\delta}}\beta^{n}}\right)+\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot\left(\sum_{n=1}^{\infty}\frac{\frac{\theta}{\delta}\beta^{2n}}{1-\frac{\theta}{\delta}\beta^{2n}}\right)$$

$$+\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot\left(\sum_{n=1}^{\infty}\frac{\frac{\sqrt{\theta}}{\sqrt{\delta}}\beta^{2n}}{1-\frac{\sqrt{\theta}}{\sqrt{\delta}}\beta^{2n}}\right)-\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot\left(\sum_{n=1}^{\infty}\frac{\frac{\theta}{\delta}\beta^{4n}}{1-\frac{\theta}{\delta}\beta^{4n}}\right).$$

By the help of Lambert series, we have

$$\sum_{n=1}^{\infty}\frac{1}{GL_{2n-1}}=-\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}},\beta\right)-\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot L\left(\frac{\theta}{\delta},\beta^{4}\right)$$
$$+\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot L\left(\frac{\theta}{\delta},\beta^{2}\right)+\frac{1}{\delta}\cdot\frac{\sqrt{\delta}}{\sqrt{\theta}}\cdot L\left(\frac{\sqrt{\theta}}{\sqrt{\delta}},\beta^{2}\right).$$

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